Part III. Teaching Arithmetic

THE LIMITS OF LEARNING RULES

Learning rules and applying them are common to many children’s experiences in mathematics. Children are taught to add the ones first and then the tens in two-place addition, to borrow in subtraction, to put a zero in the second line in multiplication, to start from the left when doing long division, to multiply across the numerators and denominators to find the product of two fractions, to divide the numerators and denominators of fractions by the same number to simplify them, to line up the decimal points before adding, to count up the decimal places to see where to place the decimal point in the answer to a decimal multiplication problem. A blatant example of teaching for “what to do” understanding is the two-line rhyme for division of fractions:

Yours is not to reason why;
Just invert and multiply.

Sometimes the rules for algorithms do not even make sense to follow, yet children are asked not to veer from the prescribed procedure. For example, it is common for second graders to learn to add two-place numbers. This is usually first presented with no required regrouping and in vertical form:

\[
\begin{array}{c}
34 \\
\hline
+21 \\
\end{array}
\]

The procedure taught is standard—first you add the ones and then you add the tens. When a teacher notices a child adding the numbers in the tens place first, the teacher will most likely redirect the child to add the ones first instead. Why? It certainly makes no difference which you add first to arrive at the correct sum.

The teaching rationale, however, is that if children learn correctly in this simpler example, they will more easily transfer their learning to the next, and more complex, task of adding when regrouping is necessary. It seems as if the decision is made in the interest of supporting the children’s learning. However, teaching “what to do” in this instance is mainly to make children’s arithmetic pursuits easier a few pages further in the book.

What do children perceive from being taught this rule? Many merely learn to add the numbers on the right first, and then the numbers on the left, without thinking that first they are adding “ones” and then “tens.” To some children, a page of such problems looks much like a page of one-digit addition problems placed vertically, except that some of the numbers have been shoved together.

There is a danger in this instructional approach, even for those children who learn rules well. Such instruction teaches a child that it is okay for something not to make sense, that it is not necessary to understand the sense in what you are expected to do. This message is counterproductive to having children approach mathematics with the goal of understanding concepts and skills.

Other second graders, however, notice that they can do the exercises either way and get the same answer, but they know that they are supposed to add the right-hand column first. If they ask why, what might the teacher say? (What would you say?) The answer most likely will be a this-is-the-way-we-do-it answer or this-will-make-your-life-easier-later answer. There is no other reason why.

Lacking in this instructional approach is support for the child who is trying to make sense out of the procedure. The most obvious message to children is that arithmetic requires that you follow rules. It would be a bleak and joyless arithmetic picture for second
graders if they imagined all the rules they have to face in the school years that stretch ahead! Fortunately for us, most children don’t take this future view of life; hopefully, those that do will be forgiving.

Teaching “what to do” in mathematics is widespread practice. Check your students’ mathematics textbooks for instances of teaching by procedural approaches. Recall your own elementary and high school mathematics learning. Can you remember being taught long division, or how to find the area of a circle, or how to multiply algebraic expressions such as \((x + 3) (2x - 5)\) and not understanding why they worked? Think back on mathematics courses you have taken. Have you ever had the experience of passing a course, perhaps even with a good grade, yet feeling you did not really understand what you were taught?

There is nothing inherently or necessarily wrong in knowing rules and being able to apply them. It is important to realize, however, that teaching procedures and teaching procedures in relation to their meaning are two very different approaches to teaching arithmetic. When children do not have the broader understanding, they may lack the flexibility to deal with situations that may differ even slightly from the particular situations learned.

The ideas in this book are firmly rooted in the belief that teaching for understanding is essential. Children must see their learning task as one of making sense of whatever they’re studying. It is an irresponsible choice to teach children how to do a procedure without teaching them how to reason. Students should not be expected to do things by rote or be made to say things they do not honestly understand.

When children subtract in an example such as this: \[\begin{array}{c}
31 \\
\hline
-16
\end{array}\]

it is not uncommon for them to arrive at the incorrect answer of 25. They have “learned” to subtract, but when confronted with a problem that requires regrouping, children will frequently just take the smaller from the larger—a procedure they have been practicing in appropriate situations. They are following a rule they’ve learned before but are applying it to the wrong situation.

The following estimation question appeared on the 1982 National Assessment of Education Progress (NAEP):

Estimate the answer to \(3.04 \times 5.4\)

<table>
<thead>
<tr>
<th></th>
<th>Age 13</th>
<th>Age 17</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>1.6</td>
<td>28%</td>
</tr>
<tr>
<td>b.</td>
<td>16</td>
<td>21%</td>
</tr>
<tr>
<td>c.</td>
<td>160</td>
<td>18%</td>
</tr>
<tr>
<td>d.</td>
<td>1600</td>
<td>23%</td>
</tr>
<tr>
<td>e.</td>
<td>I don’t know</td>
<td>9%</td>
</tr>
</tbody>
</table>

The question was given on the test taken by both 13-year-olds and 17-year-olds. The item tested the ability to estimate that a bit more than 3 multiplied by a bit more than 5 is about 16. The answer of 16 is the only possible choice that makes mathematical sense. The following shows the percentages of each answer chosen in the two age groups.
However, on items that required the students to perform computations with decimals, both 13-year-olds and 17-year-olds were able to do so with 80 percent to 90 percent success. The fact that so many of our graduating high school students demonstrate such a disparity between following a rule in one situation and using their reasoning ability in another is discouraging.

**TEACHING “WHAT TO DO” VERSES TEACHING “WHAT TO DO AND WHY”**

Why is the practice of teaching the procedures of mathematics detached from meaning and applications of these procedures so pervasive in schools? There are a host of reasons that support the emphasis on teaching procedures. These reasons have to be understood before any widespread change in this practice will occur.

1. *Learning “what to do” is usually easier than learning “what to do and why.”* Actually learning “what to do” is sometimes much easier. Take the example of learning how to divide fractions. The standard algorithm is to turn the fraction on the right-hand side upside down and then multiply across the tops and bottoms. This rule is simpler to learn (and to teach) than the principle that division is the inverse of multiplication—or that dividing by a number is the same as multiplying by its multiplicative inverse, which in the case of dividing by a fraction means multiplying by its reciprocal. If the goal of the mathematics instruction is to prepare students to produce a page of correct answers, teaching the appropriate procedures will meet this goal more quickly and easily.

2. *The textbooks emphasize the learning of procedures.* The basic goal of textbooks and workbooks is to teach students to write correct answers, rather than to think and understand. This may sound harsh. After all, no textbook publishers or authors say or even suggest that they do not care if students think as long as they get the answers on the pages correct. Teachers’ guides urge teachers to teach for the underlying understanding and often provide additional suggestions for doing so. But the reality is there, implicitly or explicitly: Children will show their understanding by being able to complete the work on the pages. Writing correct answers is the primary goal of teaching “what to do”; thinking, understanding, and seeing relationships is the primary goal of teaching “what to do and why.”

3. *The pressure of tests looms.* Standardized tests are another reality. Teachers are accountable for students’ performance, and, in some communities, test scores are published in the local newspapers. For the most part, these tests evaluate students’ proficiency with mathematical procedures. The procedures themselves are more quickly and efficiently learned by focusing specifically on how to do them. The fact that students continue to do poorly on test items that require more than rote learning does not seem to be a compelling enough reason to change the current emphasis on procedural mathematics.

4. *It is difficult to assess if students understand the “why” of arithmetic.* It is not possible to tell what a student is thinking merely from the paper-and-pencil work they submit. Think back to the work of the Red Group in the fourth-grade class. They successfully performed the algorithm for multiplication but could explain what they had done only by resorting to rules they had learned. Judging their computational work alone would not reveal this important information.
5. All teachers do not understand the difference between teaching procedures and teaching reasoning in arithmetic. Many teachers, especially elementary teachers, do not feel competent or comfortable with mathematics. Although proficient with the computational algorithms, they themselves were probably taught these algorithms without learning the underlying reasoning behind them. It has been shown that teachers generally teach as they were taught. Besides, teachers cannot teach what they do not truly understand. These circumstances seriously hinder the effort to offer children a thinking mathematics curriculum.

Other reasons could be cited as well. Parental pressure often affects what teachers teach. Changing a habit is in itself difficult, especially without consistent support. Also, students make demands. Many teachers have had the experience of explaining the meaning behind a procedure, only to have the students listen, wait patiently, and finally say, “If you’ll just show me how to do it, I’ll do it.”

In contrast, what are the reasons for teaching arithmetic in the context of meaning and application, of teaching the “why” as well as the “what to do” in mathematics? These reasons, too, are important to consider.

1. When you understand why, your understanding and skills can be applied more easily to new tasks. For example, teaching how to add, subtract, multiply, and divide decimals from the procedural approach focuses primarily on the various rules for what to do with the decimal points; when you multiply, you count up the decimal places to figure where the decimal point goes in the answer; when you divide, you move decimal points when they appear in the divisor but not when they don’t appear. Should you forget the rule for one particular operation, a rule from another has no bearing.

If, however, you learn why the rules for each of the operations makes sense—if you learn to reason with decimals as well as to compute with them—then you will understand which answers are sensible in any situation. The example from the NAEP test illustrates the limitations of students’ learning procedures without reasoning. When learning procedurally, you not only have to memorize rules, but you also have to memorize which problem the rules work for, and often you have to memorize different rules for each type of problem.

More important than the ability to follow rules, children must develop the kind of understanding that allows them to apply their learning to the new and different situations they will be sure to meet. This requires they understand why as well as knowing how. It is not an either/or situation; both are necessary. Knowing only the rules for figuring percents is not sufficient for choosing the best savings or money market accounts, the type of mortgage that makes the most sense, or the best way to finance a new car. These decisions require understanding and judgments that extend beyond algorithmic thinking.

2. Learning the meaning in arithmetic procedures makes them easier to remember. When you understand the reasons behind rules, you are not keeping a large number of unrelated rules in your memory. A common error when adding fractions is for students to add the numerators and denominators; for example: $1/2 + 1/3 = 2/5$. Students who do this are merely following a rule and, unfortunately, a rule that is totally inappropriate in this instance. Much of mathematics requires looking for the sense in the situation, not merely following a rule. It does not make sense to start with $1/2$, add to it, and wind up with an answer, $2/5$, that is less than the amount you started with. Yet many students do not notice this inconsistency or even think about looking for it.
Another estimation question on the NAEP test produced distressing results:

Estimate the answer to $12/13 + 7/8$.

a. 1  
b. 2  
c. 19  
d. 21

For the 13-year-olds, approximately equal numbers of students chose each of the answers, with 24 percent making the correct choice. For the 17-year-olds, 37 percent made the correct choice. (What was the thinking behind each of the incorrect choices? Why would so many of our students answer incorrectly?)

Objectives for arithmetic instruction are usually organized into bite-sized pieces. This structure is often seen as making curriculum goals more manageable. However, beware of such objectives. Teaching children in bite-sized pieces does not necessarily help them learn anything other than bite-sized skills. Some children do make connections between the individual pieces. However, children who have difficulty usually have to repeat their experience with the pieces, as if a second go-around, or a third, will produce eventual success. Such instruction has never worked and never will work.

Breaking the learning of mathematics into tiny pieces is like giving children a heap of graham cracker crumbs and wondering why they have no concept of the whole graham cracker. It’s an attempt to simplify learning that may seem to be in the interest of children. But it is, in fact, counterproductive and in conflict with what is known about how children do learn. It is wrong to think that children cannot deal with meaning and complexity.

3. Learning to reason is a goal effective in itself and leads to the continued support of learning. All of us (hopefully) have experienced the joy of accomplishment that comes from figuring something out in order to produce a satisfying result. When children are taught to make sense out of mathematics, they receive support for seeing connections between ideas. Their connections can lead them to further learning in ways that do not occur when learning is approached as a series of unconnected events.

In his essay, “Nature Clearly Observed” (Daedalus, Spring 1983), David Hawkins describes the distinction ancient Greeks made in their language between what they called “arithmetic” and what they called “logistic.” He writes:

Arithmetic was the investigation of the world of numbers; logistic was a set of rules, to be memorized, for doing rote sums, differences, products, quotients. Arithmetic was a kind of science, always fresh and open to endless investigation. Logistic was a dull art, needed for bookkeeping and other such practices, which you could learn by rote. If you understood something of arithmetic, you could easily master the rules of logistic; if you forgot those rules, you could reinvent them. What we mainly try to teach in all those early years of schooling is logistic, not arithmetic. We drag, not lead, and the efficiency of learning is scandalously low.

The true measure of the failure of teaching only the “what to do” is the feeling of mathematical incompetence and negativity toward mathematics experienced by so many otherwise highly educated people. This rejection of mathematics, sadly so common, is a clear indication that something is very wrong. It is not possible to appreciate something you do not truly understand, and the charge to teachers is a crucial one—to teach mathematics so that children are encouraged to make sense out of all they learn to do.