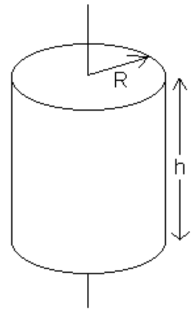
PHYS-205 (Kaldon-H14) **Spring 2004**

<u>What They Didn't Put In The Physics Textbook</u>
<u>(What? You Mean You Didn't Have It In Math, Either?)</u>



Integrals are just a fancy way of adding up things. Suppose we want to find the moment of inertia of a solid cylinder about its centerline.

$$I = \int r^2 dm$$

In order to do this integral we must: (a) determine what *dm* is, <u>and (b) look at our axis of rotation and make sure that we know what r^2 represents.</u>

We have a couple of ways of doing this:

(1) Let $\rho = \frac{m}{V}$, so $dm = \rho dV$, which for cylindrical coordinates is $dm = \rho r\, dr\, d\theta\, dz$. Since *r* represents the radial variable, and <u>is also the distance from the axis of rotation</u>, we can just multiply $r^2 dm$…

$$I = \int\int\int \rho r^3\, dr\, d\theta\, dz = \rho\int_0^R r^3 dr \int_0^{2\pi} d\theta \int_o^h dz = \rho \left.\frac{r^4}{4}\right|_0^R \theta\Big|_0^{2\pi} z\Big|_0^h = 2\pi h\rho\frac{R^4}{4} = 2\pi h\left(\frac{m}{\pi R^2 h}\right)\frac{R^4}{4} = \frac{mR^2}{2}$$

But the *z*-axis is parallel to the axis of rotation, and so nothing in the *z*-direction is going to change the distance to the axis of rotation, so we could choose to think of *dm* as little vertical slices around the cylinder, so that we are really looking at a *mass/area*.

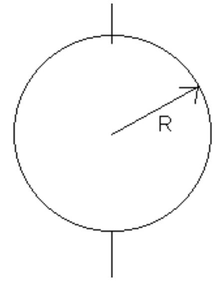
(2) Let $\sigma = \frac{m}{A}$, so $dm = \sigma dA$, which for polar coordinates is $dm = \sigma r\, dr\, d\theta$.

$$I = \int\int \sigma r^3 dr\, d\theta = \sigma\int_0^R r^3 dr \int_0^{2\pi} d\theta = \rho\left.\frac{r^4}{4}\right|_0^R \theta\Big|_0^{2\pi} = 2\pi h\sigma\frac{R^4}{4} = 2\pi\left(\frac{m}{\pi R^2}\right)\frac{R^4}{4} = \frac{mR^2}{2}$$

(3) Use Symmetry to realize that this is really the same problem as a thin disk of radius R, which has a moment of inertia $I = \frac{mR^2}{2}$.

Note that all three of these methods yield the same answer for the moment of inertia of the cylinder!

But what works for the cylinder, doesn't quite work the same way for the sphere. Read on, gentle physics student! ☞



For a Sphere, we need Spherical Coordinates

Let $\rho = \frac{m}{V}$, so $dm = \rho dV$, which for cylindrical coordinates is $dm = \rho r^2 \sin\varphi\, dr\, d\theta\, d\varphi$. (Use caution here.) [1]

r = radius

θ = theta = polar angle

ϕ = phi = azimuthal angle, the angle from +z-axis

note that r^2 here is part of the volume element dV – there is still r^2 from the *I* equation.

Now as tempting as it is to say that we are done and should just integrate to get:

$$I = \int\int\int \rho r^2 r^2 \sin\varphi\, dr\, d\theta\, d\varphi = \rho\int_0^R r^4 dr \int_0^{2\pi} d\theta \int_o^{\pi} \sin\varphi\, d\varphi = \rho\left.\frac{r^5}{5}\right|_0^R \theta\Big|_0^{2\pi}(-\cos\varphi)_0^{\pi} = 4\pi\rho\frac{R^5}{5} = 4\pi\left(\frac{m}{\frac{4}{3}\pi R^3}\right)\frac{R^5}{5} = \frac{3mR^2}{5}$$

This is neither the right Physics nor the right answer.

Instead, we should consider that distance from the axis of rotation is not *r* but *r sinφ*.. (Pointing straight up the *z*-axis is φ = 0 and down the -*z*-axis is φ = π.)

$$I = \int\int\int \rho (r\sin\varphi)^2 r^2 \sin\varphi\, dr\, d\theta\, d\varphi =$$

$$= \rho\int_0^R r^4 dr \int_0^{2\pi} d\theta \int_o^{\pi} \sin^3\varphi\, d\varphi = \rho\left.\frac{r^5}{5}\right|_0^R \theta\Big|_0^{2\pi}\left(\left. -\frac{\cos\varphi}{3}(\sin\varphi + 2)\right|_0^{\pi}\right) =$$

$$= 2\pi\rho\frac{4R^5}{3\cdot 5} = 2\pi\left(\frac{m}{\frac{4}{3}\pi R^3}\right)\frac{4R^5}{3\cdot 5} = \frac{2mR^2}{5}$$

Which is correct. My trusty <u>CRC Standard Math Tables</u>, 24th Edition, verified that:

$$\int_o^{\pi} \sin^3\varphi\, d\varphi = \left.-\frac{\cos\varphi}{3}(\sin^2\varphi + 2)\right|_0^{\pi} = -\left(-\frac{2}{3} - \frac{2}{3}\right) = \frac{4}{3}$$

An Aside: By the way, Dr. Phil fell into a trap the other year, when a student tried to argue that *I* for the hollow cylinder had to have the term $(a^2 - b^2)$, not $(a^2 + b^2)$, since you are subtracting the hole. Could the book be wrong? Well, maybe… but PTPBIP! You see, the equation has a ½ built into it – so in the case where $b = a$, which makes this a thin hoop of radius $R = a$, then the term with ½$(a^2 + b^2)$ does get the correct answer, whereas $(a^2 - b^2)$ gives zero. So the equation in the book is correct. Gotta watch those minus signs, and those factors of 2 and ½.

[1] The Ugly Truth They Didn't Want To Warn You Of: For reasons that have never made sense to Dr. Phil, there are *TWO* competing versions of Spherical Coordinates, which interchange *θ* and *φ*. The worst semester of my life as an undergrad, I had 2 advanced PHYS courses back to back – with opposing forms of spherical coordinates!