JOINT SPECTRUM AND NONISOMETRIC FUNCTIONAL CALCULUS

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Abstract. Let $T$ be a completely nonunitary contraction on a Hilbert space $\mathcal{H}$ and assume that the spectrum of $T$ contains the unit circle. We show that, in this situation, there exists a contraction $S$ that commutes with $T$, and such that the joint (Taylor) spectrum of the pair $(S, T)$ contains the two-dimensional torus, but the functional calculus generated by the pair has a nontrivial kernel. This improves a result in [12].

Let $\mathcal{H}$ be an infinite dimensional, complex, separable Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. If $S$ and $T$ belong to $\mathcal{L}(\mathcal{H})$ and commute with each other, we say that a pair $(S, T)$ is nonsingular if the sequence

$$0 \to \mathcal{H} \xrightarrow{\delta_1} \mathcal{H} \oplus \mathcal{H} \xrightarrow{\delta_2} \mathcal{H} \to 0$$

is exact, where $\delta_1(x) = (Sx, Tx)$, $x \in \mathcal{H}$ and $\delta_2(x, y) = Sy - Tx$. A pair of complex numbers $(\lambda_1, \lambda_2)$ belongs to the Taylor joint spectrum $\mathrm{Taylor}$ joint spectrum

Date: October 4, 2001.
1991 Mathematics Subject Classification. Primary: 47A15; Secondary: 47D27.

Key words and phrases. Functional calculus, Contractions, Joint spectra.
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of \((S, T)\) (notation: \((\lambda_1, \lambda_2) \in \sigma(S, T)\)), if \((S - \lambda_1, T - \lambda_2)\) is not nonsingular (cf. [8]).

Our notation is consistent with [12], [11], and [9]. As usual, \(\mathbb{C}\) is the complex plane, \(\mathbb{D}\) is the open unit disk, and \(\mathbb{T} = \partial \mathbb{D}\). Also, \(\mathbb{D}^2\) is the bidisk (the Cartesian product of two copies of \(\mathbb{D}\)) and \(\mathbb{T}^2\) is the torus (the Cartesian product of two copies of \(\mathbb{T}\)). Finally, \(H^\infty(\mathbb{D})\) [resp. \(H^\infty(\mathbb{D}^2)\)], is the Banach algebra of bounded analytic functions on \(\mathbb{D}\) [resp. \(\mathbb{D}^2\)].

It is a famous result of Sz.-Nagy and Foias (cf. [13], Theorem III.2.1) that if \(T\) is a completely nonunitary contraction in \(\mathcal{L}(\mathcal{H})\), then there exists a contractive algebra homomorphism \(\Phi_T\) from \(H^\infty(\mathbb{D})\) into \(\mathcal{L}(\mathcal{H})\)(see also [4], Theorem 4.1). This homomorphism, usually referred to as the Nagy–Foias functional calculus, has played an important role in the development of the structure theory of linear operators on Hilbert space as expounded in the classical book [13].

As a generalization of this result for a pair of commuting contractions \((S, T)\), several authors have provided conditions under which there exists an algebra homomorphism \(\Phi_{S,T}\) from \(H^\infty(\mathbb{D}^2)\) into \(\mathcal{L}(\mathcal{H})\), satisfying, at least, the following properties:

- \(\Phi_{S,T}(p) = p(S, T)\), for any complex polynomial \(p\) in two variables.
- \(\|\Phi_{S,T}(h)\| \leq \|h\|_\infty\), for any \(h \in H^\infty(\mathbb{D}^2)\).
• $\Phi_{S,T}$ is weak* continuous.

The last property means that $\Phi_{S,T}$ is continuous when $H^\infty(\mathbb{D}^2)$ and $\mathcal{L}(\mathcal{H})$ are provided with their respective weak* topologies, which arise from the fact that $\mathcal{L}(\mathcal{H})$ can be identified with the dual of the trace-class, while a predual of $H^\infty(\mathbb{D}^2)$ is described in [11].

Some sufficient conditions for the functional calculus to exist are:

(A) Both contractions are completely nonunitary (cf. [5]).

(B) The spectral measure associated with the pair is absolutely continuous with respect to some “representing” measure (cf. [9] and [11]).

(C) Each member of the pair satisfies

$$\sup_{\|p\|\leq 1} | < p(S, T)A^n x, y > | \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad \text{for all } x, y \in \mathcal{H},$$

when substituted for $A$ in the expression above (see [2] for details).

One knows (cf. [9]) that if $S$ is completely nonunitary and $T$ is absolutely continuous (i.e., the spectral measure of its minimal unitary dilation is absolutely continuous with respect to Lebesgue measure on $\mathbb{T}$), then the pair $(S, T)$ satisfies condition (B). Therefore, (B) is more general than (A). Recently, the first author and M. Kosiek have established the equivalence of (B) and (C). From now on we shall assume
that the pair \((S, T)\) satisfies one of these conditions and we shall write \(h(S, T)\) instead of \(\Phi_{S,T}(h)\).

It is an old result of Apostol (cf. [1]) that if \(T\) is an absolutely continuous contraction in \(\mathcal{L}(\mathcal{H})\) such that \(\sigma(T) \supset \mathbb{T}\) and the Nagy–Foias functional calculus is not an isometry, then \(T\) has a nontrivial hyperinvariant subspace (i.e., a closed subspace \(\mathcal{M}\) of \(\mathcal{H}\), such that \(S\mathcal{M} \subset \mathcal{M}\) for any \(S\) in the commutant of \(T\)). There is no known version of this result for a pair of commuting contractions \((S, T)\) such that \(\sigma(S, T) \supset \mathbb{T}^2\).

Actually, even the particular case when the functional calculus has a nontrivial kernel is well understood only in the one variable case. Recall that a completely nonunitary contraction \(T \in \mathcal{L}(\mathcal{H})\) belongs to the class \(C_0\) and has minimal (inner) function \(m_T \in H^\infty(\mathbb{D})\) if \(m_T(T) = 0\) and every nonzero function \(f \in H^\infty(\mathbb{D})\) such that \(f(T) = 0\) has \(m_T\) as a divisor. For a full account of this theory we recommend [3]. Sz.-Nagy and Foias ([13]) established the existence of a contraction \(T \in C_0\) such that \(\sigma(T) \supset \mathbb{T}\). In [12] the first author has proved the following extension of this result.

**Theorem 1.** Let \(T \in \mathcal{L}(\mathcal{H})\) be a completely nonunitary contraction with \(\sigma(T) \supseteq \mathbb{D}\). Then there is a completely nonunitary contraction \(S\) that commutes with \(T\) and a function \(h \in H^\infty(\mathbb{D}^2)\) such that \(\sigma(S, T) \supseteq \mathbb{T}^2\) and \(h(S, T) = 0\).
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It is the main result of this paper that the assertion above is true for a larger class of operators.

**Theorem 2.** Let $T \in \mathcal{L}(\mathcal{H})$ be a completely nonunitary contraction with $\sigma(T) \supseteq \mathbb{T}$. Then there is a completely nonunitary contraction $S$ that commutes with $T$ and a function $h \in H^\infty(\mathbb{D}^2)$ such that $\sigma(S,T) \supseteq \mathbb{T}^2$ and $h(S,T) = 0$.

Before proceeding with the proof of Theorem 2 we shall need some definitions and preliminary results. If $J$ is a subset of a metric space $(X,d)$ and $\epsilon$ is a positive number, we denote by $J_\epsilon$ the set of points $x \in X$ such that $d(x,J) < \epsilon$. The Hausdorff metric is defined on the family of compact subsets of $X$ as

$$d_H(J,K) = \inf\{\epsilon > 0 : K_\epsilon \supseteq J, J_\epsilon \supseteq K\}.$$

Let $\{S_n\}$ be a sequence of mutually commuting operators acting on $\mathcal{H}$. One knows (cf. [10]) that if $S_n \to S$ in the norm topology, then $\sigma(S_n) \to \sigma(S)$ in the Hausdorff metric. The following is a generalization of this result in two variables.

**Lemma 3.** Let $\mathcal{F} = \{S_n\} \cup \{T_n\}$ be a commuting family of operators acting on $\mathcal{H}$. Furthermore, assume that there exist $S$ and $T$ in $\mathcal{L}(\mathcal{H})$ such that $\|T_n - T\| \to 0$ and $\|S_n - S\| \to 0$. Then $\sigma(S_n, T_n) \to \sigma(S, T)$ in the Hausdorff metric.
Proof. Let $\mathcal{B}$ be the commutative Banach subalgebra of $\mathcal{L}(\mathcal{H})$ generated by $\mathcal{F}$, $S$, and $T$ and let $M$ be its maximal ideal space. By Theorem 3.3 of [8] we can associate with $\mathcal{B}$ a nonempty compact set $M_\sigma \subset M$ in such a way that for any $A, B \in \mathcal{B}$

$$\sigma(A, B) = \{(\hat{A}(\phi), \hat{B}(\phi)) : \phi \in M_\sigma\},$$

where, as usual, $\hat{A}$ and $\hat{B}$ are the Gelfand transforms of $A$ and $B$, respectively.

Since $S_n$ converges to $S$ in norm, $\hat{S}_n \to \hat{S}$ uniformly in the space of continuous functions on $M_\sigma$, denoted $C(M_\sigma)$, and the same holds for $\hat{T}_n$ and $\hat{T}$. Thus, the sequence of sets

$$\sigma(S_n, T_n) = \{(\hat{S}_n(\phi), \hat{T}_n(\phi)) : \phi \in M_\sigma\}$$

converges in the Hausdorff metric to

$$\sigma(S, T) = \{(\hat{S}(\phi), \hat{T}(\phi)) : \phi \in M_\sigma\}.$$

This completes the proof. ■

Remark. We notice that Theorem 3.3 of [8] can be used to prove an $N$–variable version of Lemma 3, for every $N \in \mathbb{N}$. Of course, when $N = 1$, the set $M_\sigma$ is exactly the maximal ideal space $M$. ■
We briefly recall that the operator $S$ in Theorem 1 is obtained as $g(T)$, where $g \in H^\infty(D)$ is any function with the property that

$$\{(g(\lambda), \lambda) : \lambda \in D\} \supseteq T^2.$$ 

Clearly, such a function is universal (i.e., does not depend on $T$). Several such functions have been constructed in the literature of Cluster Set Theory. One such construction can be found in [12].

Next we notice that the proof of Theorem 1 holds if the hypothesis $\sigma(T) \supset D$ is replaced by the weaker hypothesis that $\sigma(T)$ contains an annulus $\Omega = \{z \in D : 0 < r < |z| < 1\}$.

**Proof of Theorem 2.** Consider the sequence $\{\phi_n\}_{n=1}^\infty \subset L^\infty[0,1]$, defined as

$$\phi_n(x) = \begin{cases} 
 x + 1 - 1/n, & 0 < x \leq 1/n, \\
 1, & 1/n < x < 1,
\end{cases}$$

for every $n \in \mathbb{N}$. As usual, $M_{\phi_n}$ denotes the operator of multiplication by $\phi_n$ acting on $L^2[0,1]$. Finally, for every $n \in \mathbb{N}$, we define $T_n = M_{\phi_n} \otimes T$, a bounded linear operator acting on the Hilbert space $L^2[0,1] \otimes \mathcal{H}$. One knows (cf. [6]) that $\sigma(T_n) = \sigma(M_{\phi_n})\sigma(T)$ and thus, since $\sigma(M_{\phi_n}) = [1 - 1/n, 1]$, the set $\sigma(T_n)$ contains the annulus $\Delta_n = \{z \in \mathbb{C} : 1/n < |z| < 1\}$. Using Theorem 1, the remark above, and recalling the universal property of the function $g$, we obtain a pair
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of completely nonunitary contractions \((g(T_n), T_n)\) whose spectrum contains \(\mathbb{T}^2\). Clearly, \(T_n \to I \otimes T\) and \(g(T_n) \to g(I \otimes T) = I \otimes g(T)\), both in the norm of \(L^2[0,1] \otimes \mathcal{H}\). Now, Lemma 3 implies that \(\sigma(g(T_n), T_n) \to \sigma(I \otimes g(T), I \otimes T)\) in the Hausdorff metric and by Lemma 2.1 of [7], \(\sigma(I \otimes g(T), I \otimes T) = \sigma(g(T), T)\). Therefore, \(\sigma(g(T), T) \supset \mathbb{T}^2\), and the theorem is proved.

The authors wish to express their gratitude to the Instituto Venezolano de Investigaciones Cientificas (IVIC) and Indiana University for partial support during the period this paper was being written, as well as to Prof. Raul Curto for his guidance and help.

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