Honors Math/Physics Integrating Seminar
Worksheet on the First Kepler Law

1. (a) In the previous worksheet we have seen that \( \mathbf{p}(t) \times \mathbf{v}(t) = \mathbf{c} \). Write \( \mathbf{p}(t) = r(t) \mathbf{u}(t) \), where \( \mathbf{u}(t) \) is a vector of magnitude 1. Using properties of the cross product derive a formula for \( \mathbf{c} \) in terms of \( \mathbf{u}(t) \), \( \mathbf{u}'(t) \), and \( r(t) \) (no \( r'(t) \), please):

\[
\mathbf{c} = \quad \text{(1)}
\]

(b) By Newton’s Second Law: \( \mathbf{F} = m\mathbf{a}(t) \), and by Newton’s Law of Gravitation \( \mathbf{F} = -\frac{GMm}{||\mathbf{p}(t)||^{3}}\mathbf{p}(t) \). Combine these with (1) to obtain an expression for \( \mathbf{a}(t) \times \mathbf{c} \) in terms of \( \mathbf{u}(t) \) and \( \mathbf{u}'(t) \):

\[
\mathbf{a}(t) \times \mathbf{c} = \quad \text{(2)}
\]

(c) Use the property 1(c) and (5) (Cross Product Worksheet I) to obtain the formula for \( \mathbf{a}(t) \times \mathbf{c} \) in terms of \( \mathbf{u}'(t) \):

\[
\mathbf{a}(t) \times \mathbf{c} = \quad \text{(3)}
\]

(d) Integrate both sides of (3) to obtain the formula for \( \mathbf{v}(t) \times \mathbf{c} \). [Do not forget the constant of integration \( \mathbf{A} = (A_1, A_2, A_3)! \)]

\[
\mathbf{v}(t) \times \mathbf{c} = \quad \text{(4)}
\]
2. (a) We will choose the coordinate system so that $c$ has the direction of positive $z$-axis which means that the planet moves in the $xy$-plane. In addition, we will set the positive $x$-axis in the direction of $A$. We will denote by $\theta(t)$ the angle between the $x$-axis and $p(t)$ measured counterclockwise. Using (4) write an expression for the scalar quantity $p(t) \cdot (v(t) \times c)$ and solve it for $r = r(t)$:

\[
\begin{align*}
(5) & \quad r = \\
(6) & \quad r = \frac{\epsilon d}{1 + \epsilon \cos \theta}
\end{align*}
\]

(b) Show that for any 3 vectors $a, b, c$ we have $a \cdot (b \times c) = (a \times b) \cdot c$.

(c) Use (b) to obtain that

\[
\begin{align*}
(6) & \quad r = \frac{\epsilon d}{1 + \epsilon \cos \theta}
\end{align*}
\]

for some constants $d$ and $e$. 
3. (a) We will show that the equation (6) represents an ellipse with a focus in the origin. Start by drawing a point \((x, y)\) in the plane, and label \(x, y, r, \) and \(\theta\) in your picture.

(b) Using the picture express \(r\) and \(\theta\) in terms of \(x\) and \(y\):

\[
\begin{align*}
\theta &= \\
r &= 
\end{align*}
\]

(c) Using formulas in (b) write (6) in the form \(\frac{(x - h)^2}{a^2} + \frac{y^2}{b^2} = 1\) and express \(a, b,\) and \(h\) in terms of \(d\) and \(e\).

\[
\begin{align*}
a &= \\
b &= \\
h &= 
\end{align*}
\]

(d) Now that we know that the planet follows an ellipse let us show that the sun is in a focus. Compute \(c = \sqrt{a^2 - b^2}\) and show that it equals \(-h\).