

1. What is the value of c if $\sum_{n=2}^{\infty} (1+c)^{-n} = 2$?
2. Determine whether the series $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$ is convergent or divergent.
3. Determine whether the series $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}$ is convergent or divergent.
4. Approximate to 4 decimal places the sum of the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n n!}$.
5. Find the sum of the series $\sum_{n=1}^{\infty} (-1)^n \frac{x^n}{2^{2n} n!}$.
6. Write the MacLaurin series for $f(x) = \tan^{-1} x$.
7. Use series to solve the initial value problem $y'' - xy' - y = 0$, $y(0) = 1$, $y'(0) = 0$.
8. How many terms are needed in the MacLaurin series for $\ln(1+x)$ to estimate $\ln 1.4$ within 0.001?
9. Find the limit $\lim_{x \rightarrow 0} \frac{x - \tan^{-1} x}{x^3}$.
10. Find the domain of $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(n+1)!2^{2n+1}}$.

Solutions

1. Converges for $|1 + c| < 1$ to $\frac{(1+c)^{-2}}{1 - (1+c)^{-1}} = \frac{1}{c(1+c)}$. Solve $\frac{1}{c(1+c)} = 2$. Obtain $c = \frac{-1 \pm \sqrt{3}}{2}$. Only $c = \frac{1 - \sqrt{3}}{2}$ satisfies $|1 + c| < 1$.

2. Using L'Hopital's Rule $\lim_{x \rightarrow \infty} \frac{\sin(1/x)}{1/x} = 1$, so $\lim_{n \rightarrow \infty} \frac{\sin(1/n)}{1/n} = 1$. Use the Limit Comparison Theorem and the fact that the harmonic series diverges to conclude that the given series diverges.

3. The function $f(x) = \frac{\ln x}{x}$ has derivative $f'(x) = \frac{1 - \ln x}{x^2} < 0$ for $x > e$. Also $\lim_{x \rightarrow \infty} f(x) = 0$. Thus, $b_n = \frac{\ln n}{n}$ is a decreasing sequence with limit 0. By the Alternating series test, the given series converges.

4. This is an alternating series so need $b_{n+1} < 0.5 \cdot 10^{-5}$, i.e. $\frac{1}{2^{n+1}(n+1)!} < 0.5 \cdot 10^{-5}$.

From TI-83: $n \geq 5$. With $n = 5$ obtain $\sum_{n=1}^5 (-1)^n \frac{1}{2^n n!} \approx -0.3935$.

5. This is $\sum_{n=1}^{\infty} \frac{(-x/4)^n}{n!} = e^{-x/4} - 1$.

6. $f'(x) = \frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-x^2)^n$. Therefore, $f(x) = \int \sum_{n=0}^{\infty} (-x^2)^n dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C$. Since $f(0) = 0$ we obtain $C = 0$. Thus $f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$.

7. Write $y = c_0 + c_1x + c_2x^2 + \dots$. Conditions $y(0) = 1$ and $y'(0) = 0$ yield $c_0 = 1$ and $c_1 = 0$. Now $y'' - xy' - y = (2c_2 - c_0) + (2 \cdot 3c_3 - 2c_1)x + (3 \cdot 4c_4 - 3c_2)x^2 + (4 \cdot 5c_5 - 4c_3)x^3 + \dots$, so $c_2 = 1/2$, $c_3 = 0$, $c_4 = 1/(2 \cdot 4)$, $c_5 = 0$, $c_6 = 1/(2 \cdot 4 \cdot 6)$, etc. Thus $y = 1 + \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 2!} + \frac{x^6}{2^3 \cdot 3!} + \dots = e^{x^2/2}$.

8. If $f(x) = \ln(1+x)$ then $|f^{(n)}(x)| = \left| \frac{(n-1)!}{(1+x)^n} \right| \leq (n-1)!$ for $0 < x < 1$. With $M = (n-1)!$, $a = 0$, and $x = 0.4$, we get $|R_n(0.4)| \leq \frac{(0.4)^{n+1}}{n(n+1)}$. Using TI-83, the right hand side is smaller than 0.001 when $n \geq 4$.

9. From Pbm 6, $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$. Thus $\frac{x - \tan^{-1} x}{x^3} =$

$$\frac{x - (x - x^3/3 + x^5/5 - \dots)}{x^3} = \frac{x^3/3 - x^5/5 + \dots}{x^3} \rightarrow \frac{1}{3} \text{ when } x \rightarrow 0.$$

10. Using the Ratio Test: $\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^2}{4(n+1)(n+2)} \right| \rightarrow 0$ when $n \rightarrow \infty$. Thus the series converges for all x , and the domain of f is the set of all real numbers.