ON EXTENDED EIGENVALUES OF OPERATORS

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ABSTRACT. A complex number $\lambda$ is an extended eigenvalue of an operator $A$ if there is a nonzero operator $X$ such that $AX = \lambdaXA$. We characterize the set of extended eigenvalues for operators acting on finite dimensional spaces, finite rank operators, Jordan blocks, and $C_0$ contractions. We also describe the relationship between the extended eigenvalues of an operator $A$ and its powers. We derive some applications of this result to the commutant of an operator.

1. Introduction and Preliminaries

Let $\mathcal{H}$ be a complex Hilbert space, and denote by $L(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. It was shown in [10] that to each operator $A$ in $L(\mathcal{H})$ one can associate an algebra $B_A$ (a spectral algebra in the terminology of [10]) with some interesting properties. In particular, when $K$ is a compact operator, the algebra $B_K$ has a nontrivial invariant subspace (n. i. s.). This result can be compared to the celebrated theorem of Lomonosov [13] since $B_A$ always contains the commutant $\{A\}'$ of $A$. An important question left open in [10] was whether this inclusion is, in the case of compact operators, proper. It was noticed in the same paper that, if $\lambda$ is a complex number such that $|\lambda| \leq 1$, and if there is an operator $X$ such that

\begin{equation}
AX = \lambdaXA
\end{equation}

then $X \in B_A$. Thus, one way of making progress on the question above is to study the equation (1.1), especially in the case when $A$ is compact.

In this paper we make a modest contribution towards this goal by considering equation (1.1) when $A$ belongs to one of the classes that are related to compact operators: operators on a finite dimensional space, finite rank operators, Jordan blocks, or $C_0$ contractions (see definitions below). In view of the terminology of [10], a scalar $\lambda$ is called an extended eigenvalue of $A$, if there exists a nonzero $X$ in $L(\mathcal{H})$ satisfying (1.1). We refer to $X$ as the corresponding eigenoperator. The set of all extended eigenvalues of $A$ will be called the extended point spectrum and it will be denoted as $E(A)$. Clearly, 1 always belongs to $E(A)$ and therefore, we will refer to any complex number $\lambda$ belonging to the set $E(A) \setminus \{1\}$ as a nontrivial extended eigenvalue. In this terminology, we will characterize the extended point spectrum of an operator $A$ in each of the mentioned classes.

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Equation (1.1) has been studied in the work of S. Brown [6] and, independently, of Kim, Pearcy, and Shields [16]. Both of these papers deal with the case when $A$ is a compact operator that possesses a nontrivial extended eigenvalue. More recently, Brooke, Busch and Pearson [5] studied (1.1) and showed that in case $A$ is selfadjoint, there exists a nonzero selfadjoint solution $X$ to (1.1) if and only if $\lambda = \pm 1$, i.e., one has either commutation or anti-commutation relations. More recent work on this topic includes [12], [17], [7], [8], [3], and [9].

We start by giving a complete characterization of $E(A)$ when $A$ is an operator on a finite dimensional space. We show that in this case there are 3 different scenarios depending on the spectrum $\sigma(A)$ of $A$ (Corollaries 2.6–2.8). A nontrivial situation occurs when $A$ is an invertible operator with more than one eigenvalue. In this case, $\lambda$ is an extended eigenvalue if and only if it is a quotient of two eigenvalues of $A$. The results of this section also yield the description of the extended point spectrum of finite rank operators (Proposition 2.9).

In Section 3 we study Jordan blocks. Recall that a Jordan block $S(\theta)$ associated with an inner function $\theta$ is a compression of the unilateral shift $S$ to the subspace $\mathcal{H}(\theta) \equiv H^2 \ominus \theta H^2$. Since $\mathcal{H}(\theta)$ is invariant under $S^*$, we see that $S$ is an isometric lifting of $S(\theta)$. (An operator $\tilde{T} \in \mathcal{L}(\mathcal{H})$ is a lifting of $T$ acting on $\mathcal{M} \subset \mathcal{H}$ if $P\tilde{T} = P\tilde{T}P = TP$, where $P$ is the orthogonal projection in $\mathcal{L}(\mathcal{H})$ with range $\mathcal{M}$.) We will establish a connection between the extended point spectra of an operator and of its isometric lifting and exploit properties of the shift $S$ to obtain a description of $E(S(\theta))$. One knows that every inner function admits a unique factorization (up to a constant factor of absolute value 1) into a product $b(z)s(z)$, where $b$ is a Blaschke product and $s$ is a singular function. We will refer to this factorization as the canonical factorization. Our main result of Section 3 is a characterization of the extended point spectrum of a Jordan block $S(\theta)$ in terms of the zeros of $b$ and the singular measure $\mu$ corresponding to $s$ (Theorem 3.10). Just as in the case of operators on a finite dimensional space, quotients of eigenvalues of $S(\theta)$ (and the latter are precisely zeros of $b$) belong to $E(S(\theta))$. However, in this case, part of the extended point spectrum may be present due to the properties of $\mu$. Namely, using notation $\mu_\lambda(E) \equiv \mu(\lambda E)$, if $\mu$ and $\mu_\lambda$ are not mutually singular then $\lambda \in E(S(\theta))$.

It is well known (cf., [1, Proposition II.1.18]) that each Jordan block belongs to the class of $C_0$ contractions. Recall that a contraction $A$ is said to be of class $C_0$ if there exists a nonzero function $m$ in $H^\infty$ satisfying $m(A) = 0$. In fact, there is a unique inner function $m_A$ with this property that is minimal, i.e., if $m(A) = 0$ for another inner function $m$ then $m_A$ divides $m$. In particular, $\theta$ is the minimal function for $S(\theta)$. Thus, results of Section 3 can be viewed as a characterization of the extended point spectrum in terms of the minimal function. In Section 4 we obtain an analogous characterization of the extended point spectrum for arbitrary $C_0$ contractions. We employ the fact that a $C_0$ contraction $A$ is quasisimilar to a direct sum of operators, one of which is a Jordan block. Recall that operators $A$ and $B$ are quasisimilar if there exist quasiaffinities $Y$ and $Z$ such that $AY = YB$ and $ZA = BZ$. (An operator is a quasiaffinity if it is injective and has dense range.) We
show that the extended point spectrum is invariant under a quasisimilarity. This allows us to extend the results of Section 3 to all $C_0$ contractions (Theorem 4.5).

Finally, in Section 5 we study the relationship between the extended point spectrum of an operator $A$ and its powers $A^n$, $n = 1, 2, \ldots$. The motivation for this study comes from the fact that there are examples of operators that are not compact but some of their powers may be. We show that $E(A^n) = (E(A))^n$ (Theorem 5.2). It turns out that this result has some unexpected applications to the commutant of an operator. Namely, if at least one of the operators $A$ and $A^*$ has trivial kernel, we characterize the equality of the commutants of $A$ and $A^n$ in terms of $E(A)$ (Theorem 5.6). We mention that the question as to when $A^n B = B A^n$ implies $AB = BA$ can be considered as a special case of the following general problem: Assume that $f(A)$ and $g(B)$ are defined for two functions $f$ and $g$. When does $f(A)g(B) = g(B)f(A)$ imply that $AB = BA$? In case $f(x) = g(x) = e^x$, this problem has been of interest in control theory (cf. [2]) and sufficient conditions, which guarantee this implication, have been discussed by several authors (cf. [20], [21], [19], [15]).

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2. Finite Dimensional Hilbert Space

In this section, we consider finite dimensional Hilbert space $H$ and the extended point spectra of operators on $H$. We will obtain a characterization of $E(A)$ for an arbitrary operator $A$. As we will see, a major role in this context is being played by the eigenvalues of $A$, and the assumption on the dimension of $H$ is used to guarantee their existence. The following theorem of Rosenblum [18] gives a necessary condition for the existence of an eigenoperator.

**Theorem 2.1.** If $A$ and $B$ are two operators on Hilbert space such that $\sigma(A) \cap \sigma(B) = \emptyset$, then $X = 0$ is the only solution to the operator equation $AX - XB = 0$.

By using $B = \lambda A$ we get an immediate consequence.

**Proposition 2.2.** For any operator $A$ we have

$$E(A) \subset \{ \lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset \}.$$

In particular, if $\sigma(A) = \{ \alpha \}$ for some nonzero complex number $\alpha$, then $E(A) = \{ 1 \}$.

One knows that Theorem 2.1 and, hence, Proposition 2.2 holds when $H$ is infinite dimensional. That way we get a generalization of the example in [12] related to the Volterra operator.

**Corollary 2.3.** If $A$ is a quasinilpotent operator on Hilbert space then, for $\alpha \neq 0$, $E(\alpha + A) = \{ 1 \}$. 

Thus, \( E \) to kernel of \( A \).

Proof. First we consider the case when \( A \) is invertible. In this situation both \( A \) and \( A^* \) have nontrivial kernels. Let \( X' \) be a nonzero operator from kernel of \( A^* \) to kernel of \( A \). Define \( X = X'P \) where \( P \) denotes the orthogonal projection on kernel of \( A^* \). Clearly, \( X \neq 0 \). Note further that, in this case, \( AX \) and \( XA \) are both zero and thus \( AX = AXA \) for any \( \lambda \in \mathbb{C} \). Consequently, \( \mathbb{E}(A) = \mathbb{C} \). On the other hand, since \( A \) is not invertible, for any complex number \( \lambda \), \( 0 \in \sigma(A) \cap \sigma(\lambda A) \). Thus, \( \mathbb{E}(A) = \mathbb{C} = \{ \lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset \} \).

Now assume that \( A \) is invertible so that \( 0 \notin \sigma(A) \). In view of Proposition 2.2 it suffices to show that \( \{ \lambda \in \mathbb{C} : \sigma(A) \cap \sigma(\lambda A) \neq \emptyset \} \subset \mathbb{E}(A) \). So suppose that \( \mu \) is a (necessarily nonzero) complex number such that \( \mu \in \sigma(A) \) and \( \mu \notin \sigma(\lambda A) \). Since \( \mu \in \sigma(A) \) there exists a vector \( u \) such that \( Au = \mu u \). On the other hand, \( \mu \in \sigma(\lambda A) \) implies that \( \lambda \neq 0 \) so \( \mu/\lambda \in \sigma(A) \). Therefore, \( (\mu/\lambda) \in \sigma(A^*) \) and there is a vector \( v \) such that \( A^*v = (\mu/\lambda)v \). Let \( X = u \otimes v \). Then \( AX = \lambda XA \) and consequently \( \lambda \in \mathbb{E}(A) \). \( \square \)

From this theorem we can derive some simple consequences that we deem worth stating. In each of the following results \( A \) is an operator acting on a finite dimensional Hilbert space.

Corollary 2.6. If \( A \) is invertible then \( \mathbb{E}(A) = \{ \lambda/\mu : \lambda, \mu \in \sigma(A) \} \).

Corollary 2.7. \( \mathbb{E}(A) = \{ 1 \} \) if and only if \( \sigma(A) = \{ \alpha \} \), \( \alpha \neq 0 \).

Corollary 2.8. \( \mathbb{E}(A) = \mathbb{C} \) if and only if \( 0 \notin \sigma(A) \).

As mentioned in the introduction we are interested to what extent can the results of this section be extended to operators on infinite dimensional Hilbert space. Unfortunately, much of the analysis is based on the presence of eigenvalues which an operator need not have in general. In fact, neither Theorem 2.5 nor Corollaries 2.6 – 2.8 carry over. Indeed, one knows that the spectrum of the Volterra operator \( V \) on \( L^2(0, 1) \) is \( \{ 0 \} \) but it was shown in [3] that \( \mathbb{E}(V) = (0, \infty) \). Since, as it is easy to see, \( \{ \lambda \in \mathbb{C} : \sigma(V) \cap \sigma(\lambda V) \neq \emptyset \} = \mathbb{C} \), we see that the inclusion in Proposition 2.2 can be proper. Similarly, the aforementioned example from [4] shows that the equality in Corollary 2.6 is reduced to an inclusion, while the equivalence in Corollary 2.7 is an implication. Finally, Volterra operator shows that \( 0 \notin \sigma(A) \) need not imply \( \mathbb{E}(A) = \mathbb{C} \). The converse is true since, when \( A \) is invertible, \( \sigma(A) \) and \( \sigma(\lambda A) \) are disjoint when \( |\lambda| \) is large enough. Of course, one can generalize Corollary 2.8 to infinite dimensional Hilbert space if one is willing to make a stronger assumption.
Proposition 2.9. Suppose $A$ is an operator on a Hilbert space such that both $A$ and $A^*$ have nontrivial kernel. Then, $E(A) = \mathbb{C}$.

Remark 2.10. Although the hypotheses of Proposition 2.9 may seem too strong, we mention that it settles the question of the extended point spectrum for all finite rank operators.

3. Jordan blocks

In this section, we characterize the extended point spectrum of a Jordan block $S(\theta)$ in terms of the inner function $\theta$. First we will show that there is a strong connection between the extended point spectrum of a contraction and its isometric lifting (Theorem 3.2). Since $S(\theta)$ has the unilateral shift $S$ as an isometric lifting we will solve equation (1.1) when $A = S$. Finally, we will give a description of $E(S(\theta))$ in terms of the inner function $\theta$.

In this section we will work with equation

\[XA = \beta AX,\]

instead of (1.1) with the assumption that $\beta \neq 0$. Once we characterize complex numbers $\beta$ satisfying (3.1), the description of extended eigenvalues of $S(\theta)$ will follow immediately. The reason for switching to (3.1) is the lack of symmetry in the following modification of the Sz.-Nagy–Foias commutant lifting theorem ([14]).

Theorem 3.1. Let $T_1$ and $T_2$ be two contractions on spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. Let $U_1$ on $\mathcal{K}_1(\supset \mathcal{H}_1)$ be an isometric lifting of $T_1$, and let $U_2$ on $\mathcal{K}_2(\supset \mathcal{H}_2)$ be a contractive lifting of $T_2$. If $B : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is a contraction satisfying $BT_1 = T_2B$, then there exists a contraction $\hat{B} : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ such that

\[\hat{B}U_1 = U_2\hat{B}, P_2\hat{B} = BP_1,\]

where $P_i$ is the orthogonal projection on $\mathcal{H}_i$ in $\mathcal{L}(\mathcal{K}_i), i = 1, 2$.

Proof. Let $U'_2$ on $\mathcal{K}_2(\supset \mathcal{K}_2)$ be an isometric lifting of the contraction $U_2$. If $P_{\mathcal{K}_2}$ is the projection on $\mathcal{K}_2$ in $\mathcal{L}(\mathcal{K}_2')$ then $P_{\mathcal{K}_2}U'_2 = U_2P_{\mathcal{K}_2}$. Notice that $U'_2$ is an isometric lifting of $T_2$. By the commutant lifting theorem, there exists a contraction $\hat{B}' : \mathcal{K}_1 \rightarrow \mathcal{K}_2'$ satisfying $\hat{B}'U_1 = U'_2\hat{B}'$ and $P_2\hat{B}' = BP_1$. Let $\hat{B} : \mathcal{K}_1 \rightarrow \mathcal{K}_2$ be the contraction defined by $\hat{B} = P_{\mathcal{K}_2}\hat{B}'$. Since $\mathcal{H}_2 \subset \mathcal{K}_2'$, we have

\[P_2\hat{B} = P_2P_{\mathcal{K}_2}\hat{B}' = P_2\hat{B}' = BP_1.\]

Furthermore,

\[\hat{B}U_1 = P_{\mathcal{K}_2}\hat{B}'U_1 = P_{\mathcal{K}_2}U'_2\hat{B}' = U_2P_{\mathcal{K}_2}\hat{B}' = U_2\hat{B},\]

and the proof is complete. \qed

Now we can establish a connection between extended eigenvectors of a contraction and those of its isometric lifting.
**Theorem 3.2.** Let $S$ on $K(\supset H)$ be an isometric lifting of a contraction $A \in \mathcal{L}(H)$, let $P$ denote the projection on $H$ in $\mathcal{L}(K)$, and suppose that $|\beta| \leq 1$. There exists a nonzero contraction $X \in \mathcal{L}(H)$ satisfying (3.1) if and only if there exists a nonzero contraction $\hat{X} \in \mathcal{L}(K)$ satisfying

\begin{equation}
\hat{X}S = \beta S\hat{X}, \quad P\hat{X} = P\hat{X}P, \quad P\hat{X} \neq 0.
\end{equation}

**Proof.** First, assume that there exists a nontrivial contractive solution $X$ to equation (3.1). Note that the operator $\beta S$ is a contractive lifting of the contraction $\beta A$. Then, taking $T_1 = A$, $T_2 = \beta A$, $B = X$, $U_1 = S$ and $U_2 = \beta S$ in Theorem 3.1, it follows that there exists a contraction $\hat{X}$ which satisfies $\hat{X}S = \beta S\hat{X}$ and $P\hat{X} = XP$. Clearly, $\hat{X}$ satisfies all three conditions in (3.2). In particular, $P\hat{X} \neq 0$ shows that $\hat{X} \neq 0$.

Conversely, assume that there exists a contraction $\hat{X} \in \mathcal{L}(K)$ satisfying (3.2). Then $X = P\hat{X}|H$ is a nonzero operator satisfying (3.1). Indeed, $AP = PS$ so

\[ \beta AX = \beta AP\hat{X}|H = \beta PS\hat{X}|H = P\hat{X}S|H = P\hat{X}PS|H = XPS|H = XA. \]

Finally, $X \neq 0$. Indeed, $P\hat{X} = P\hat{X}P \neq 0$, so there is $f \in H$ such that $P\hat{X}Pf \neq 0$. Then $X(Pf) \neq 0$ and the theorem is proved. \qed

**Remark 3.3.** Condition $P\hat{X} = P\hat{X}P$ in (3.2) means that the subspace $K \ominus H$ is invariant under $\hat{X}$. Condition $P\hat{X} \neq 0$, on the other hand, means that $\text{Ran}\, \hat{X}$ is not a subspace of $K \ominus H$.

Since the unilateral shift $S$ is an isometric lifting of $S(\theta)$ we now turn our attention to the extended point spectrum of $S$. It was shown in [11] that $\mathcal{E}(S) = \{\lambda : |\lambda| \geq 1\}$. Regarding the equation $XS = \beta SX$ this means that there is a nonzero solution $X$ for each $\beta$ in the closed unit disk. We will give a function theoretic description of such a solution $X$ by making the usual identification of $S$ with the multiplication by $z$ on the Hardy space $H^2$.

**Definition 3.4.** For a function $\phi \in H^2$ and a complex number $\beta$ with $|\beta| \leq 1$, we define the operator $\hat{X}_\phi$ as

\begin{equation}
(\hat{X}_\phi g)(z) = \phi(z)g(\beta z)
\end{equation}

for all $z \in \mathbb{D}$ and all polynomials $g$.

An easy calculation shows that $\hat{X}_\phi S = \beta S\hat{X}_\phi$ on polynomials. In order to extend $\hat{X}_\phi$ to all of $H^2$ we need a test for its continuity.

**Proposition 3.5.** If $|\beta| = 1$, then the operator $\hat{X}_\phi$ defined in (3.3) is bounded if and only if $\phi$ is in $H^\infty$. In case $|\beta| < 1$, $\hat{X}_\phi$ is bounded for any $\phi$ in $H^2$.

**Proof.** Let $\phi \in H^2$. In case $|\beta| = 1$, the mapping $g(\cdot) \to g(\beta \cdot)$ defines an unitary operator on $H^2$. Multiplication by $\phi$ defines a bounded operator on $H^2$ precisely when $\phi \in H^\infty$. Consequently, in case $|\beta| = 1$, the operator $\hat{X}_\phi$ defined in (3.3) defines a bounded operator if and only if $\phi \in H^\infty$. \qed
Suppose now that $|\beta| < 1$ and $\phi \in H^2$. If $g(z) = \sum_{n=0}^{N} g_n z^n$ is a polynomial then
\[ \hat{X}_\phi g = \hat{X}_\phi (\sum_{n=0}^{\infty} g_n S^n 1) = \sum_{n=0}^{N} \beta^n g_n S^n \hat{X}_\phi 1 = \sum_{n=0}^{N} \beta^n g_n S^n \phi. \]

Therefore,
\[
\|\hat{X}_\phi g\| \leq \sum_{n=0}^{N} |\beta|^n |g_n| \|S^n \phi\| \leq \|\phi\| \left( \sum_{n=0}^{N} |\beta|^{2n} \right)^{1/2} \left( \sum_{n=0}^{N} |g_n|^2 \right)^{1/2} \leq \|\phi\| \left( \frac{1}{1-|\beta|^2} \right)^{1/2} \|g\|.
\]
This shows that if $\phi \in H^2$ and $|\beta| < 1$, the operator $\hat{X}_\phi$ is bounded on $H^2$. \hfill \square

Now we can describe solutions of the equation $XS = \beta SX$. The proof is an immediate consequence of Proposition 3.5, so we omit it.

**Theorem 3.6.** Let $S$ denote the unilateral shift on $H^2$ and let $|\beta| \leq 1$. An operator $\hat{X}$ on $H^2$ satisfies the relation $\hat{X}S = \beta SX$ if and only if there exists a function $\phi$ in $H^2$ (in $H^\infty$ when $|\beta| = 1$) such that $\hat{X} = \hat{X}_\phi$.

In order to describe the extended point spectrum of $S(\theta)$ we will need the following technical result.

**Lemma 3.7.** Let $\theta$ be an inner function and let $A = S(\theta)$ be the Jordan block corresponding to $\theta$. Suppose that $\beta$ is a nonzero complex number such that $|\beta| \leq 1$ and let $\theta_\beta \in H^\infty$ be defined by $\theta_\beta(z) = \theta(\beta z)$, $z \in \mathbb{D}$. Then equation (3.1) has a nonzero solution if and only if there exists a nonconstant inner function which divides both $\theta_\beta$ and $\theta$ in the algebra $H^\infty$.

**Proof.** Suppose that there exists a nonconstant inner function $c \in H^\infty$ that divides both $\theta_\beta$ and $\theta$. Then, there exist functions $a$ and $b$ in $H^\infty$ such that $\theta = ac$ and $\theta_\beta = bc$. Clearly, $a$ is inner. Furthermore, $\theta$ does not divide $a$, that is, $a \not\in \theta H^\infty$. Notice that $X_a \theta(z) = a(z) \theta(\beta z) = \theta(z) b(z)$, which implies that the subspace $\mathcal{M} \equiv \theta H^2$ is invariant under $X_a$. Next we show that $a \not\in \mathcal{M}$. Indeed, if $a(z) = \theta(z) c(z)$ for some $c \in H^2$ then, using the fact that both $a$ and $\theta$ are inner, it would follow that $c$ is inner, hence in $H^\infty$. Since $a \not\in \theta H^\infty$ we see that $a \not\in \mathcal{M}$ so $\text{Ran } X_a \not\subseteq \mathcal{M}$. Finally, $S$ is an isometric lifting of $A$ and $X_a S = \beta S X_a$, so Theorem 3.2 and Remark 3.3 show that equation (3.1) has a nonzero solution given by $X = P_{\mathcal{H}(\theta)} X_a |\mathcal{H}(\theta)|$.

Conversely, suppose that equation (3.1) has a nonzero solution $X$. By Theorem 3.2 there exists an operator $\hat{X}$ satisfying (3.2). Moreover, Theorem 3.6 shows that there exists a function $\phi \in H^2$ such that $\hat{X} = \hat{X}_\phi$. In view of Remark 3.3 the subspace $\mathcal{M} \equiv \theta H^2$ is invariant under $\hat{X}_\phi$, so $\hat{X}_\phi \theta \in \mathcal{M}$. Thus,

\[ \phi(z) \theta(\beta z) = \theta(z) g(z), \]
for some $g \in H^2$. Let $\phi(z) = a(z)a_o(z)$ be the inner-outer factorization of $\phi$. By a theorem of Beurling (cf., [14, Proposition III.1.2]) there is a sequence of polynomials $p_n \in H^2$ such that $a_o p_n \to 1$ in the norm of $H^2$. If we multiply both sides of (3.4) by $p_n$ and let $n$ go to infinity we obtain that $\theta(z) g(z) p_n(z)$ converges in $H^2$ to $\theta(\beta z) a(z)$. Since multiplication by $\theta$ is an isometry on $H^2$, this implies that $g p_n$ converges to some function $b$ in $H^2$ and we obtain that $\theta \beta a = \theta b$. Moreover, $\theta$ and $a$ are inner and $\theta \beta \in H^\infty$, so $b$ is in $H^\infty$ as well. If $\theta$ and the inner part of $\theta \beta$ have no common inner divisor, then $a$ must be divisible by $\theta$. In other words, there is a function $h \in H^\infty$ such that $a(z) = \theta(z) h(z)$. In that case, factorization formula $\phi = aa_o$ shows that $\phi$ is a multiple of $\theta$ and it would follow that $\text{Ran} (\hat{X}_\phi) \subset \mathcal{M}$. By Remark 3.3 this would contradict the last condition in (3.2). This finishes the proof of the lemma.

The previous lemma shows that solving (3.1) requires understanding the divisibility of inner functions. The following result is straightforward to verify (cf., [14, Page 107]).

**Lemma 3.8.** Suppose that, for $i = 1, 2$, $\phi_i = b_i s_i$ is an inner function whose Blaschke factor $b_i$ has a zero set $Z_i$, and the singular function $s_i$ corresponds to a measure $\mu_i$. Then $\phi_1$ divides $\phi_2$ if and only if $Z_1 \subset Z_2$ and $\mu_1(E) \leq \mu_2(E)$ whenever $E$ is a Borel subset of $\mathbb{T}$.

Now we can prove the main result of this section.

**Theorem 3.9.** Let $S(\theta)$ be the Jordan block corresponding to the inner function $\theta$ and let $\beta$ be a nonzero complex number. Let $\theta(z) = b(z)s(z)$ be the canonical factorization, let $\{\alpha_n\}_{n=1}^\infty$ be the zeros of the Blaschke product $b$, and let $\mu$ be the finite, positive, singular measure on the circle corresponding to the singular function $s$.

If $|\beta| \neq 1$ then the equation $XS(\theta) = \beta S(\theta) X$ has a nontrivial solution if and only if there exists $m, n \in \mathbb{N}$ such that $\beta \alpha_m = \alpha_n$.

If $|\beta| = 1$, then the equation $XS(\theta) = \beta S(\theta) X$ has a nontrivial solution if and only if either (a) or (b) holds:

a) there exists $m, n \in \mathbb{N}$ such that $\beta \alpha_m = \alpha_n$.

b) The measures $\mu$ and $\mu_\beta$ are not mutually singular.

**Proof.** First we consider the case $|\beta| < 1$. In view of Lemma 3.7 we are interested in the inner factor of $\theta \beta$. In this situation, the function $s_\beta(z) \equiv s(\beta z)$ is an outer function. Indeed, it is straightforward to verify that

$$\left| \frac{1}{s_\beta(z)} \right| \leq \frac{1 + |\beta|}{e^{1 - |\beta|}} \mu(\mathbb{T})$$

so $1/s_\beta \in H^\infty$. By considering the inner-outer factorizations $s_\beta = f_1 g_1$ and $1/s_\beta = f_2 g_2$ we obtain that $1 = (f_1 f_2)(g_1 g_2)$. It follows that $f_1 f_2 = 1$ so that the inner part $f_1$ of $s_\beta$ must be constant and, hence, $s_\beta$ is an outer function. Next, we notice that $b_\beta(z) = 0$ if and only if $z = \alpha_n / \beta$ for some $n \in \mathbb{N}$. When $z \in \mathbb{D}$ this means that $|\alpha_n| < |\beta|$. Since zeros of a Blaschke product cannot have an accumulation
point in the disk of radius $|\beta| < 1$ it follows that the inner part of $\theta_\beta$ is a finite Blaschke product with zeros of the form $\alpha_n/\beta$ for $n$ in a finite subset of $\mathbb{N}$. Now, if $c \in H^\infty$ is a common divisor of $\theta_\beta$ and $\theta$ then $c$ is an inner function (since $\theta$ is inner) and it is a finite Blaschke product (since the inner part of $\theta_\beta$ is). Also $c(z) = 0$ if and only if $\theta_\beta(z) = 0$ and $\theta(z) = 0$ which means that $z = \alpha_m/\beta$ and $z = \alpha_n$. In other words, $\alpha_m = \beta \alpha_n$ for some positive integers $m$ and $n$.

When $|\beta| > 1$, we consider equation $X^*S(\theta)^* = (1/\beta)S(\theta)^*X^*$. One knows (cf., [1, Corollary III.1.7]) that $S(\theta)^*$ is unitarily equivalent to $S(\bar{\theta})$, where $\bar{\theta}(z) = \overline{\theta(z)}$. By the previous, a nonzero solution $X^*$ exists if and only if there are positive integers $m$ and $n$ such that $(1/\beta)\tilde{\alpha}_m = \tilde{\alpha}_n$, where $\tilde{\alpha}_m$ and $\tilde{\alpha}_n$ are two zeros of the inner function $\bar{\theta}$. Of course, $\bar{\theta}(z) = 0$ if and only if $\bar{\theta}(\bar{z}) = 0$ so we obtain once again that $\alpha_m = \beta \alpha_n$.

Consider now the case $|\beta| = 1$. It is easy to see that the function $b_\beta$ is a Blaschke product with zeros $\{\alpha_n/\beta\}_{n \in \mathbb{N}}$, and that $s_\lambda$ is a singular function corresponding to the measure $\mu_\beta$. By Lemma 3.8, if $c \in H^\infty$ is a common divisor of inner functions $\theta$ and $\theta_\beta$ then $c$ must be an inner function as well. Moreover, if $c = \hat{b} \hat{s}$ (with $\hat{b}$ the Blaschke factor of $c$) then $\hat{b}$ divides $b$ and $b_\beta$, and $\hat{s}$ divides $s$ and $s_\lambda$. By Lemma 3.8, this means that either there is a common zero of $b$ and $b_\beta$ (which is equivalent to (a)), or there exists a measure $\nu$ that is a common minorant of $\mu$ and $\mu_\beta$ (which is equivalent to (b)).

We close this section by characterizing the extended point spectrum $\mathbb{E}(S(\theta))$ in terms of the inner function $\theta$.

**Theorem 3.10.** Let $S(\theta)$ be the Jordan block corresponding to the inner function $\theta$ and let $\lambda$ be a nonzero complex number. Let $\theta(z) = b(z)s(z)$ be the canonical factorization, let $\{\alpha_n\}_{n=1}^\infty$ be the zeros of the Blaschke product $b$, and let $\mu$ be the finite, positive, singular measure on the circle corresponding to the singular function $s$.

- If $|\lambda| \neq 1$ then the equation $S(\theta)X = \lambda XS(\theta)$ has a nontrivial solution if and only if there exist $m, n \in \mathbb{N}$ such that $\alpha_m = \lambda \alpha_n$.

- If $|\lambda| = 1$, then the equation $S(\theta)X = \lambda XS(\theta)$ has a nontrivial solution if and only if either (a) or (b) holds:
  a) there exist $m, n \in \mathbb{N}$ such that $\alpha_m = \lambda \alpha_n$.
  b) The measures $\mu$ and $\mu_\lambda$ are not mutually singular.

**Proof.** The case $|\lambda| \neq 1$ follows immediately from Theorem 3.9. We only mention that 0 cannot be an extended eigenvalue of $S(\theta)$ since a Jordan block is always injective (cf., [1, Corollary III.1.12]).

When $|\lambda| = 1$, Theorem 3.9 shows that there is a measure $\nu$ that is dominated by both $\mu$ and $\mu_{1/\lambda}$. It is easy to see that this is equivalent to $\nu_\lambda$ being a common minorant of $\mu$ and $\mu_\lambda$. \[\Box\]

4. $C_0$ contractions

In this section we extend the results of Section 3 to $C_0$ contractions. In order to do that we will take advantage of the fact that a quasisimilarity model for a $C_0$
contraction involves Jordan blocks. Therefore, we first need to establish that the extended point spectrum is invariant under a quasisimilarity.

**Proposition 4.1.** Suppose that operators $A$ and $B$ are quasisimilar. Then $E(A) = E(B)$.

*Proof.* It suffices to show that $E(A) \subset E(B)$. So, suppose that $\lambda \in E(A)$. Then there is a nonzero operator $X$ satisfying (1.1). By assumption there exist quasi-affinities $Y$ and $Z$ such that $AY = YB$ and $ZA = BZ$. Multiplying (1.1) by $Y$ from the right and by $Z$ from the left we obtain that $ZAXY = \lambda ZXAY$ and, hence, that $BZXY = \lambda ZXYB$. Now $X \neq 0$, $Z$ is injective, and $Y$ has dense range, so $ZXY \neq 0$ and, consequently, $\lambda \in E(B)$ with the corresponding eigenoperator $ZXY$. \qed

The following result gives a connection between a $C_0$ contraction with a minimal function $\theta$ and a Jordan block $S(\theta)$.

**Proposition 4.2.** Let $\theta$ be the minimal function of a $C_0$ contraction $A$. There exists a $C_0$ contraction $A'$ on a Hilbert space $\mathcal{H}'$ such that the operator $S(\theta) \oplus A'$ acting on $\mathcal{H}(\theta) \oplus \mathcal{H}'$ is quasisimilar to $A$.

*Proof.* See [1, Corollary III.3.5] and the paragraph immediately following its proof. \qed

Proposition 4.2 shows that if a complex number $\lambda$ is an extended eigenvalue of $S(\theta)$ then it is in the extended point spectrum of $A$. Therefore, it is natural to ask whether the converse is true. In order to answer this question we will need information about the behavior of a $C_0$ contraction when restricted to a cyclic invariant subspace.

**Proposition 4.3.** Let $A$ be a $C_0$ contraction with the minimal function $m_A$. Suppose that $\mathcal{G}$ is a cyclic invariant subspace of $A$. In this case, $A|\mathcal{G}$ is also of class $C_0$ and its minimal function $\theta$ divides $m_A$. Moreover, $A|\mathcal{G}$ is quasisimilar to $S(\theta)$.

*Proof.* See [1, Proposition II.4.3 and Theorem III.2.3]. \qed

We have mentioned that a Jordan block is always injective. This is not true for an arbitrary $C_0$ contraction and we will have to address this possibility when characterizing its extended point spectrum. The following result will be useful in this direction.

**Proposition 4.4.** Let $A$ be a $C_0$ contraction with the minimal function $m_A$. Then its point spectrum $\sigma_p(A)$ is the set of zeros of $m_A$ in the open unit disk. In particular, $A$ has a nontrivial kernel if and only if 0 is a zero of $m_A$.

*Proof.* See [1, Theorem II.4.11]. \qed

We are now ready to prove the main result of this paper.

**Theorem 4.5.** Let $A$ be a $C_0$ contraction with the minimal function $m_A$ and let $m_A(z) = b(z)s(z)$ be the canonical factorization, let $\{\alpha_n\}_{n=1}^\infty$ be the zeros of the
Blaschke product \( b \), and let \( \mu \) be the finite, positive, singular measure on the circle corresponding to the singular function \( s \).

If \( |\lambda| \neq 1 \) then the equation \( AX = \lambda XA \) has a nontrivial solution if and only if there exist \( m, n \in \mathbb{N} \) such that \( \alpha_m = \lambda \alpha_n \).

If \( |\lambda| = 1 \), then the equation \( AX = \lambda XA \) has a nontrivial solution if and only if either (a) or (b) holds:

a) there exist \( m, n \in \mathbb{N} \) such that \( \alpha_m = \lambda \alpha_n \).

b) The measures \( \mu \) and \( \mu_\lambda \) are not mutually singular.

**Proof.** First we assume that condition (a) holds (with \( |\lambda| \neq 1 \)), or that either (a) or (b) holds (with \( |\lambda| = 1 \)). By Proposition 4.4, \( A \) is quasisimilar to \( S(m_A) \oplus A' \).

Furthermore, by Theorem 3.10, \( \lambda \in \mathcal{E}(S(m_A)) \) so there is a nonzero operator \( X \) such that \( S(m_A)X = \lambda X S(m_A) \). Clearly, \( (S(m_A) \oplus A')(X \oplus 0) = \lambda (X \oplus 0)(S(m_A) \oplus A') \) so Proposition 4.1 implies that \( \lambda \in \mathcal{E}(A) \).

To prove the converse, suppose that \( AX = \lambda XA \) for a complex number \( \lambda \) and a nonzero operator \( X \). Notice that, by Proposition 4.4, if \( AX = 0 \) then \( 0 \) is a zero of \( m_A \). By taking \( \alpha_m = \alpha_n = 0 \) we obtain that the condition (a) in the statement of the theorem holds for any \( \lambda \). Thus we may assume that \( AX \neq 0 \) (and, hence, that \( \lambda \neq 0 \)). Then, there exists \( \xi \) such that \( AX \xi \neq 0 \) and, obviously, \( X \xi \neq 0 \). Now \( X \) maps the cyclic subspace \( \mathcal{H}_1 \) of \( A \) generated by \( \xi \) into the cyclic subspace \( \mathcal{H}_2 \) of \( A \) generated by \( X \xi \). By Proposition 4.3, there exists an inner function \( \theta \) that divides \( m_A \) and such that \( A|\mathcal{H}_1 \) is quasisimilar to \( S(\theta) \). By Theorem 3.10 either condition (a) or (b) holds, with the distinction that each \( \alpha \) is a zero of \( \theta \) and the measure \( \mu \) corresponds to the singular part of \( \theta \). However, \( \theta \) divides \( m_A \) and, by Lemma 3.8, each \( \alpha \) is a zero of \( m_A \) and if \( \nu \) is a common minorant of \( \mu \) and \( \mu_\lambda \) it is all the more so when \( \mu \) is the measure associated with \( m_A \). This completes the proof. \( \square \)

## 5. Powers of operators

In this section we study the relationship between the extended point spectra of an operator \( A \) and its powers \( A^n \), for \( n = 1, 2, \ldots \). In one direction, it is easy to see that if \( \lambda \in \mathcal{E}(A) \) then \( \lambda^n \in \mathcal{E}(A^n) \). The main result of this section is that the converse is true (Theorem 5.2) below. First we need to establish a “factorization” result. Let \( A \) be a fixed operator in \( \mathcal{L}(\mathcal{H}) \). For a complex number \( \lambda \) we define a map \( F_\lambda : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{H}) \) as \( F_\lambda(X) = AX - \lambda XA \). We will use the term the primitive \( n \)th root of \( 1 \) to denote \( e^{2\pi i/n} \).

**Theorem 5.1.** Let \( n \in \mathbb{N} \), let \( \lambda \in \mathbb{C} \), let \( \mu \) be any \( n \)th root of \( \lambda \), and let \( \omega \) be the primitive \( n \)th root of \( 1 \). Then

\[
A^n X - \lambda X A^n = F_{\mu \omega^{n-1}} \circ F_{\mu \omega^{n-2}} \circ \cdots \circ F_{\mu \omega} \circ F_\mu (X).
\]

**Proof.** First we establish the fact that, for any nonnegative integer \( k \), there exist complex numbers \( \alpha_{k0}, \alpha_{k1}, \ldots, \alpha_{k,k+1} \) such that

\[
F_{\mu \omega^k} \circ F_{\mu \omega^{k-1}} \circ \cdots \circ F_{\mu \omega} \circ F_\mu (X) = \sum_{j=0}^{k+1} \alpha_{kj} A^{k+1-j} X A^j.
\]
We use induction to prove this. The case $k = 0$ is obvious, so we concentrate on the inductive step. We have that

$$F_{\mu \omega^{k+1}} \circ F_{\mu \omega^k} \circ \cdots \circ F_{\mu \omega} \circ F_\mu(X)$$

$$= AF_{\mu \omega^k} \circ F_{\mu \omega^{k-1}} \circ \cdots \circ F_{\mu \omega} \circ F_\mu(X) - \mu \omega^{k+1} F_{\mu \omega^k} \circ F_{\mu \omega^{k-1}} \circ \cdots \circ F_{\mu \omega} \circ F_\mu(X)A$$

$$= A \sum_{j=0}^{k+1} \alpha_{kj} A^{k+1-j} X A^j - \mu \omega^{k+1} \sum_{j=0}^{k+1} \alpha_{kj} A^{k+1-j} X A^j$$

$$= \sum_{j=0}^{k+1} \alpha_{kj} A^{k+2-j} X A^j - \mu \omega^{k+1} \sum_{j=0}^{k+1} \alpha_{kj} A^{k+1-j} X A^{j+1}$$

and a simple change of the index of summation in the second sum establishes (5.1).

Next, we will show that the coefficients above satisfy a recursive relation. Namely

$$\alpha_{k+1,j} = \alpha_{kj} - \mu \omega^{k+1} \alpha_{k,j-1},$$

with $\alpha_{00} = 1$, $\alpha_{01} = -\mu$, and $\alpha_{0s} = 0$ for $s \geq 2$. Indeed, using the calculation above we get

$$F_{\mu \omega^{k+1}} \circ F_{\mu \omega^k} \circ \cdots \circ F_{\mu \omega} \circ F_\mu(X) = \sum_{j=0}^{k+1} \alpha_{kj} A^{k+2-j} X A^j - \mu \omega^{k+1} \sum_{j=1}^{k+2} \alpha_{k,j-1} A^{k+2-j} X A^j$$

$$= \alpha_{k0} A^{k+2} X - \mu \omega^{k+1} \alpha_{k,k+1} X A^{k+2} + \sum_{j=1}^{k+1} (\alpha_{kj} - \mu \omega^{k+1} \alpha_{k,j-1}) A^{k+2-j} X A^j.$$

Thus $\alpha_{k+1,0} = \alpha_{k0}$, $\alpha_{k+1,k+2} = -\mu \omega^{k+1} \alpha_{k,k+1}$, and $\alpha_{k+1,j} = \alpha_{kj} - \mu \omega^{k+1} \alpha_{k,j-1}$ for $1 \leq j \leq k + 1$. If we adopt the convention that $\alpha_{mn} = 0$ for $n < 0$ and $n \geq m + 2$ we see that there is a unique recursive relation

$$\alpha_{k+1,j} = \alpha_{kj} - \mu \omega^{k+1} \alpha_{k,j-1},$$

for $j \geq 0$.

The last relation can be written in a more suitable form. Let us use the notation $\alpha_i$ for the sequence $(\alpha_{i0}, \alpha_{i1}, \alpha_{i2}, \ldots)$. If $S$ is the unilateral shift then the previous expression becomes

$$\alpha_{k+1} = \alpha_k - \mu \omega^{k+1} S \alpha_k = (I - \mu \omega^{k+1} S) \alpha_k.$$

Therefore,

$$\alpha_{k+1} = (I - \mu \omega^{k+1} S)(I - \mu \omega^k S) \cdots (I - \mu \omega^2 S)(I - \mu \omega S) \alpha_0,$$

where $\alpha_0 = (1, -\mu, 0, 0, \ldots)$. Using the obvious identification between $\ell^2$ and $H^2$, under which $S$ becomes identified with the operator of multiplication by $z$, we obtain that the function $\alpha_{n-1}(z)$ equals

$$(1 - \mu \omega^{n-1} z)(1 - \mu \omega^{n-2} z) \cdots (1 - \mu \omega^2 z)(1 - \mu \omega z)(1 - \mu z).$$

Since, by assumption, $\omega$ is the primitive $n$th root of 1, and $\mu^n = \lambda$, the last expression is just $1 - \lambda z^n$. In other words, the sequence $\alpha_{n-1}$ is $(1, 0, 0, \ldots, 0, -\lambda, 0, \ldots)$ where $-\lambda$ is in the position $n + 1$. Using (5.1) we get the desired result. \qed
Our main result of this section is an easy consequence of Theorem 5.1.

**Theorem 5.2.** A complex number $\lambda$ is an extended eigenvalue of $A^n$ for some positive integer $n$ if and only if there is a complex number $\mu \in \mathbb{E}(A)$ with the property that $\mu^n = \lambda$.

**Proof.** It is obvious that if $\mu^n = \lambda$ and $AX = \mu X A$ then $A^n X = \lambda X A^n$. Thus we concentrate on the converse. Suppose that there exists a nonzero operator $X$ such that $A^n X = \lambda X A^n$. Let $\mu$ satisfy $\mu^n = \lambda$ and let $\omega = e^{2\pi i/n}$. By Theorem 5.1, $0 = A^n X - \lambda X A^n = F_{\mu^{n-1}} \circ F_{\mu^{n-2}} \circ \cdots \circ F_{\mu} (X)$. If $F_{\mu}(X) = 0$ then $\mu \in \mathbb{E}(A)$. So assume that $F_{\mu}(X) \neq 0$ and let $m$ be the smallest integer such that $F_{\mu^m} \circ F_{\mu^{m-1}} \circ \cdots \circ F_{\mu} (X) = 0$. Then $Y \equiv F_{\mu^{m-1}} \circ F_{\mu^{m-2}} \circ \cdots \circ F_{\mu} (X) \neq 0$ and $F_{\mu} (Y) = 0$ so $\mu \omega^m \in \mathbb{E}(A)$. Since $(\mu \omega^m)^n = \lambda$ the theorem is proved.

It was shown in [3] that, if $V$ is the Volterra integral operator on $L^2(0,1)$, then $\mathbb{E}(V^n) = (0,\infty)$. Based on Theorem 5.2, we obtain the extended point spectrum of $V^n$, for any $n \in \mathbb{N}$.

**Proposition 5.3.** Let $n \in \mathbb{N}$. Then $\mathbb{E}(V^n) = (0,\infty)$.

**Remark 5.4.** Proposition 5.3 remains true if the Volterra operator $V$ is replaced by any operator with extended point spectrum $(0,\infty)$.

Although the results of this section were motivated by the study of the extended point spectrum, they can be applied to the commutant of an operator as well. By taking $\lambda = 1$ in Theorem 5.2 we obtain an immediate consequence.

**Proposition 5.5.** Let $n \in \mathbb{N}$ and suppose that the extended point spectrum of an operator $A$ does not contain an $n$th root of 1 different from 1. Then $A$ and $A^n$ must have the same commutant.

The converse of Proposition 5.5 is not true. For example, the operator $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has $\mathbb{E}(A) = \mathbb{C}$ yet $A = A^n$ so the commutant is the same. Our next result shows that this phenomenon is due to the presence of the nontrivial kernel.

**Theorem 5.6.** Let $A$ be an operator such that either $A$ has trivial kernel or the range of $A$ is dense. Then the commutant of $A^n$ coincides with that of $A$ if and only if $\mathbb{E}(A)$ does not contain any $n$th root of 1 other than 1.

**Proof.** The “if” part follows from Proposition 5.5. To prove the “only if” part, suppose that there exists a complex number $\lambda \neq 1$ such that $\lambda^n = 1$ and a nonzero operator $X$ satisfying $AX = \lambda X A$. Clearly, $X$ commutes with $A^n$. We will demonstrate that $AX \neq X A$. Indeed, if $AX = X A$ then (using $AX = \lambda X A$ and $\lambda \neq 1$) we obtain that $X A = 0$, and also that $AX = 0$. Since $X \neq 0$, the former equality implies that the range of $A$ is not dense, while the latter implies that the kernel of $A$ is nontrivial. This contradicts the assumption on $A$ and the proof is complete.

The Volterra operator $V$ on $L^2(0,1)$ is an example of an operator with trivial kernel and without a nontrivial root of unity (since $\mathbb{E}(V) = (0,\infty)$). So we get an immediate consequence.
Proposition 5.7. \( \{V^n\}' = \{V\}' \) for all \( n \in \mathbb{N}, n \geq 1 \).

The following application of the extended point spectrum reveals another curious property of \( V \).

Proposition 5.8. The sequence \( V^n/\|V^n\| \) converges to 0 in the weak operator topology.

Proof. Since the operators \( V^n/\|V^n\| \) are all in the closed unit ball of \( \mathcal{L}(\mathcal{H}) \) which is weakly compact, there has to be at least one accumulation point \( B \). Our claim is that \( B = 0 \).

Suppose \( B \neq 0 \). There is a subsequence \( V^{n_k}/\|V^{n_k}\| \) that converges weakly to \( B \). Let \( \lambda \in (0,1) \). Since \( \lambda \in \mathbb{E}(V) \) there exists an operator \( X \in \mathcal{L}(\mathcal{H}) \) such that \( VX = \lambda XV \). Moreover, it was shown in [3] that \( X \) can be selected to have dense range. For instance, one can take \( X \) to be the operator given by

\[
(Xf)(t) = \int_0^{t/\lambda} f(s)ds.
\]

It is easy to see that

\[
\frac{V^{n_k}}{\|V^{n_k}\|} X = \frac{\lambda^{n_k}X V^{n_k}}{\|V^{n_k}\|}.
\]

By letting \( k \to \infty \) we see that \( BX = 0 \). Since \( X \) has dense range it follows that \( B = 0 \). Since 0 is the only weak accumulation point of the bounded sequence \( V^n/\|V^n\| \), the sequence converges weakly to 0. \( \square \)

Remark 5.9. An inspection of the proof shows that the same conclusion is valid for any operator with at least one extended eigenvalue in the open unit disk, provided that the appropriate eigenoperator can be selected so that it has dense range.

References


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