Trend Following Trading under a Regime Switching Model*

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Abstract

This paper is concerned with the optimality of a trend following trading rule. The idea is to catch a bull market at its early stage, ride the trend, and liquidate the position at the first evidence of the subsequent bear market. We characterize the bull and bear phases of the markets mathematically using the conditional probabilities of the bull market given the up to date stock prices. The optimal buying and selling times are given in terms of a sequence of stopping times determined by two threshold curves. Numerical experiments are conducted to validate the theoretical results and demonstrate how they perform in a marketplace.

Keywords: Optimal stopping time, regime switching model, Wonham filter, trend following trading rule

AMS subject classifications: 91G80, 93E11, 93E20

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1 Introduction

Trading in organized exchanges has increasingly become an integrated part of our life. Big moves of market indices of major stock exchanges all over the world are often the headlines of news media. By and large, active market participants can be classified into two groups according to their trading strategies: those who trade contra-trend and those who follow the trend. On the other hand, there are also passive market participants who simply buy and hold for a long period of time (often indirectly through mutual funds). Within each group of strategies there are numerous technical methods. Much effort has been devoted to theoretical analysis of these strategies.

Using optimal stopping time to study optimal exit strategy for stock holdings has become standard textbook examples. For example, Øksendal [24, Examples 10.2.2 and 10.4.2] considered optimal exit strategy for stocks whose price dynamics were modeled by a geometric Brownian motion. To maximize an expected return discounted by the risk free interest rate, the analysis in [24] showed that if the drift of the geometric Brownian motion was not high enough in comparison to the discount of interest rate then one should sell at a given threshold. Although the model of a single geometric Brownian motion with a constant drift was somewhat too simplistic, this result well illustrated the flaw of the so-called “buy and hold” strategy, which worked only in limited situations. Stock selling rules under more realistic models have gained increasing attention. For example, Zhang [30] considered a selling rule determined by two threshold levels, a target price and a stop-loss limit. Under a regime switching model, optimal threshold levels were obtained by solving a set of two-point boundary value problems. In Guo and Zhang [13], the results of Øksendal [24] were extended to incorporate a model with regime switching. In addition to these analytical results, various mathematical tools have been developed to compute these threshold levels. For example, a stochastic approximation technique was used in Yin et al. [29]; a linear programming approach was developed in Helmes [14]; and the fast Fourier transform was used in Liu et al. [20]. Furthermore, consideration of capital gain taxes and transaction costs in connection with selling can be found in Cadenillas and Pliska [3], Constantinides [4], and Dammon and Spatt [9] among others.
Recently, there has been an increasing volume of literature concerning with trading rules that involved both buying and selling. For instance, Zhang and Zhang [32] studied the optimal trading strategy in a mean reverting market, which validated a well known contratrend trading method. In particular, they established two threshold prices (buy and sell) that maximized overall discounted return if one traded at those prices. In addition to the results obtained in [32] along this line of research, an investment capacity expansion/reduction problem was considered in Merhi and Zervos [22]. Under a geometric Brownian motion market model, the authors used the dynamic programming approach and obtained an explicit solution to the singular control problem. A more general diffusion market model was treated by Løkka and Zervos [21] in connection with an optimal investment capacity adjustment problem. More recently, Johnson and Zervos [16] studied an optimal timing of investment problem under a general diffusion market model. The objective was to maximize the expected cash flow by choosing when to enter an investment and when to exit the investment. An explicit analytic solution was obtained in [16].

However, a theoretical justification of trend following trading methods is missing despite that they are widely used among professional traders (see e.g. [26]). It is the purpose of this paper to fill this void. We adopt a finite horizon regime switching model for the stock price dynamics. In this model the price of the stock follows a geometric Brownian motion whose drift switches between two different regimes representing the up trend (bull market) and down trend (bear market), respectively, and the exactly switching times between the different trends are not directly observable as in the real markets. We model the switching as an unobservable Markov chain. Our trading decisions are based on current information represented by both the stock price and historical information with the probability in the bull phase conditioning to all available historical price levels as a proxy. Assuming trading one share with a fixed percentage transaction cost, we show that the strategy that optimizes the discounted expected return is a simple implementable trend following system. This strategy is characterized by two time dependent thresholds for the conditional probability in a bull regime signaling buy and sell, respectively. The main advantage of this approach is that the conditional probability in a bull market can be obtained directly using actual historical stock price data through a differential equation.
The derivation of this result involves a number of different technical tools. One of the main difficulties in handling the regime switching model is that the Markov regime switching process is unobservable. Following Rishel and Helmes [25] we use the optimal nonlinear filtering technical (see e.g. [18, 28]) regarding the conditional probability in a bull regime as an observation process. Combining with the stock price process represented in terms of this observing process, we obtain an optimal stopping problem with complete observation. Our model involves possibly infinitely many buy and sell operations represented by sequences of stopping times and it is not a standard stopping time problem. As in Zhang and Zhang [32] we introduce two optimal value functions that correspond to starting net position being either flat or long. Using a dynamic programming approach, we can formally derive a system of two variational inequalities. A verification theorem justifies that the solutions to these variational inequalities are indeed the optimal value functions. It is interesting that we can show that this system of variational inequalities leads to a double obstacle problem satisfied by the difference of the two value functions. Since the solution and properties of double obstacle problems are well understood, this conversion simplifies the analysis of our problem considerably. Accompanying numerical procedure is also established to determine the thresholds involved in our optimal trend following strategy.

Numerical experiments have been conducted for a simple trend following trading strategy that approximates the optimal one. We test our strategy using both simulation and actual market data for the NASDAQ, SP500 and DJIA indices. Our trend following trading strategy outperforms the buy and hold strategy with a huge advantage in simulated trading. This strategy also significantly prevails over the buy and hold strategy when tested with the real historical data for the NASDAQ, SP500 and DJIA indices.

The rest of the paper is arranged as follows. We formulate our problem and present its theoretical solutions in the next section. Numerical results for optimal trading strategy are presented in Section 3. We conduct extensive simulations and tests on market data in Section 4, and conclude in Section 5. Details of data and results related to simulations and market tests are collected in the Appendix.
2 Problem formulation

Let \( S_r \) denote the stock price at time \( r \) satisfying the equation

\[
dS_r = S_r [\mu(\alpha_r) dr + \sigma dB_r], \quad S_t = S, \quad t \leq r \leq T < \infty
\]

where \( \mu(i) = \mu_i, \ i = 1, 2 \), are the expected return rates, \( \alpha_r \in \{1, 2\} \) is a two-state Markov chain, \( \sigma > 0 \) is the volatility, \( B_r \) is a standard Brownian motion, and \( T \) is a finite time.

The process \( \alpha_r \) represents the market mode at each time \( r \): \( \alpha_r = 1 \) indicates a bull market and \( \alpha_r = 2 \) a bear market. Naturally, we assume \( \mu_1 > \mu_2 \).

Let \( \lambda_i > 0, \ i = 1, 2 \) denote the generator of \( \alpha_r \). We assume that \( \{\alpha_r\} \) and \( \{B_r\} \) are independent.

Let

\[
t \leq \tau_1 \leq v_1 \leq \tau_2 \leq v_2 \leq \cdots \leq T, \ a.s.,
\]

denote a sequence of stopping times. Note that one may construct a sequence of stopping times satisfying the above inequalities from any monotone sequence of stopping times truncated at time \( T \). A buying decision is made at \( \tau_n \) and a selling decision is at \( v_n \), \( n = 1, 2, \ldots \).

We consider the case that the net position at any time can be either flat (no stock holding) or long (with one share of stock holding). Let \( i = 0, 1 \) denote the initial net position. If initially the net position is long \( (i = 1) \), then one should sell the stock before acquiring any share. The corresponding sequence of stopping times is denoted by \( \Lambda_1 = (v_1, \tau_2, v_2, \tau_3, \ldots) \). Likewise, if initially the net position is flat \( (i = 0) \), then one should first buy a stock before selling any shares. The corresponding sequence of stopping times is denoted by \( \Lambda_0 = (\tau_1, v_1, \tau_2, v_2, \ldots) \).

Let \( 0 < K < 1 \) denote the percentage of slippage (or commission) per transaction. Given the initial stock price \( S_t = S \), initial market trend \( \alpha_t = \alpha \in \{1, 2\} \), and initial net position \( i = 0, 1 \), the reward functions of the decision sequences, \( \Lambda_0 \) and \( \Lambda_1 \), are given as follows:

\[
J_i(S, \alpha, t, \Lambda_i) = \begin{cases} 
E_t \left\{ \sum_{n=1}^{\infty} \left[ e^{-\rho(v_n-t)} S_{v_n} (1 - K) - e^{-\rho(\tau_n-t)} S_{\tau_n} (1 + K) \right] I_{\{\tau_n < T\}} \right\}, & \text{if } i = 0, \\
E_t \left\{ e^{-\rho(v_1-t)} S_{v_1} (1 - K) \right. \\
+ \sum_{n=2}^{\infty} \left[ e^{-\rho(v_n-t)} S_{v_n} (1 - K) - e^{-\rho(\tau_n-t)} S_{\tau_n} (1 + K) \right] I_{\{\tau_n < T\}} \right\}, & \text{if } i = 1,
\end{cases}
\]
where $\rho > 0$ is the discount factor. Here, term $E \sum_{n=1}^{\infty} \xi_n$ for random variables $\xi_n$ is interpreted as $\limsup_{N \to \infty} E \sum_{n=1}^{N} \xi_n$. Our goal is to maximize the reward function.

To exclude trivial cases,\(^1\) we always assume

$$\mu_2 < \rho < \mu_1.$$  

**Remark 1** Note that the indicator function $I_{\{\tau_n < T\}}$ is used in the definition of the reward functions $J_i$. This is to ensure that if the last buy order is entered at $t = \tau_n$, then the position will be sold at $v_n \leq T$.

The indicator function $I$ confines the effective part of the sum to a finite time horizon so that the reward functions are bounded above.

Note that only the stock price $S_r$ is observable at time $r$ in marketplace. The market trend $\alpha_r$ is not directly available. Thus, it is necessary to convert the problem into a completely observable one. One way to accomplish this is to use the Wonham filter [28]; See also [18] and [31] for recent development and references therein in connection with Wonham filters.

Let $p_r = P(\alpha_r = 1 | S_r)$ denote the conditional probability of $\alpha_r = 1$ (bull market) given the filtration $\mathcal{S}_r = \sigma \{ S_u : 0 \leq u \leq r \}$. Then we can show (see Wonham [28]) that $p_r$ satisfies the following SDE

$$dp_r = \left[-(\lambda_1 + \lambda_2) p_r + \lambda_2\right] dr + \frac{(\mu_1 - \mu_2)p_r(1-p_r)}{\sigma} d\hat{B}_r,$$

where $\hat{B}_r$ is the innovation process (a standard Brownian motion; see e.g., Øksendal [24]) given by

$$d\hat{B}_r = \frac{d \log(S_r) - [(\mu_1 - \mu_2)p_r + \mu_2 - \sigma^2/2] dr}{\sigma}.$$

Given $S_t = S$ and $p_t = p$, the problem is to choose $\Lambda_i$ to maximize the discounted return

$$J_i(S, p, t, \Lambda_i) = J_i(S, \alpha, t, \Lambda_i),$$

subject to

$$\begin{cases} 
    dS_r = S_r [(\mu_1 - \mu_2) p_r + \mu_2] dr + S_r \sigma d\hat{B}_r, & S_t = S, \\
    dp_r = \left[-(\lambda_1 + \lambda_2) p_r + \lambda_2\right] dr + \frac{(\mu_1 - \mu_2)p_r(1-p_r)}{\sigma} d\hat{B}_r, & p_t = p.
\end{cases}$$

\(^1\)Intuitively, one should never buy stock if $\rho \geq \mu_1$ and never sell stock if $\rho \leq \mu_2$, which coincides with Lemma 4.
Indeed, this new problem is a completely observable one because the conditional probability can be obtained using the stock price up to time \( r \).

For \( i = 0, 1 \), let \( V_i(S, p, t) \) denote the value functions with the states \((S, p)\) and net positions \( i = 0, 1 \) at time \( t \). That is,

\[
V_i(S, p, t) = \sup_{\Lambda_i} J_i(S, p, t, \Lambda_i).
\]

The following lemma gives the upper bounds of the values functions.

**Lemma 1** We have

\[
0 \leq V_0(S, p, t) \leq S[e^{(\mu_1 - \rho)(T - t)} - 1], \\
0 \leq V_1(S, p, t) \leq S[2e^{(\mu_1 - \rho)(T - t)} - 1].
\]

Proof. It is clear that the nonnegativity of \( V_i \) follows from their definition. It remains to show their upper bounds. First, given \( \Lambda_0 \), we have

\[
e^{-\rho v_n}S_v - e^{-\rho \tau_n}S_{\tau_n} = \int_{\tau_n}^{\nu_n} e^{-\rho r}S_r((\mu_1 - \mu_2)p_r + \mu_2 - \rho)dr + \int_{\tau_n}^{\nu_n} e^{-\rho r}S_r\sigma d\tilde{B}_r.
\]

Note that

\[
E \left[ I_{\{\tau_n < T\}} \int_{\tau_n}^{\nu_n} e^{-\rho r}S_r\sigma d\tilde{B}_r \right] = E \left[ I_{\{\tau_n < T\}} E \left[ \int_{\tau_n}^{\nu_n} e^{-\rho r}S_r\sigma d\tilde{B}_r | \tau_n \right] \right] = 0.
\]

It follows that

\[
e \left[ I_{\{\tau_n < T\}} \int_{\tau_n}^{\nu_n} e^{-\rho r}S_r((\mu_1 - \mu_2)p_r + \mu_2 - \rho)dr \right]
\]

\[
\leq (\mu_1 - \rho) \int_{\tau_n}^{\nu_n} e^{-\rho r}ES_rdr.
\]

Using the definition of \( J_0(S, p, t, \Lambda_0) \), we have

\[
J_0(S, p, t, \Lambda_0) \leq \sum_{n=1}^{\infty} E \left( e^{-\rho (\nu_n - t)}(S_{\nu_n} - e^{-\rho (\tau_n - t)}S_{\tau_n}) \right) I_{\{\tau_n < T\}}
\]

\[
\leq (\mu_1 - \rho) \sum_{n=1}^{\infty} e^{\rho t} \int_{\tau_n}^{\nu_n} e^{-\rho r}S_rdr
\]

\[
\leq (\mu_1 - \rho)e^{\rho t} \int_{t}^{T} e^{-\rho r}ES_rdr.
\]

It is easy to see using Gronwall’s inequality that \( ES_r \leq Se^{\mu_1(r-t)} \). It follows that

\[
J_0(S, p, t, \Lambda_0) \leq (\mu_1 - \rho)e^{\rho t} S \int_{t}^{T} e^{-\rho r + \mu_1(r-t)}dr = S[e^{(\mu_1 - \rho)(T-t)} - 1].
\]
This implies that $0 \leq V_0(x) \leq S[e^{(\mu_1 - \rho)(T-t)} - 1]$.

Similarly, we have the inequality

$$J_1(S, p, t, \Lambda_1) \leq E e^{-\rho(v_1-t)} S v_1 (1 - K) + S[e^{(\mu_1 - \rho)(T-t)} - 1].$$

Moreover, note that

$$E e^{-\rho(v_1-t)} S v_1 - S \leq (\mu_1 - \rho)e^{\rho t} \int_t^T e^{-\rho r} S_r dr \leq S[e^{(\mu_1 - \rho)(T-t)} - 1].$$

This implies that

$$V_1(S, p, t) \leq S + 2S[e^{(\mu_1 - \rho)(T-t)} - 1] = S[2e^{(\mu_1 - \rho)(T-t)} - 1].$$

Therefore, $0 \leq V_1(S, p, t) \leq S[2e^{(\mu_1 - \rho)(T-t)} - 1]$. This completes the proof. □

Let

$$A = \frac{1}{2} \left( \frac{(\mu_1 - \mu_2)p(1-p)}{\sigma} \right)^2 \partial_{pp} + \frac{1}{2} \sigma^2 S^2 \partial_{SS} + S [(\mu_1 - \mu_2)p(1-p)] \partial_p$$
$$+ [- (\lambda_1 + \lambda_2) + \lambda_2] \partial_p + S [(\mu_1 - \mu_2)p + \mu_2] \partial_S - \rho.$$

Then the HJB equations associated with our optimal stopping time problem can be given formally as follows:

$$\min \{-\partial_t V_0 - AV_0, V_0 - V_1 + S(1 + K)\} = 0,$$
$$\min \{-\partial_t V_1 - AV_1, V_1 - V_0 - S(1 - K)\} = 0,$$

in $(0, +\infty) \times (0, 1) \times [0, T)$, with the terminal conditions

$$V_0(S, p, T) = 0,$$
$$V_1(S, p, T) = S(1 - K).$$

The terminal condition implies that at the terminal time $T$, the net position must be flat.

**Remark 2** In this paper, we restrict the state space of $p$ to $(0, 1)$ because both $p = 0$ and $p = 1$ are entrance boundaries (see Karlin and Taylor [17] for definition and discussions). Such boundaries cannot be reached from the interior of the state space. If the process begins there, it quickly moves to the interior and never returns. We next show that $p = 0$ is indeed an entrance boundary. The case when $p = 1$ is similar. It is easy to see that the speed density (see [17]) can be given by

$$m(x) = \frac{\sigma^2}{(\mu_1 - \mu_2)^2 x^2 (1 - x)^2 s(x)^3}.$$
where
\[ s(x) = \exp \left\{ \frac{2\sigma^2}{(\mu_1 - \mu_2)^2} \left[ \frac{\lambda_2}{x} + \frac{\lambda_1}{1-x} + (\lambda_2 - \lambda_1) \log \frac{x}{1-x} \right] \right\}. \]

To show \( p = 0 \) is the entrance boundary, it suffices ([17]) to show that, for any \( 0 < a < 1 \),
\[ \lim_{\delta \to 0^+} \int_a^\delta \left( \int_\delta^\xi s(\eta) d\eta \right) m(\xi) d\xi = \infty, \] (3)

and
\[ \lim_{\delta \to 0^+} \int_a^\delta \left( \int_\delta^a s(\eta) d\eta \right) m(\xi) d\xi < \infty. \] (4)

Now, (3) follows the fact that \( \lim_{\delta \to 0^+} \int_\delta^\xi s(\eta) d\eta = \infty \), for any \( \xi > 0 \). Moreover, note that for any \( A > 0 \), after change of variables,
\[ \int_0^a \left( \int_\xi^a e^{A/\eta} d\eta \right) e^{-A/\xi} d\xi = \int_1^\infty \left( \int_1^a e^{A/\eta} d\eta \right) e^{-A\xi} d\xi = \int_1^\infty \left( \int_a^\xi e^{A/\eta} d\eta \right) e^{-A\xi} d\xi < \infty. \]

Using this estimate, it is not difficult to see (4).

It is easy to show that the value functions \( V_0 \) and \( V_1 \) are linear in \( S \). This motivates us to adopt the following transformation: \( U_0(p,t) = V_0(S,p,t)/S \) and \( U_1(p,t) = V_1(S,p,t)/S \). Then the HJB equations (1) with the terminal condition (2) can be reduced to
\[ \min \{ -\partial_t U_0 - \mathcal{L}U_0, U_0 - U_1 + (1 + K) \} = 0, \]
\[ \min \{ -\partial_t U_1 - \mathcal{L}U_1, U_1 - U_0 - (1 - K) \} = 0, \] (5)
in \((0,1) \times [0,T)\), with the terminal conditions
\[ U_0(p,T) = 0, \]
\[ U_1(p,T) = 1 - K, \] (6)

where
\[ \mathcal{L} = \frac{1}{2} \left( \frac{(\mu_1 - \mu_2)p(1-p)}{\sigma^2} \right)^2 \partial_{pp} + \left[ -(\lambda_1 + \lambda_2) p + \lambda_2 + (\mu_1 - \mu_2)p(1-p) \right] \partial_p + (\mu_1 - \mu_2) p + \mu_2 - \rho. \]

Thanks to Lemma 1, we will focus on the bounded solutions of problem (5)-(6).

**Lemma 2** Problem (5)-(6) has a unique bounded strong solution \( (U_0, U_1) \), where \( U_i \in W_{q,1}^2([\varepsilon, 1-\varepsilon] \times [0,T]) \), for any \( \varepsilon \in (0,1/2), q \in [1, +\infty) \). Moreover,
\[ -\partial_t U_0 - \mathcal{L}U_0 = (-\partial_t Z - \mathcal{L}Z)^-, \] (7)
\[ -\partial_t U_1 - \mathcal{L}U_1 = (-\partial_t Z - \mathcal{L}Z)^+, \] (8)
where $Z(p,t) \equiv U_1(p,t) - U_0(p,t)$ is the unique strong solution to the following double obstacle problem:

$$\min \{ \max \{ -\partial_t Z - \mathcal{L}Z, Z - (1 + K) \}, Z - (1 - K) \} = 0,$$

or equivalently,

$$-\partial_t Z - \mathcal{L}Z = 0 \text{ if } 1 - K < Z < 1 + K,$$

$$-\partial_t Z - \mathcal{L}Z \geq 0 \text{ if } Z = 1 - K,$$

$$-\partial_t Z - \mathcal{L}Z \leq 0 \text{ if } Z = 1 + K,$$

in $(0, 1) \times [0, T)$, with the terminal condition $Z(p,T) = 1 - K$. Furthermore,

$$\partial_p Z \geq 0, \text{ and }$$
$$\partial_t Z \leq 0.$$

Proof: For any strong solution $(U_0, U_1)$ of problem (5)-(6), we first show $2Z(p,t) \equiv U_1(p,t) - U_0(p,t)$ satisfies (9)-(11). Indeed, if $1 - K < Z(p,t) < 1 + K$, then

$$-\partial_t U_0 - \mathcal{L}U_0|_{(p,t)} = -\partial_t U_1 - \mathcal{L}U_1|_{(p,t)} = 0,$$

which gives $-\partial_t Z - \mathcal{L}Z|_{(p,t)} = 0$. If $Z(p,t) = 1 - K$ or $U_1(p,t) - U_0(p,t) - (1 - K) = 0$, then $U_0(p,t) = U_1(p,t) + 1 + K = 2K > 0$, from which we infer $-\partial_t U_0 - \mathcal{L}U_0|_{(p,t)} = 0$. On the other hand, we always have $-\partial_t U_1 - \mathcal{L}U_1|_{(p,t)} \geq 0$, so that $-\partial_t Z - \mathcal{L}Z|_{(p,t)} \geq 0$. Similarly we can deduce $-\partial_t Z - \mathcal{L}Z \leq 0$ if $Z = 1 + K$.

By the penalization method (cf. Friedman [12]), we can show that the double obstacle problem has a unique strong solution

$$Z(p,t) \in W^{2,1}_q([\varepsilon, 1 - \varepsilon] \times [0, T]),$$

for any $\varepsilon \in (0, 1/2)$, $q \in [1, +\infty)$. To show the regularity and uniqueness of bounded solution to problem (5)-(6), it suffices to show that the solution $(U_0, U_1)$ to problem (5)-(6) satisfies (7)-(8). Let us prove (7) first. When $U_0(p,t) - U_1(p,t) > -(1 + K)$ or $Z(p,t) < 1 + K$, we

\footnote{After finishing the paper, we found that the connection between a system of variational inequalities and a double obstacle problem was first revealed in Nakoulima [23].}
have
\[-\partial_t U_0 - \mathcal{L}U_0 = 0,\]
\[-\partial_t Z - \mathcal{L}Z \geq 0,\]
from which we can see (7) holds. As a result, it suffices to show that (7) remains valid when
\[U_0(p,t) - U_1(p,t) = -Z(p,t) = -(1 + K).\] In this case
\[U_1(p,t) - U_0(p,t) = Z(p,t) > 1 - K\]
and
\[-\partial_t U_1 - \mathcal{L}U_1 = 0,\]
\[-\partial_t Z - \mathcal{L}Z \leq 0.\]

It follows that
\[0 \geq -\partial_t Z - \mathcal{L}Z = (-\partial_t U_1 - \mathcal{L}U_1) - (-\partial_t U_0 - \mathcal{L}U_0)\]
\[= -(-\partial_t U_0 - \mathcal{L}U_0),\]
which implies the desired result (7). Equation (8) can be proved in a similar way.

Now let us prove (12). We need only to restrict attention to
\[NT \equiv \{(p,t) \in (0,1) \times [0,T) : 1 - K < Z(p,t) < 1 + K\},\]
in which \(-\partial_t Z - \mathcal{L}Z = 0\). Differentiating the equation w.r.t. \(p\), we have
\[-\partial_t (\partial_p Z) - \mathcal{T} [\partial_p Z] = (\mu_1 - \mu_2) Z,\]
where \(\mathcal{T}\) is another differential operator. Owing to \((\mu_1 - \mu_2) Z \geq 0\) in \(NT\) and \(\partial_p Z = 0\)
on the boundary of \(NT\setminus\{p = 0, 1\}\),\(^3\) we then get (12) by using the maximum principle. It
remains to prove (13). Clearly \(\partial_t Z|_{t=T} \leq 0\), which yields the desired result again by the
maximum principle. \(\Box\)

The region \(NT\) defined in (14) refers to the no-trading region. In terms of the solution
\[Z(p,t)\] to the double obstacle problem (9)-(11) with the terminal condition \(Z(p,T) = 1 - K\),
we can define the buying region (\(BR\)) and the selling region (\(SR\)) as follows:
\[BR = \{(p,t) \in (0,1) \times [0,T) : Z(p,t) = 1 + K\},\]
\[SR = \{(p,t) \in (0,1) \times [0,T) : Z(p,t) = (1 - K)\}.\]

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\(^3\)On \(p = 0\) or \(1\), no boundary conditions are required due to the degeneracy of the differential operator.
We aim to characterize these regions through the study of the double obstacle problem. To begin with, we prove the connectivity of any $t$-section of $BR$ or $SR$.

**Lemma 3** For any $t \in [0, T)$,

i) if $(p_1, t) \in BR$ and $p_2 \geq p_1$, then $(p_2, t) \in BR$;

ii) if $(p_1, t) \in SR$ and $p_2 \leq p_1$, then $(p_2, t) \in SR$.

Proof: We only prove part i) since the proof of part ii) is similar. Since $Z(p_1, t) = 1 + K$, we infer by (12) that $Z(p_2, t) \leq Z(p_1, t) = (1 + K)$. On the other hand, $Z(p_2, t) \geq 1 + K$. So we must have

$$Z(p_2, t) = 1 + K,$$

i.e. $(p_2, t) \in BR$, as desired. □

The lemma below implies that $BR$ shrinks and $SR$ expands as $t$ approaches the terminal time $T$.

**Lemma 4** For any $p \in (0, 1)$,

i) if $(p, t_1) \in BR$ and $t_2 \leq t_1$, then $(p, t_2) \in BR$;

ii) if $(p, t_1) \in SR$ and $t_2 \geq t_1$, then $(p, t_2) \in SR$.

Moreover

$$BR \subset \{(p, t) \in (0, 1) \times [0, T) : p \geq \frac{\rho - \mu_2}{\mu_1 - \mu_2}\}; \quad (15)$$

$$SR \subset \{(p, t) \in (0, 1) \times [0, T) : p \leq \frac{\rho - \mu_2}{\mu_1 - \mu_2}\}. \quad (16)$$

Proof. In view of (13), the proofs of part i) and ii) are similar to that of Lemma 3. It remains to show (15) and (16). Due to (10), we infer that if $(p, t) \in BR$, then

$$0 \leq (-\partial_t - \mathcal{L}) (1 - K) = -[(\mu_1 - \mu_2)p + \mu_2 - \rho],$$

namely,

$$p \leq \frac{\rho - \mu_2}{\mu_1 - \mu_2},$$

which gives (15). By (11), we can similarly get (16). □
Combining $BR \cap SR = \emptyset$ with (15) and (16), we deduce that any $t$-section of $NT$ is non-empty. In view of Lemma 3, we can define two free boundaries:

$$p_s^*(t) = \inf\{p \in (0, 1) : (p, t) \in NT\} \quad (17)$$
$$p_b^*(t) = \sup\{p \in (0, 1) : (p, t) \in NT\} \quad (18)$$

for any $t \in [0, T)$. They are the threshold for sell and buy, respectively.

**Theorem 5** Let $p_s^*(t)$ and $p_b^*(t)$ be as given in (17)-(18). Then

i) $p_b^*(t)$ and $p_s^*(t)$ are monotonically increasing in $t$, and

$$p_b^*(t) \geq \frac{\rho - \mu_2}{\mu_1 - \mu_2} \geq p_s^*(t)$$

for all $t \in [0, T)$. Moreover, $p_s^*(t), p_b^*(t) \in C^\infty(0, T)$;

ii) $p_s^*(T) := \lim_{t \to T^-} p_s^*(t) = \frac{\rho - \mu_2}{\mu_1 - \mu_2}$;

iii) There is a $\delta > 0$ such that $(1, t) \in NT$ for all $t \in (T - \delta, T)$, namely,

$$p_s^*(t) = 1 \text{ for } t \in (T - \delta, T).$$

Moreover,

$$\delta \geq \frac{1}{\mu_1 - \rho} \log \frac{1 + K}{1 - K}. \quad (19)$$

Proof: The monotonicity in Part i) is a corollary of Lemmas 3 and 4. The proof for the smoothness of $p_s^*(t)$ and $p_b^*(t)$ is somewhat technical and is placed in Appendix. To show part ii), we use the method of contradiction. Suppose not. Due to $p_s^*(t) \leq \frac{\rho - \mu_2}{\mu_1 - \mu_2}$, we would have

$$p_s^*(T) < \frac{\rho - \mu_2}{\mu_1 - \mu_2}.$$  

Then for any $p \in (p_s^*(T), \frac{\rho - \mu_2}{\mu_1 - \mu_2})$,

$$\partial_t Z|_{t=T} = -\mathcal{L} Z|_{t=T} = -\mathcal{L} (1 - K) = -[(\mu_1 - \mu_2) p + \mu_2 - \rho] > 0,$$

which is in contradiction with (13).

It remains to show part iii). Owing to $Z|_{t=T} = 1 - K$ and (12), the existence of $\delta$ is apparent. We only need to show (19). On $p = 1$, problem (9)-(11) is reduced to

$$\begin{cases}
-Z_t + (\rho - \mu_1) Z|_{p=1} = -\lambda_1 \partial_p Z \text{ if } Z < 1 + K, \\
-Z_t + (\rho - \mu_1) Z|_{p=1} \leq -\lambda_1 \partial_p Z \text{ if } Z = 1 + K,
\end{cases} \quad (20)$$
in \( t \in (0, T) \), with the terminal condition \( Z(1, T) = 1 - K \), where we have excluded the lower obstacle according to (16). Due to \(-\lambda_1 \partial_t Z \leq 0\), problem (20) has a supersolution \( \overline{Z}(t) \) satisfying

\[
\left\{ \begin{array}{ll}
-Z_t + (\rho - \mu_1) Z &= 0 \quad \text{if } \overline{Z}(t) < 1 + K, \\
-Z_t + (\rho - \mu_1) Z &\leq 0 \quad \text{if } \overline{Z}(t) = 1 + K,
\end{array} \right.
\]

in \( t \in (0, T) \), with \( \overline{Z}(T) = 1 - K \). It is easy to verify

\[
\overline{Z}(t) = \left\{ \begin{array}{ll}
e^{(\mu_1 - \rho)(T-t)} (1 - K) & \text{if } t > T - \frac{1}{\mu_1 - \rho} \log \frac{1 + K}{1 - K}, \\
1 + K & \text{if } t \leq T - \frac{1}{\mu_1 - \rho} \log \frac{1 + K}{1 - K}.
\end{array} \right.
\]

We then infer that

\[
Z(1, t) \leq \overline{Z}(t) = e^{(\mu_1 - \rho)(T-t)} (1 - K) < 1 + K \quad \text{for } t > T - \frac{1}{\mu_1 - \rho} \log \frac{1 + K}{1 - K},
\]

which implies \((1, t) \in NT\) for any \( t > T - \frac{1}{\mu_1 - \rho} \log \frac{1 + K}{1 - K} \). Then (19) follows. \( \Box \)

**Remark 3** Part iii) indicates that there is a critical time after which it is never optimal to buy stock. This is an important feature when transaction costs are involved in a finite horizon model. Similar results were obtained in the study of finite horizon portfolio selection with transaction costs (cf. [5], [7], [8], and [19]). The intuition is that if the investor does not have a long enough time horizon to recover at least the transaction costs, then s/he should not initiate a long position (bear in mind that the terminal position must be flat).

**Remark 4** Using the maximum principle, it is not hard to show that \( Z(\cdot, \cdot; \lambda_1, \lambda_2, \rho) \) is a decreasing function of \( \lambda_1, \rho \), and an increasing function of \( \lambda_2 \). As a consequence, \( p_s^*(\cdot; \lambda_1, \lambda_2, \rho) \) and \( p_b^*(\cdot; \lambda_1, \lambda_2, \rho) \) are also increasing functions of \( \lambda_1, \rho \), and decreasing functions of \( \lambda_2 \).

By Lemma 2, Sobolev embedding theorem (cf. [12]), and the smoothness of free boundaries, the solutions \( U_0 \) and \( U_1 \) of problem (5)-(6) belong to \( C^1 \) in \((0, 1) \times [0, T)\). Furthermore, it is easy to show that the solutions are sufficiently smooth (i.e., at least \( C^2 \)) except at the free boundaries \( p_s^*(t) \) and \( p_b^*(t) \). These enable us to establish a verification theorem to show that the solutions \( U_0 \) and \( U_1 \) of problem (5)-(6) are equal to the value functions \( V_0/S \) and \( V_1/S \), respectively, and sequences of optimal stopping times can be constructed by using \( (p_s^*, p_b^*) \).
This theorem gives sufficient conditions for optimality of the trading rules in terms of the stopping times \( \{\tau_n, v_n\} \). The construction procedure will be used in the next sections to develop numerical solutions in various scenarios.

**Theorem 6** (Verification Theorem) Let \((U_0, U_1)\) be the unique bounded strong solution to problem (5)-(6) and \(p_n^*(t)\) and \(p_n^*(t)\) be the associated free boundaries. Then, \(w_0(S,p,t) \equiv SU_0(p,t)\) and \(w_1(S,p,t) \equiv SU_1(p,t)\) are equal to the value functions \(V_0(S,p,t)\) and \(V_1(S,p,t)\), respectively.

Moreover, let
\[
\Lambda_0^* = (\tau_1^*, v_1^*, \tau_2^*, v_2^*, \ldots),
\]
where the stopping times \(\tau_1^* = T \wedge \inf\{r \geq t : p_r \geq p_0^*(r)\}\), \(v_n^* = T \wedge \inf\{r \geq \tau_n^* : p_r \leq p_0^*(r)\}\), and \(\tau_{n+1}^* = T \wedge \inf\{r > v_n^* : p_r \geq p_0^*(r)\}\) for \(n \geq 1\), and let
\[
\Lambda_1^* = (v_1^*, \tau_2^*, v_2^*, \tau_3^*, \ldots),
\]
where the stopping times \(v_1^* = T \wedge \inf\{r \geq t : p_r^* \leq p_s^*(r)\}\), \(\tau_n^* = T \wedge \inf\{r > v_{n-1}^* : p_r \geq p_0^*(r)\}\), and \(v_n^* = T \wedge \inf\{r \geq \tau_n^* : p_r \leq p_0^*(r)\}\) for \(n \geq 2\). If \(v_n^* \to T\), a.s., as \(n \to \infty\), then \(\Lambda_0^*\) and \(\Lambda_1^*\) are optimal.

Proof: The proof is divided into two steps. In the first step, we show that \(w_i(S,p,t) \geq J_i(S,p,t,\Lambda_i)\) for all \(\Lambda_i\). Then in the second step, we show that \(w_i(S,p,t) = J_i(S,p,t,\Lambda_i^*)\). Therefore, \(w_i(S,p,t) = V_i(S,p,t)\) and \(\Lambda_i^*\) is optimal.

Using \((-\partial_t w_i - \mathcal{L} w_i) \geq 0\), Dynkin’s formula and Fatou’s lemma as in Øksendal [24, p. 226], we have, for any stopping times \(t \leq \theta_1 \leq \theta_2\), a.s.,
\[
E e^{-\rho(\theta_1-t)} w_i(S_{\theta_1}, p_{\theta_1}, \theta_1) I_{\{\theta_1 < a\}} \geq E e^{-\rho(\theta_2-t)} w_i(S_{\theta_2}, p_{\theta_2}, \theta_2) I_{\{\theta_1 < a\}},
\]
(22)
for any \(a\) and \(i = 0, 1\).

Note that \(w_0 \geq w_1 - S(1 + K)\). Given \(\Lambda_0 = (\tau_1, v_1, \tau_2, v_2, \ldots)\), by (22), and noting that \(w_0(S,p,T) = 0\), we have
\[
w_0(S,p,t) \geq E e^{-\rho(\tau_1-t)} w_0(S_{\tau_1}, p_{\tau_1}, \tau_1) = E e^{-\rho(\tau_1-t)} w_0(S_{\tau_1}, p_{\tau_1}, \tau_1) I_{\{\tau_1 < T\}} \geq E e^{-\rho(\tau_1-t)} (w_1(S_{\tau_1}, p_{\tau_1}, \tau_1) - S_{\tau_1}(1 + K)) I_{\{\tau_1 < T\}} = E e^{-\rho(\tau_1-t)} w_1(S_{\tau_1}, p_{\tau_1}, \tau_1) I_{\{\tau_1 < T\}} - E e^{-\rho(\tau_1-t)} S_{\tau_1}(1 + K) I_{\{\tau_1 < T\}}.
\]

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Using (22) again with $i = 1$ and noticing that $v_1 \geq \tau_1$ and $w_1 \geq w_0 + S(1 - K)$, we have

$$w_0(S, p, t) \geq E e^{-\rho(v_1 - t)} w_1(S_{v_1}, p_{v_1}, v_1) I_{(\tau_1 < T)} - E e^{-\rho(\tau_1 - t)} S_{\tau_1} (1 + K) I_{(\tau_1 < T)}$$

$$\geq E e^{-\rho(v_1 - t)} (w_0(S_{v_1}, p_{v_1}, v_1) + S_{v_1} (1 - K)) I_{(\tau_1 < T)} - E e^{-\rho(\tau_1 - t)} S_{\tau_1} (1 + K) I_{(\tau_1 < T)}$$

$$= E e^{-\rho(v_1 - t)} w_0(S_{v_1}, p_{v_1}, v_1) I_{(\tau_1 < T)}$$

$$+ E \left[ e^{-\rho(v_1 - t)} S_{v_1} (1 - K) - e^{-\rho(\tau_1 - t)} S_{\tau_1} (1 + K) \right] I_{(\tau_1 < T)}$$

$$= E e^{-\rho(v_1 - t)} w_0(S_{v_1}, p_{v_1}, v_1)$$

$$+ E \left[ e^{-\rho(v_1 - t)} S_{v_1} (1 - K) - e^{-\rho(\tau_1 - t)} S_{\tau_1} (1 + K) \right] I_{(\tau_1 < T)}.$$

Continue this way and recall $w_1(S, p, t) \geq 0$ to obtain

$$w_0(S, p, t) \geq E \sum_{n=1}^N \left[ e^{-\rho(v_n - t)} S_{v_n} (1 - K) - e^{-\rho(\tau_n - t)} S_{\tau_n} (1 + K) \right] I_{(\tau_n < T)}.$$ 

Sending $N \to \infty$, we have $w_0(S, p, t) \geq J_0(S, p, t, \Lambda_0)$ for all $\Lambda_0$. This implies that $w_0(S, p, t) \geq V_0(S, p, t)$. Similarly, we can show that $w_1(S, p, t) \geq V_1(S, p, t)$.

We next establish the equalities. For given $t$, define

$$\tau_1^* = \left\{ \begin{array}{ll} t & \text{if } p \geq p_b^*(t), \\
T \land \inf \{ r \geq t : p_r = p_b^*(r) \} & \text{if } p < p_b^*(t). \end{array} \right.$$ 

Using Dynkin’s formula, we have

$$w_0(S, p, t) = E e^{-\rho(\tau_1^* - t)} w_0(S_{\tau_1^*}, p_{\tau_1^*}, \tau_1^*) I_{(\tau_1^* < T)}$$

$$= E e^{-\rho(\tau_1^* - t)} w_0(S_{\tau_1^*}, p_{\tau_1^*}, \tau_1^*) I_{(\tau_1^* < T)}$$

$$= E e^{-\rho(\tau_1^* - t)} (w_1(S_{\tau_1^*}, p_{\tau_1^*}, \tau_1^*) - S_{\tau_1^*} (1 + K)) I_{(\tau_1^* < T)}$$

$$= E e^{-\rho(\tau_1^* - t)} w_1(S_{\tau_1^*}, p_{\tau_1^*}, \tau_1^*) I_{(\tau_1^* < T)} - E e^{-\rho(\tau_1^* - t)} S_{\tau_1^*} (1 + K) I_{(\tau_1^* < T)}.$$ 

Let $v_1^* = T \land \inf \{ r \geq \tau_1^* : p_r = p_b^*(r) \}$. Noticing that $w_1(S, p, T) = S(1 - K)$, we have

$$E e^{-\rho(\tau_1^* - t)} w_1(S_{\tau_1^*}, p_{\tau_1^*}, \tau_1^*) I_{(\tau_1^* < T)}$$

$$= E e^{-\rho(v_1^* - t)} w_1(S_{v_1^*}, p_{v_1^*}, v_1^*) I_{(\tau_1^* < T)}$$

$$= E e^{-\rho(v_1^* - t)} w_1(S_{v_1^*}, p_{v_1^*}, v_1^*) I_{(\tau_1^* < T)} + E e^{-\rho T} S_{T - t} (1 - K) I_{(\tau_1^* < T)} I_{(v_1^* = T)}$$

$$= E e^{-\rho(v_1^* - t)} (w_0(S_{v_1^*}, p_{v_1^*}, v_1^*) + S_{v_1^*} (1 - K)) I_{(\tau_1^* < T)} I_{(v_1^* = T)}$$

$$+ E e^{-\rho T} S_{T - t} (1 - K) I_{(\tau_1^* < T)} I_{(v_1^* = T)}$$

$$= E e^{-\rho(v_1^* - t)} w_0(S_{v_1^*}, p_{v_1^*}, v_1^*) I_{(\tau_1^* < T)} I_{(v_1^* = T)} + E e^{-\rho(\tau_1^* - t)} S_{\tau_1^*} (1 - K) I_{(\tau_1^* < T)} I_{(v_1^* < T)}$$

$$+ E e^{-\rho T} S_{T - t} (1 - K) I_{(\tau_1^* < T)} I_{(v_1^* = T)}$$

$$= E e^{-\rho(v_1^* - t)} w_0(S_{v_1^*}, p_{v_1^*}, v_1^*) + E e^{-\rho(\tau_1^* - t)} S_{\tau_1^*} (1 - K) I_{(\tau_1^* < T)} I_{(v_1^* < T)}$$

$$+ E e^{-\rho T} S_{T - t} (1 - K) I_{(\tau_1^* < T)} I_{(v_1^* = T)}$$

$$= E e^{-\rho(v_1^* - t)} w_0(S_{v_1^*}, p_{v_1^*}, v_1^*) + E e^{-\rho(\tau_1^* - t)} S_{\tau_1^*} (1 - K) I_{(\tau_1^* < T)} I_{(v_1^* < T)}.$$ 

It follows that

$$w_0(S, p, t) = E e^{-\rho(v_1^* - t)} w_0(S_{v_1^*}, p_{v_1^*}, v_1^*) + E \left[ e^{-\rho(v_1^* - t)} S_{v_1^*} (1 - K) - e^{-\rho(\tau_1^* - t)} S_{\tau_1^*} (1 + K) \right] I_{(\tau_1^* < T)}.$$
Continue the procedure to obtain
\[ w_0(S, p, t) = E e^{-\rho(v_n^*-t)} w_0(S_{v_n^*}, p_{v_n^*}, v_n^*) + E \sum_{k=1}^{n} \left[ e^{-\rho(v_k^*-t)} S_{v_k^*} (1-K) - e^{-\rho(\tau_k^*-t)} S_{\tau_k^*} (1+K) \right] I_{(\tau_k^* < T)}. \]

Similarly, we have
\[ w_1(S, p, t) = E e^{-\rho(v_n^*-t)} w_0(S_{v_n^*}, p_{v_n^*}, v_n^*) + E e^{-\rho(v_1^*-t)} S_{v_1^*} (1-K) \]
\[ + E \sum_{k=2}^{n} \left[ e^{-\rho(v_k^*-t)} S_{v_k^*} (1-K) - e^{-\rho(\tau_k^*-t)} S_{\tau_k^*} (1+K) \right] I_{(\tau_k^* < T)}. \]

Recall that \( v_n^* \to T \). Sending \( n \to \infty \) and noticing \( w_0(S, p, T) = 0 \), we obtain the equalities. This completes the proof. □

### 3 Numerical Results for Optimal Trading Strategy

The theoretical analysis in Section 2 shows that \( p_s^*(t) \) and \( p_b^*(t) \) are thresholds for the optimal trend following trading strategy: buy the stock when \( p_t \) cross \( p_s^*(t) \) from below and sell the stock when \( p_t \) cross \( p_b^*(t) \) from above. Knowing the parameters of our regime switching model, we can numerically solve the double obstacle problem (9)-(11) to derive approximations of those thresholds. To do that, we employ the penalization method with a finite difference discretization (see Dai et al. [6] and Forsyth and Vetzal [11]). The penalized approximation to the double obstacle problem is
\[ -\partial_t Z - L Z = \beta (1-K-Z)^+ - \beta (Z-1-K)^+, \]
where \( \beta \) is the penalty parameter. In our numerical examples, we choose \( \beta = 10^7 \). The right-hand side of the approximation is linearized by using a non-smooth version of the Newton iteration. Then the resulting equations are discretized by the standard finite difference method.

We take \( T = 1 \), and use model parameters in Table 1 based on the statistics for DJIA. Figure 1 represents \( p_s^*(\cdot) \) and \( p_b^*(\cdot) \) as functions of time \( t \). We see that \( p_s^*(t) \) approaches the theoretical value \( (\rho - \mu_2)/ (\mu_1 - \mu_2) = (0.0679 + 0.77)/(0.18 + 0.77) = 0.882 \), as \( t \to T = 1 \). Also, we observe that there is a \( \delta > 0 \) such that \( p_b^*(t) = 1 \) for \( t \in [T-\delta, T] \), which indicates that it is never optimal to buy stock when \( t \) is very close to \( T \). Using Theorem 5, the lower
bound of $\delta$ is estimated as $\frac{1}{\mu_1 - \rho} \log \frac{1+K}{1-K} = \frac{1}{0.18 - 0.0679} \log \frac{1.001}{0.999} = 0.0178$, which is consistent with the numerical result.

<table>
<thead>
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<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
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<th>$\mu_2$</th>
<th>$\sigma$</th>
<th>$K$</th>
<th>$\rho$</th>
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<td>0.18</td>
<td>-0.77</td>
<td>0.184</td>
<td>0.001</td>
<td>0.0679</td>
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</table>

Table 1. Parameter values

Figure 2 illustrates the impact of parameters $\lambda$ and $\rho$ on the optimal thresholds $p^*_s(\cdot)$ and $p^*_b(\cdot)$. We can see that they are increasing functions of $\rho$ and decreasing functions of $\lambda_2$ as indicated in Remark 2. Moreover, properties i)–iii) of $p^*_s(\cdot)$ and $p^*_b(\cdot)$ stated in Theorem 5 are also visible.

In Figure 3, we examine the impact of transaction cost $K$ on the optimal thresholds. It is observed that the no-trading regions expands as we increase the transaction cost from 0.001 to 0.005. This is consistent with our intuition that increasing transaction costs can decrease trading frequency.

## 4 Simulations and Market Tests

To evaluate how well our trend following trading strategy would work in practice, we conduct experiments using both simulations and market historical data. The tests are oriented towards the practicality of using $p_t$ to detect the regime switches. Thus, we focus on an
Figure 2: Effects of parameters $\lambda$ and $\rho$ on optimal trading strategy

Figure 3: Effects of transaction cost $K$ on optimal trading strategy
implementable trend following strategy that approximates the optimal one and test its robustness.

4.1 Method

From Fig. 1-3 we can see that the thresholds $p^*_s(\cdot)$ and $p^*_b(\cdot)$ are almost constant except when $t$ gets very close to the terminal time $T$. In our experiments, $T$ is always more than ten years which is relatively large. As a result, the contribution of the trading near $T$ to the reward function is small. Thus, we will approximate $p^*_s(\cdot)$ and $p^*_b(\cdot)$ with two constant threshold values $p^*_s = \lim_{t \to 0^+} p^*_s(t)$ and $p^*_b = \lim_{t \to 0^+} p^*_b(t)$. Then we estimate $p_t$, the conditional probability in a bull market at time $t$ and check it against the thresholds to determine whether to buy, hold or sell. The trend following trading strategy we will use is buy the stock when $p_t$ crosses $p^*_b$ from below for the first time and convert to the bond when $p_t$ crosses $p^*_s$ from above for the first time. Moreover, we always liquidate our holdings of stock or bond at $T$.

Some qualitative analysis of $p_t$ is helpful before experiments. Using the observation SDE equations we see that $p_t$ is related to the stock price $S_t$ by

$$dp_t = f(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1 - p_t)}{\sigma^2}d\log(S_t),$$

(23)

where $f$ is a third order polynomial of $p_t$ given by

$$f(p_t) = -(\lambda_1 + \lambda_2)p_t + \lambda_2 - \frac{(\mu_1 - \mu_2)p_t(1 - p_t)((\mu_1 - \mu_2)p_t + \mu_2 - \sigma^2/2)}{\sigma^2}.$$ 

It is easy to check that $f(0) = \lambda_2 > 0$ and $f(1) = -\lambda_1 < 0$. Moreover, as $p_t$ approaches $\pm\infty$ so does $f(p_t)$. Thus, $f$ has exactly one root $\xi$ in $[0,1]$. When the stock prices stay constant $p_t$ is attracted to $\xi$. This attractor is an unbiased choice for $p_0$. Since

$$\frac{(\mu_1 - \mu_2)p_t(1 - p_t)}{\sigma^2} \geq 0,$$

$p_t$ moves in the same direction as the stock prices. This is also intuitive since the stock price movements indicate trends. The magnitude of the impact varies though. Fixing all the parameters, the impact of stock price movement becomes relatively small when $p_t$ is getting close to 0 or 1. Among the parameters $\mu_1, \mu_2$ and $\sigma$ the latter has a larger impact, since it
appears in square in the formula. A smaller $\sigma$ magnifies the impact of stock movement and tends to cause more frequent trading and a larger $\sigma$ does the opposite. We will estimate $p_t$ simply by replacing the differential in (23) with a difference using trading day as step size on a finite time horizon $[0, Ndt]$:

$$p_{t+1} = p_t + f(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1 - p_t)}{\sigma^2} \log(S_{t+1}/S_t),$$

where $dt = 1/250$ and $t = 0, dt, 2dt, \ldots, Ndt$.

If everything changes continuously then $p_t \in [0, 1]$. Indeed, simply changing the differential equation to a difference equation works extremely well for simulated paths. However, this could be violated in approximation when $S_t$ jumps. This would happen, for example, in testing the SP500: during the 1987 crash on the downside and recently on the upside. We have to restrict $p_t \in [0, 1]$ in implementing the approximation. Thus, in simulation and testing we calculate $p_t$ iteratively with

$$p_{t+1} = \min\left(\max\left(p_t + f(p_t)dt + \frac{(\mu_1 - \mu_2)p_t(1 - p_t)}{\sigma^2} \log(S_{t+1}/S_t), 0\right), 1\right).$$

4.2 Simulation

In our analysis the choice of the objective function is partly due to tractability. Nevertheless, this gives us a way of recognizing trends by relatively high probabilities in the bull or bear regimes, respectively. How effective is this trend recognizing method? We check it by simulation. We use the same parameter values as in Table 1.

Numerically solving the obstacle problem for $t \to 0+$ yields thresholds $p_s^* = 0.768$ and $p_b^* = 0.934$ for down and up trends, respectively. A typical 20 year sample path is given in Figure 4, in which the prices $S_t$ when we are long the stock according to our strategy ($p(t) \in BR \cup NT$ after crossing $p_b^*$ from below) are marked in blue and those when we are flat ($p(t) \in SR \cup NT$ after crossing $p_s^*$ from above) in red. We can see that on this typical path, the signals are quite effective in detecting the regime switching.

A natural trading strategy of using these signals is to buy stock in the beginning of an up trend as signaled by $p_t$ crossing the upper threshold $p_b^*$ from below and switch to bond when $p_t$ crosses the lower threshold $p_s^*$ from above. We simulate this trend following strategy against the buy and hold strategy by using a large number of simulated paths. The average
return of the trend following (TF) strategy on one unit invested on simulation paths are listed in Table 2, along with the average number of trades on each path. We also list the average return of the buy and hold (BH) strategy for comparison.

<table>
<thead>
<tr>
<th>No. of Simulations</th>
<th>TF</th>
<th>BH</th>
<th>TF/BH</th>
<th>No. of Trades</th>
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</tr>
<tr>
<td>10000</td>
<td>73.6</td>
<td>5.45</td>
<td>13.50</td>
<td>37.64</td>
</tr>
<tr>
<td>50000</td>
<td>74.5</td>
<td>5.59</td>
<td>13.33</td>
<td>37.56</td>
</tr>
</tbody>
</table>

Table 2. Simulation results

Judging from the ratio TF/BH, the simulation tends to stabilize around 5000 rounds. We run the 5000 round simulation for 10 times and summarize the mean and standard deviation in Table 3, which confirms our observation.

<table>
<thead>
<tr>
<th></th>
<th>Trend Following</th>
<th>Buy and Hold</th>
<th>No. of Trades</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>74.6</td>
<td>5.72</td>
<td>37.55</td>
</tr>
<tr>
<td>Stdev</td>
<td>1.64</td>
<td>0.31</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Table 3. Statistics of ten 5000-path simulations
These simulations show that the trend following strategy has a distinctive advantage over the buy and hold strategy and is quite stable in both return and trading frequency. What if the parameters are perturbed? It turns out that the thresholds are not sensitive to the parameters as summarized in Table 4.

Next we test the robustness of the trend following trading strategy against the perturbation of the thresholds. This is important because we are using the limits of the optimal thresholds $p_s^\ast(\cdot)$ and $p_b^\ast(\cdot)$ as $t \to 0^+$ to approximate them. We perturb the constant thresholds $p_s^\ast$ and $p_b^\ast$ both by shifting and by altering the spreads. The results are summarized in Tables 5 and 6.

<table>
<thead>
<tr>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>$p_b^\ast$</th>
<th>$p_s^\ast$</th>
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</thead>
<tbody>
<tr>
<td>0.36</td>
<td>2.53</td>
<td>0.18</td>
<td>-0.77</td>
<td>0.184</td>
<td>0.067</td>
<td>0.768</td>
<td>0.934</td>
</tr>
<tr>
<td>0.36</td>
<td>2.53</td>
<td>0.18</td>
<td>-0.77</td>
<td>0.184</td>
<td>0.062</td>
<td>0.761</td>
<td>0.931</td>
</tr>
<tr>
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<td>0.18</td>
<td>-0.77</td>
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<td>0.072</td>
<td>0.775</td>
<td>0.938</td>
</tr>
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<td>2.53</td>
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<td>-0.77</td>
<td>0.184</td>
<td>0.174</td>
<td>0.762</td>
<td>0.936</td>
</tr>
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<td>0.194</td>
<td>0.773</td>
<td>0.933</td>
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<td>0.184</td>
<td>0.067</td>
<td>0.774</td>
<td>0.935</td>
</tr>
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<td>2.53</td>
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<td>0.184</td>
<td>0.067</td>
<td>0.762</td>
<td>0.934</td>
</tr>
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<td>0.184</td>
<td>0.067</td>
<td>0.769</td>
<td>0.935</td>
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<td>-0.77</td>
<td>0.184</td>
<td>0.067</td>
<td>0.767</td>
<td>0.935</td>
</tr>
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<td>0.36</td>
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<td>0.18</td>
<td>-0.77</td>
<td>0.184</td>
<td>0.067</td>
<td>0.767</td>
<td>0.934</td>
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<td>0.36</td>
<td>2.53</td>
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<td>0.935</td>
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<td>0.18</td>
<td>-0.77</td>
<td>0.184</td>
<td>0.067</td>
<td>0.775</td>
<td>0.936</td>
</tr>
<tr>
<td>0.36</td>
<td>2.53</td>
<td>0.18</td>
<td>-0.77</td>
<td>0.184</td>
<td>0.067</td>
<td>0.773</td>
<td>0.933</td>
</tr>
<tr>
<td>0.36</td>
<td>2.53</td>
<td>0.18</td>
<td>-0.77</td>
<td>0.184</td>
<td>0.067</td>
<td>0.762</td>
<td>0.934</td>
</tr>
<tr>
<td>0.36</td>
<td>2.53</td>
<td>0.18</td>
<td>-0.77</td>
<td>0.184</td>
<td>0.067</td>
<td>0.769</td>
<td>0.935</td>
</tr>
<tr>
<td>0.36</td>
<td>2.53</td>
<td>0.18</td>
<td>-0.77</td>
<td>0.184</td>
<td>0.067</td>
<td>0.767</td>
<td>0.934</td>
</tr>
<tr>
<td>0.36</td>
<td>2.53</td>
<td>0.18</td>
<td>-0.77</td>
<td>0.184</td>
<td>0.067</td>
<td>0.769</td>
<td>0.935</td>
</tr>
</tbody>
</table>

Table 4. Thresholds corresponding to different parameters

Next we test the robustness of the trend following trading strategy against the perturbation of the thresholds. This is important because we are using the limits of the optimal thresholds $p_s^\ast(\cdot)$ and $p_b^\ast(\cdot)$ as $t \to 0^+$ to approximate them. We perturb the constant thresholds $p_s^\ast$ and $p_b^\ast$ both by shifting and by altering the spreads. The results are summarized in Tables 5 and 6.

<table>
<thead>
<tr>
<th>$p_b^\ast$</th>
<th>$p_s^\ast$</th>
<th>TF</th>
<th>BH</th>
<th>TF/BH</th>
<th>No. of Trades</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.75900</td>
<td>0.92500</td>
<td>74.407</td>
<td>5.6478</td>
<td>13.174</td>
<td>37.039</td>
</tr>
<tr>
<td>0.76300</td>
<td>0.92900</td>
<td>76.523</td>
<td>5.9231</td>
<td>12.919</td>
<td>37.139</td>
</tr>
<tr>
<td>0.76700</td>
<td>0.93300</td>
<td>75.467</td>
<td>5.8328</td>
<td>12.938</td>
<td>37.371</td>
</tr>
<tr>
<td>0.77100</td>
<td>0.93700</td>
<td>74.232</td>
<td>5.6367</td>
<td>13.169</td>
<td>37.648</td>
</tr>
<tr>
<td>0.77500</td>
<td>0.94100</td>
<td>75.986</td>
<td>5.5975</td>
<td>13.575</td>
<td>37.766</td>
</tr>
<tr>
<td>0.77900</td>
<td>0.94500</td>
<td>77.029</td>
<td>6.0613</td>
<td>12.708</td>
<td>37.923</td>
</tr>
</tbody>
</table>

Table 5. Shifting the thresholds
Table 6. Changing the spreads of the thresholds

We can see from Table 5 that shifting the thresholds has little impact. Table 6 shows that the average number of trades in the trend following trading strategy is inversely correlated to the spreads of the thresholds. However, the relative advantage of the trend following strategy over the buy and hold strategy is not sensitive to the perturbation of the thresholds.

In the tests discussed above the trading cost $K = 0.001$ is fixed. Increasing $K$, the spread of the optimal thresholds will also increasing as indicated in Fig 3. Simulations summarized in Table 6 indicate that this will reduce the trading frequency which is consistent with intuition. Simulating 5000 rounds for each of the trading cost levels $K = 0.005, 0.01, 0.02$ shows that the advantage of the trend following methods decreases as $K$ increases as expected. However, even at $K = 0.02$ the advantage is still quite obvious. The testing results are summarized in Table 7.

Table 7. Changing the trading cost

The simulations convince us that the trend following trading system of using $p_t$ crossing the constant thresholds $p^*_s$ and $p^*_b$ to detect the trend of the stock price movement in a regime switching model is effective and robust.

4.3 Testing in stock markets

Does this trend following strategy work in real markets? We test it on the historical data of SP500, DJIA and NASDAQ indices. The SP500 index started active trading in 1962
and NASDAQ in 1991 and we test them up to the end of 2008. We also test DJIA from 1962-2008.

First we need to determine the parameters. We regard a decline of more than 19% as a bear market and a rally of 24% or more as a bull market. Statistics of bull and bear markets for SP500 index and DJIA in the 47 years from 1962 to 2008 and NASDAQ from 1991-2008 (see Tables A.1–A.3 in the Appendix) gives us

<table>
<thead>
<tr>
<th>Index</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$\mu_1$</th>
<th>$\mu_2$</th>
<th>$\sigma_1$</th>
<th>$\sigma_2$</th>
<th>$\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>SP500 (62-08)</td>
<td>0.353</td>
<td>2.208</td>
<td>0.196</td>
<td>-0.616</td>
<td>0.135</td>
<td>0.211</td>
<td>0.173</td>
</tr>
<tr>
<td>DJIA (62-08)</td>
<td>0.36</td>
<td>2.53</td>
<td>0.18</td>
<td>-0.77</td>
<td>0.144</td>
<td>0.223</td>
<td>0.184</td>
</tr>
<tr>
<td>NASDAQ (91-08)</td>
<td>2.158</td>
<td>2.3</td>
<td>0.875</td>
<td>-1.028</td>
<td>0.273</td>
<td>0.35</td>
<td>0.31</td>
</tr>
</tbody>
</table>

Table 8. Statistics of bull and bear markets

Here $\sigma_1$ and $\sigma_2$ are standard deviation corresponding to the bull and bear markets, respectively, and $\sigma = (\sigma_1 + \sigma_2)/2$. Currently, retail discount brokers usually charge $2.5-10 per trade for unlimited number of shares (e.g. Just2Trade $2.5 per trade. ScotTrade $7 per trade, ETrade $10 per trade and TDAmeritrade $10 per trade). For professional traders who deal with clearing houses directly, trading a $100000 standard lot usually cost less than $1. Assuming $10 per trade for an account of size $10000, we choose $K = 0.001$ to simulate our strategy. This is close to the actual cost for a typical individual investor now or for a professional trader before the emergence of online discount brokers. We choose 10 year treasury bond as the alternative risk free investment instrument and use the annual yield released on the Federal Reserve Statistical Release web site [10]. This gives us an average yield of 6.7% per year from 1962-2008 and 5.4% from 1991-2008. However, to be more realistic we use the actual yield (see Table A.4. in the Appendix) when holding the bonds. We also take advantage of knowing $\sigma_1$ and $\sigma_2$ for the bull and bear markets. Solving the obstacle problems we derive for each index two sets of thresholds corresponding to $\sigma_1$ and $\sigma_2$, respectively, as listed in Table 7. Since when holding bonds and looking for signal to switch to a stock position, we anticipate entering a bull market whose volatility is better represented by $\sigma_1$. Therefore, it is reasonable to choose the upper threshold related to $\sigma_1$. Similarly, for a signal to exit a stock position we should use the lower threshold corresponding to $\sigma_2$. Thus, in conducting our test we use the boldfaced thresholds in the Table 7.
<table>
<thead>
<tr>
<th>Index</th>
<th>lower thresholds</th>
<th>upper thresholds</th>
</tr>
</thead>
<tbody>
<tr>
<td>SP500 $\sigma_1$</td>
<td>0.69</td>
<td>0.91</td>
</tr>
<tr>
<td>SP500 $\sigma_2$</td>
<td>0.74</td>
<td>0.90</td>
</tr>
<tr>
<td>DJIA $\sigma_1$</td>
<td>0.74</td>
<td>0.94</td>
</tr>
<tr>
<td>DJIA $\sigma_2$</td>
<td>0.78</td>
<td>0.93</td>
</tr>
<tr>
<td>NASDAQ $\sigma_1$</td>
<td>0.43</td>
<td>0.69</td>
</tr>
<tr>
<td>NASDAQ $\sigma_2$</td>
<td>0.45</td>
<td>0.67</td>
</tr>
</tbody>
</table>

Table 9. Thresholds for Indices

The discussions so far are based on the raw index values. This is appropriate since when pursuing market trend the raw price is what market agents can directly observe and react upon. However, for a long investment horizon we need also to consider the effect of dividend. In this aspect the three indices are different. The DJIA already includes dividend in its computation while NASDAQ and SP500 do not. The dividend paid by companies listed on NASDAQ is small, for example, the estimated 2009 average dividend for the 100 largest companies (who usually pay more dividend compare to smaller ones) listed in the NASDAQ is only 0.68% (see [15]). Moreover, the NASDAQ is tested for a shorter period of time. Thus, omitting dividend in the test for the NASDAQ index does not skew the comparison much. This is not the case for the SP500 index which averages an annual dividend of about 2%. We use the annual dividend compiled in [1] (see Table A.5.) to compensate the raw gains for using the trend following strategy to trade the SP500 index.

The testing results for trading the NASDAQ, SP500 and DJIA indices are summarized in Table 10 and the trading details for the trend following strategy are contained in Tables A.6–A.8 in the Appendix, respectively. Taking the NASDAQ as an example, using our trend following trading strategy one dollar invested in the beginning of 1991 returns 8.82 at the end of 2008. Buy comparison, one dollar invested in the NASDAQ index using the buy and hold strategy in the same period returns only 4.24 while invested in 10 year bonds returns 2.63. The stories for the SP-500 and DJIA are similar.
Table 10. Testing results for trend following trading strategies

<table>
<thead>
<tr>
<th>Index (time frame)</th>
<th>TF</th>
<th>BH</th>
<th>10y bonds</th>
<th>No. Trades</th>
<th>G</th>
</tr>
</thead>
<tbody>
<tr>
<td>NASDAQ (1991-2008)</td>
<td>8.82</td>
<td>4.24</td>
<td>2.63</td>
<td>66</td>
<td>3.35</td>
</tr>
<tr>
<td>SP500 (1962-2008)</td>
<td>64.98</td>
<td>33.5</td>
<td>23.44</td>
<td>80</td>
<td>5.36</td>
</tr>
<tr>
<td>DJIA (1962-2008)</td>
<td>26.03</td>
<td>12.11</td>
<td>23.44</td>
<td>80</td>
<td>4.16</td>
</tr>
</tbody>
</table>

Legend: TF – trend following, BH – Buy and hold, and G–average % gain per trade

The average percentage gains per trade listed in Table 10 indicate that there is room for the trend following method to absorb a higher trading cost and still outperform the buy and hold method. Using the NASDAQ test as an example, the same 66 trades with a trading cost 1% per trade, the total return of trend following method will be 4.64, still higher than the 4.24 from the buy and hold method. In theory, changing $K$ to recalculate the buy and sell thresholds will yield even better returns and this is confirmed also by the simulation reported in Table 7. However, this is not the case here. In fact, using thresholds corresponding to $K = 0.01$ to test the NASDAQ data we get a total return of only 4.04 with 22 trades. On the other hand, the same test with $K = 0.0001$ yields a total return of 10.8 with 130 trades. This corresponds to an average gain of 1.847% per trade. Had a 0.001 trading cost were charged on these same trades, we would end up with a total return of 9.5 which is higher than the return of 8.82 tested with the “optimized” thresholds corresponding to $K = 0.001$. Moreover, the relative advantage of the trend following method over the buy and hold in the testing results for the stock indices is not as good as those from the simulations in the previous subsection. These indicate that the regime switching geometric Brownian motion model with those parameter values is only an approximation of the real markets.

5 Conclusion

We show that under a regime switching model, a trend following trading system can be justified as an optimal trading strategy with a discounted reward function of trading one share of stock with a fixed percentage transaction cost. The optimal trading strategy has a simple implementable approximation. Extensive simulations and tests on historical stock market data show that this ‘optimal’ trend following trading strategy (using constant $p^*_b(t)$ and $p^*_s(t)$) significantly outperforms the buy and hold strategy and is robust when parameters
are perturbed. This investigation provides a useful theoretical framework for the widely used trend following trading methods.

6 Appendix

6.1 The smoothness of the free boundaries \( p_0^*(t) \) and \( p_s^*(t) \)

By changing variables

\[ y = \log \left( \frac{p}{p - 1} \right), \quad w(y, t) = Z(p, t), \]

the double obstacle problem (9)-(11) become

\[
\min \{ \max \{ -\partial_t w - L_1 w, w - (1 + K) \}, w - (1 - K) \} = 0
\]

in \((-\infty, +\infty) \times [0, T)\), with the terminal condition \( w(y, T) = (1 - K) \), where

\[
L_1 = \frac{(\mu_1 - \mu_2)^2}{2\sigma^2} \partial_{yy} + \left[ \left( \mu_1 - \mu_2 - \lambda_2 + \lambda_2 - \frac{(\mu_1 - \mu_2)^2}{2\sigma^2} \right) + \frac{(\mu_1 - \mu_2)^2}{\sigma^2} \frac{e^y}{e^y + 1} \right] \partial_y
\]

\[ + \left( \mu_1 \frac{e^y}{e^y + 1} + \mu_2 \frac{1}{e^y + 1} - \rho \right). \]

To show the smoothness of the free boundaries, it suffices to verify the so-called cone property (cf. [7] or [27]):

\[(T - t) \partial_t w + C \partial_y w \geq 0\]

with some constant \( C > 0 \), locally uniformly for \( y \in (-\infty, +\infty) \). For illustration\(^\text{4}\), let us restrict attention to the region \( \{(y, t) : 1 - K < w(y, t) < 1 + K\} \) in which

\[-\partial_t w - L_1 w = 0.\]

Differentiating the above equation w.r.t. \( y \), we have

\[
(-\partial_t - L_2)(\partial_y w) = \frac{e^y}{(e^y + 1)^2} \left( \frac{(\mu_1 - \mu_2)^2}{\sigma^2} \partial_y w + (\mu_1 - \mu_2) w \right)
\]

\[ \geq \frac{e^y}{(e^y + 1)^2} (\mu_1 - \mu_2)(1 - K) \]

where

\[
L_2 = L_1 - (\lambda_1 e^y + \lambda_2 e^{-y}).
\]

\(^4\text{A rigorous proof needs the use of penalization method (cf. [12]).}\)
On the other hand,

\[ (-\partial_t - L_2)((T - t) \partial_t w) = \left[ (\lambda_1 e^y + \lambda_2 e^{-y}) (T - t) + 1 \right] \partial_t w. \]

It is easy to see that \( \partial_t w \) is uniformly bounded and \( \frac{e^y}{(e^y + 1)^2} \) has a local positive lower bound. By using the auxiliary function \( \psi(y, t; y_0) = e^{C_1 t} (y - y_0)^2 \), with some positive constant \( C_1 \), as adopted in [7] and [27], we can infer from the maximum principle

\[ (T - t) \partial_t w + C \partial_y w + \psi(y, t; y_0) \geq 0 \]

for an appropriate \( C > 0 \). The desired result follows by taking \( y = y_0 \).

### 6.2 Statistics for bull and bear markets of NASDAQ, SP500 and DJIA indices

<table>
<thead>
<tr>
<th>top/bottom</th>
<th>Index</th>
<th>%move</th>
<th>mean</th>
<th>stdev</th>
<th>duration</th>
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<td></td>
</tr>
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<td>339</td>
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Table A.2. Statistics of SP500 Bull and Bear Markets (1962-2009)

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6.3 Yield for ten year US treasury bonds

The table below summarize the annual yield of ten year US treasury bonds as released on the Federal Reserve Statistical Release web site [10].

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Table A.4. Yield of Ten Year Bonds (1962-2008)

6.4 Annual dividend for SP500 index

Annual dividend for SP500 index is given in Table A.5. below. This table is quoted from Damodaran Online, [1].

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Table A.5. Annual dividend for SP500 index (1962-2008)
6.5 Transactions for the trend following strategy on NASDAQ, SP500 and DJIA indices

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