1. What is the value of \( c \) if \( \sum_{n=2}^{\infty} (1 + c)^{-n} = 2? \)

2. Determine whether the series \( \sum_{n=1}^{\infty} \sin \left( \frac{1}{n} \right) \) is convergent or divergent.

3. Determine whether the series \( \sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n} \) is convergent or divergent.

4. Approximate to 4 decimal places the sum of the series \( \sum_{n=1}^{\infty} (-1)^n \frac{1}{2^n n!} \).

5. Find the sum of the series \( \sum_{n=1}^{\infty} (-1)^n \frac{x^n}{2^{2n} n!} \).

6. Write the MacLaurin series for \( f(x) = \tan^{-1} x \).

7. Use series to solve the initial value problem \( y'' - xy' - y = 0, \ y(0) = 1, \ y'(0) = 0 \).

8. How many terms are needed in the MacLaurin series for \( \ln(1 + x) \) to estimate \( \ln 1.4 \) within 0.001?

9. Find the limit \( \lim_{x \to 0} \frac{x - \tan^{-1} x}{x^3} \).

10. Find the domain of \( f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{n!(n + 1)!2^{2n+1}} \).
Solutions

1. Converges for |1 + c| < 1 to \( \frac{(1 + c)^{-2}}{1 - (1 + c)^{-1}} = \frac{1}{c(1 + c)} \). Solve \( \frac{1}{c(1 + c)} = 2 \). Obtain 
   
   \[ c = \frac{-1 \pm \sqrt{3}}{2} \]. Only \( c = \frac{1 - \sqrt{3}}{2} \) satisfies |1 + c| < 1.

2. Using L'Hopital’s Rule \( \lim_{x \to \infty} \frac{\sin(1/x)}{1/x} = 1 \), so \( \lim_{n \to \infty} \frac{\sin(1/n)}{1/n} = 1 \). Use the Limit Comparison Theorem and the fact that the harmonic series diverges to conclude that the given series diverges.

3. The function \( f(x) = \frac{\ln x}{x} \) has derivative \( f'(x) = \frac{1 - \ln x}{x^2} < 0 \) for \( x > e \). Also \( \lim_{x \to \infty} f(x) = 0 \). Thus, \( b_n = \frac{\ln n}{n} \) is a decreasing sequence with limit 0. By the Alternating series test, the given series converges.

4. This is an alternating series so need \( b_{n+1} < 0.5 \cdot 10^{-5} \), i.e. \( \frac{1}{2n+1(n+1)!} < 0.5 \cdot 10^{-5} \).

From TI-83: \( n \geq 5 \). With \( n = 5 \) obtain \( \sum_{n=1}^{5} (-1)^n \frac{1}{2^n n!} \approx -0.3935 \).

5. This is \( \sum_{n=1}^{\infty} \frac{(-x/4)^n}{n!} = e^{-x/4} - 1 \).

6. \( f'(x) = \frac{1}{1 + x^2} = \sum_{n=0}^{\infty} (-x^2)^n \). Therefore, \( f(x) = \int \sum_{n=0}^{\infty} (-x^2)^n \ dx = \sum_{n=0}^{\infty} \int (-1)^n x^{2n} \ dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C \). Since \( f(0) = 0 \) we obtain \( C = 0 \). Thus \( f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \).

7. Write \( y = c_0 + c_1 x + c_2 x^2 + \ldots \). Conditions \( y(0) = 1 \) and \( y'(0) = 0 \) yield \( c_0 = 1 \) and \( c_1 = 0 \). Now \( y'' - xy' - y = (2c_2 - c_0) + (2 \cdot 3c_3 - 2c_1)x + (3 \cdot 4c_4 - 3c_2)x^2 + (4 \cdot 5c_5 - 4c_3)x^3 + \ldots \), so \( c_2 = 1/2, c_3 = 0, c_4 = 1/(2 \cdot 4), c_5 = 0, c_6 = 1/(2 \cdot 4 \cdot 6) \), etc. Thus \( y = 1 + \frac{x^2}{2!} + \frac{x^4}{3!} + \ldots = e^{x^2/2} \).

8. If \( f(x) = \ln(1 + x) \) then \( |f^{(n)}(x)| = \frac{(n-1)!}{(1 + x)^n} \leq (n-1)! \) for \( 0 < x < 1 \). With \( M = (n-1)!, a = 0, \) and \( x = 0.4 \), we get \( |R_n(0.4)| \leq \frac{(0.4)^{n+1}}{n(n+1)} \). Using TI-83, the right hand side is smaller than 0.001 when \( n \geq 4 \).

9. From Pbm 6, \( \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \ldots \). Thus \( \frac{x - \tan^{-1} x}{x^3} = \frac{x^3/3 - x^5/5 + \ldots}{x^3} \to \frac{1}{3} \) when \( x \to 0 \).
10. Using the Ratio Test: \[
\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^2}{4(n+1)(n+2)} \right| \to 0 \text{ when } n \to \infty. \]
Thus the series converges for all \( x \), and the domain of \( f \) is the set of all real numbers.