Section 1.5: Vectors in Three-Dimensional Space

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- A unit vector is a vector with magnitude 1

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In physics, representing force using vectors is an important example. The magnitude and the direction of the vector represents the corresponding attributes of the force. Moving force to different action points of a solid does not influence its effect justifies vectors are allow to free moving.

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In geometry, vectors are used to represent sides of polygons. Move a polygon relative to a coordinate system does not change its property is consistent with free vectors.

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- To compute, if $\vec{a} = (a_1, a_2, a_3)$ and $\vec{b} = (b_1, b_2, b_3)$ then

$$\vec{a} + \vec{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$$

and

$$r\vec{a} = (ra_1, ra_2, ra_3).$$

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- (associative) $\vec{a} + (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + \vec{c}$
- (distributive) $k\vec{a} + k\vec{b} = k(\vec{a} + \vec{b})$

Examples

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- Let A, B, C, and D be four points in space such that no three are colinear. Show that segments joining the midpoints of the segments AB, BC, CD, and DA form a parallelogram.

Linearly (In)dependent

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implies that $c_1 = c_2 = 0$.

Example for linear independent

Show that the diagonals of a parallelogram bisect one another.

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  - $\overrightarrow{a} + \overrightarrow{b} = (a_1 + b_1)\widehat{\imath} + (a_2 + b_2)\widehat{\jmath} + (a_3 + b_3)\widehat{k}$
  - $c\,\overrightarrow{a} = (ca_1)\widehat{\imath} + (ca_2)\widehat{\jmath} + (ca_3)\widehat{k}$
  - $\|\overrightarrow{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

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  - $\overrightarrow{a} + \overrightarrow{b} = (a_1 + b_1)\widehat{\imath} + (a_2 + b_2)\widehat{\jmath} + (a_3 + b_3)\widehat{k}$
  - $c\,\overrightarrow{a} = (ca_1)\widehat{\imath} + (ca_2)\widehat{\jmath} + (ca_3)\widehat{k}$
  - $\|\overrightarrow{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
  - $\overrightarrow{a} = \overrightarrow{b}$ if and only if $a_1 = b_1$, $a_2 = b_2$, and $a_3 = b_3$.

Another Example

Find the vector that points from $(0, 5, 10)$ to $(-4, 11, 1)$. Then find its magnitude.