# Answers to Worksheet II, Math 272

1. Find the cylindrical and spherical coordinates for the point $(-1, 1, -1)$ with the rectangular coordinates.

   **Solution:** The cylindrical coordinate $(r, \theta, w)$ for the point $(-1, 1, -1)$ are

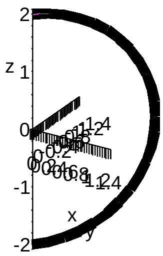
$$\begin{aligned} r &= \sqrt{(-1)^2 + 1^2} = \sqrt{2} \\ \tan\theta &= \frac{1}{-1} = -1 \quad \text{i.e.} \quad \theta = \frac{3}{4}\pi \\ w &= -1 \end{aligned}$$

   The spherical coordinate $(\rho, \theta, \phi)$ for the point $(-1, 1, -1)$ are

$$\begin{aligned} \rho &= \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3} \\ \tan\theta &= \frac{1}{-1} = -1 \quad \text{i.e.} \quad \theta = \frac{3}{4}\pi \\ \cos\phi &= \frac{-1}{\sqrt{3}} \quad \text{i.e.} \quad \phi = \arccos\frac{-1}{\sqrt{3}} \end{aligned}$$

2. Describe the points specifies by the set $\{(\rho, \theta, \phi) | \rho = 2, \theta = \frac{3\pi}{4}\}$ of spherical coordinates.

   **Solution:** It is a half-circle. See the following figure




3. Show that if a vector $\vec{a}$ forms angles $\alpha$, $\beta$, and $\gamma$ with the coordinate axes, then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

   **Solution:** Let the non-zero vector $\vec{a} = a\vec{i} + b\vec{j} + c\vec{k}$. Notice that

$$\begin{aligned} a &= \vec{a} \cdot \vec{i} = \|\vec{a}\|\|\vec{i}\| \cos\alpha \\ b &= \vec{a} \cdot \vec{j} = \|\vec{a}\|\|\vec{j}\| \cos\beta \\ c &= \vec{a} \cdot \vec{k} = \|\vec{a}\|\|\vec{k}\| \cos\gamma. \end{aligned}$$

Squaring above three equations from both sides and adding them together, we have

$$a^2 + b^2 + c^2 = \|\vec{a}\|^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma).$$

In above equastion, we use the fact $\|\vec{i}\| = 1$, $\|\vec{j}\| = 1$, and $\|\vec{k}\| = 1$.

Because of $a^2 + b^2 + c^2 = \|\vec{a}\|^2$, we have

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

This completes our proof.

4. Let $\vec{a} = 2\vec{i}$ and $\vec{b} = \vec{i} + \vec{j}$ be two vectors in $\mathbb{R}^2$.

   (a) Sketch $\vec{a}$, $\vec{b}$, and $\vec{a} - \vec{b}$ on a single set of coordinate axes.

   **Solution:** Omitted.

   (b) Determine the angle $\theta$ between $\vec{a}$ and $\vec{b}$

   **Solution:** Since

$$\begin{aligned}\cos\theta &= \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} \\ &= \frac{2\times 1 + 0\times 1}{\sqrt{2^2+0^2}\times\sqrt{1^2+1^2}} \\ &= \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}\end{aligned}$$

   so,

$$\theta = \frac{\pi}{4}$$

   (c) Use the definition of the dot product to determine $\vec{a}\cdot\vec{b}$

   **Solution:**

$$\vec{a}\cdot\vec{b} = 2\times 1 + 0\times 1 = 2$$

   (d) Find the area of the parallelogram formed by the vectors $\vec{a}$ and $\vec{b}$.

   **Solution:**

$$\text{Area} = \left|\det\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\right)\right| = |2\times 1 - 0\times 1| = 2$$

   (e) Compute $\text{proj}_{\vec{b}}\vec{a}$

   **Solution:**

$$\text{proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b} = \frac{2}{2}\vec{b} = \vec{i} + \vec{j}$$

(f) Without using your calculator, please determine if the angle formed by $\vec{b}$ and $2\vec{a} - \vec{b}$ is acute or not.

**Solution:** Because

$$2\vec{a} - \vec{b} = 4\vec{i} - (\vec{i} + \vec{j}) = 3\vec{i} - \vec{j}$$

and

$$\cos\theta = \frac{\vec{b} \cdot (2\vec{a} - \vec{b})}{\|\vec{b}\|\|(2\vec{a} - \vec{b})\|} = \frac{1 \times 3 + 1 \times (-1)}{\sqrt{2} \times \sqrt{3^2 + (-1)^2}} = \frac{2}{2\sqrt{5}} > 0.$$

It implies that the angle $\theta$ formed by $\vec{b}$ and $2\vec{a} - \vec{b}$ is acute.

5. If $\vec{a} = \frac{1}{\sqrt{2}}\hat{i} - \sqrt{2}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}$ and $\vec{b} = \hat{i} + \hat{j} + \hat{k}$, then

(a) Find $\sqrt{2}\vec{a} - \vec{b}$

**Solution:**

$$\sqrt{2}\vec{a} - \vec{b} = (\vec{i} - 2\vec{j} + \vec{k}) - (\hat{i} + \hat{j} + \hat{k}) = -3\vec{j}$$

(b) Find $\|\vec{a}\|$ and $\|\vec{b}\|$

**Solution:**

$$\|\vec{a}\| = \sqrt{(1/\sqrt{2})^2 + (-\sqrt{2})^2 + (1/\sqrt{2})^2} = \sqrt{3}$$

$$\|\vec{b}\| = \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

(c) Find a unit vector that has the opposite direction as $\vec{a}$.

**Solution:** It is

$$-\frac{\vec{a}}{\|\vec{a}\|} = -\frac{1}{\sqrt{3}}(\frac{1}{\sqrt{2}}\hat{i} - \sqrt{2}\hat{j} + \frac{1}{\sqrt{2}}\hat{k}) = -\frac{1}{\sqrt{6}}\hat{i} + \frac{\sqrt{2}}{\sqrt{3}}\hat{j} - \frac{1}{\sqrt{6}}\hat{k}$$

(d) Find a vector of magnitude 7 that has the same direction to $\vec{a}$.

**Solution:** It is

$$7\frac{\vec{a}}{\|\vec{a}\|} = \frac{7}{\sqrt{6}}\hat{i} - \frac{7\sqrt{2}}{\sqrt{3}}\hat{j} + \frac{7}{\sqrt{6}}\hat{k}$$

(e) Compute $\vec{a} \cdot \vec{b}$

**Solution:**

$$\vec{a} \cdot \vec{b} = \frac{1}{\sqrt{2}} \times 1 + (-\sqrt{2}) \times 1 + \frac{1}{\sqrt{2}} \times 1 = 0$$

(f) Show that $\vec{a}$ and $\vec{b}$ are linearly independent.

**Solution:** From previous result, $\vec{a}$ and $\vec{b}$ are orthogonal. Hence, $\vec{a}$ and $\vec{b}$ are linearly independent.