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I: THEAETETUS DEFENDS HIS AXIOM OF PROBABILITY

SOCRATES: My dear Theodorus, I seem to have the time to accept your kind invitation . . .

THEODORUS: What! Socrates!

SOCRATES: . . . to chat some more with your brilliant pupils, the one with my snub nose and protruding eyes, the one with my name, and the other "me" I also recognize, even if not quite as in a mirror.

THEODORUS: Socrates, you old codger. You haven't been away long enough to skin a rabbit, no less arrange for an amicable settlement of your disagreement with Meletus.

SOCRATES: I gave the matter as much time as it was worth, and along the way I fell into an interesting conversation on piety.

THEODORUS: With whom?

SOCRATES: An authority on the subject.

THEODORUS: So now you know all about piety.

SOCRATES: No, I'm sorry to say. He refused to tell me.

THEODORUS: I can well imagine the scene. You did have time after all to skin a rabbit. And you've made another convert to Meletus's side.

SOCRATES: Theodorus, dear friend, indulge an old man. "Give me conversation or give me death" in the words of an old patriot. What's this? The boys are leaving.

THEODORUS: Yes. Actually, we hadn't expected to see you again today. I must be off too.

SOCRATES: Where are you off to?

THEODORUS: Piraeus.

SOCRATES: What on earth for?

THEODORUS: It's a private matter, Socrates.

SOCRATES: Well, I'll join you for the stroll anyway. I had been looking forward to continuing our discussion of knowledge with the boys. I'm disappointed. Why do they walk so far ahead of us?

THEODORUS: There's a good reason for it.

SOCRATES: I've never felt we kept secrets from each other. What's the reason?

THEODORUS: You wouldn't understand. And please don't come with us.

SOCRATES: You're being much too mysterious, Theodorus. Tell me what you're up to. When you try to hide what you're doing, you suggest it might be shameful. I find it hard to believe that you and the boys are off to shameful doings in Piraeus. But I'll believe it if you want me to.

THEODORUS: No, no, no. I just don't want to get into an argument with you about it. That's all.

SOCRATES: How curious! Something's fishy in Piraeus. I'll tag along to see for myself. Should I walk with you, or stalk you from behind?

THEODORUS: Look here, Socrates. I'll explain it all to you; just you try to understand our point of view. OK?

SOCRATES: OK.

[Axioms and Conventions]
[In Six Parts]

1. *What is an axiom? What is the axiom of probability?*

THEODORUS: We're studying probability, that's all. Theaetetus has lots of ideas about it, as you know. He's been quite successful in using the methods of geometers to prove theorems about probability. As in geometry, he needs to start off with axioms. His starting point must be truths so obviously true that any proof of them would be an appeal to the less obvious in support of the more obvious.

SOCRATES: Obvious truths are a drachma a dozen: "1 = 1," and so on.

THEODORUS: No; he wants his to imply all other truths of probability.

SOCRATES: Aha! The mother of all the other truths of probability.

THEODORUS: Yes, very pregnant. But what should the axioms of probability be? Theaetetus prefers just one axiom about propositions that as a group represent all possibilities.

SOCRATES: "As a group"?

THEODORUS: Yes, when taken together. One or other of them must be true, but no two of them could be true together. They're each mutually exclusive of each of the others, and they jointly exhaust all possibilities. The axiom says their probabilities sum to the maximum probability.

SOCRATES: The axiom is . . . that the probabilities of each of a group of mutually exclusive and jointly exhaustive propositions sum to the maximum probability.

THEODORUS: You got it.

2. *What are mutually exclusive propositions? What are jointly exhaustive propositions?*

SOCRATES: Well . . . I notice you're admiring those geranium flowers too. The topmost flower's got ten petals or more; how many exactly I'm not sure. But I can express propositions that are exclusive of each other, such as, "the flower has exactly ten petals," "the flower has exactly eleven petals," "the flower has exactly twelve petals," and so on. No two of these propositions can be true together. Is that what you mean by "mutually exclusive" propositions?

THEODORUS: Yes. They're mutually exclusive of each other.

SOCRATES: What about the proposition, "the flower has eleven petals at least"?

THEODORUS: It excludes the proposition that the flower has exactly ten petals, but not the other two propositions.

SOCRATES: So we can't put it in our group of mutually exclusive propositions?

THEODORUS: Right. Can you make the group exhaustive of all possibilities?

SOCRATES: That flower will've died long before I could give you the whole group of them! I'd start with the possibility that it has zero petals, then one petal, then two, and so on, and I don't think I'd ever stop, because by "possibility" I suppose you mean whatever can be imagined to be true without self-contradiction. Is that right?

THEODORUS: That's correct.

SOCRATES: So, in order for my group of propositions to be exhaustive of all possibilities, it would have to include such propositions as, "the flower has ten thousand million petals."

THEODORUS: Yes.

3. *Let's only consider finite collections of propositions.*

SOCRATES: And an infinity of possibilities!

THEODORUS: We needn't go that far.

SOCRATES: Their average probability would be . . . indulge me, Theodorus. Would their average probability be 1 or more than one or less than one?

THEODORUS: Oh, much less than one. But you have an extravagant imagination.

SOCRATES: And might not the average case be the actual one—each possibility represented by a proposition with a probability of . . . would it be ever so close to zero?

THEODORUS: I suppose so. Dividing infinity into a finite maximum would yield a decimal number that began with zeroes in the tenths, hundredths, and thousandths place and would still be zero in any place one could reach in a finite number of steps.

SOCRATES: Then I'm puzzled. It seems that this fraction is equal to a decimal number that we could never finish writing. It would be a decimal point followed by an infinity of zeroes. And if each proposition's probability were that decimal number with zeroes forever, and I added all these decimal numbers up as your axiom says, then . . . My dear Theodorus, I'm embarrassed to say this. I think your maximum probability must be zero.

THEODORUS: Oh, you do, do you? Well, you're wrong. And don't think you're going to trap me in contradictions. I'm just going to rule that our axiom applies for the time being only to finite collections of propositions.

SOCRATES: But how can you do that and still have a collection of propositions that exhausts all possibilities?

THEODORUS: Fortunately for us, we can always collect possibilities together into a few groups. Consider that flower back there . . .

SOCRATES: Right here. I picked it.

THEODORUS: Oh. All its possibilities are expressed in just three mutually exclusive propositions: "the flower has less than ten petals," "the flower has from ten to twenty petals," or "the flower has more than twenty petals." They're jointly exhaustive of all possibilities.

SOCRATES: If you count this half-petal as one. Otherwise there are twenty and one-half petals.

THEODORUS: Count any part of a petal as one petal, you petal-splitter, you. The three propositions are jointly exhaustive. Since each is mutually exclusive of each other, the axiom applies to them. Whatever the probability of truth you think each has, the sum of them must equal the maximum probability.

SOCRATES: Is there a place for impossibility here? Take, for example, the proposition that the flower has twenty one petals and does not have twenty one petals.

THEODORUS: You take it. I don't want it.

SOCRATES: Patience, Theodorus. I only want to know how you treat propositions that affirm the impossible.

THEODORUS: Take a set of propositions which are mutually exclusive and jointly exhaustive . . .

SOCRATES: With pleasure.

THEODORUS: Hmph. Now add to that set another proposition which affirms the impossible. Is the enlarged set also a set of propositions which are mutually exclusive and jointly exhaustive? Yes, because the impossible proposition, being never true, is never true together with any other proposition, and the new set still covers all the possibilities.

SOCRATES: So the impossible proposition can be in a set of mutually exclusive and jointly exhaustive propositions?

THEODORUS: Yes. And the axiom still applies. The probabilities of each of the propositions in the set must sum to the maximum.

SOCRATES: So you'll accept the proposition that the flower both has twenty one petals and does not have twenty one petals?

THEODORUS: With pleasure. In fact, I remember Theaetetus used this gimmick in proving a theorem from his axiom.[1]

4. *What's axiomatically true about the axiom of probability?*

SOCRATES: His axiom says two things, if I understand you. First, there's a maximum probability, and, secondly, you reach it by adding the probabilities of each of a group of mutually exclusive and jointly exhaustive propositions. Why assume either?

THEODORUS: Is the idea of a maximum probability so dubious?

SOCRATES: Well, you've said it's not zero. So maybe it's infinity? I couldn't doubt that was maximum.

THEODORUS: No, no, no. A maximum is a finite limit, a cap.

SOCRATES: Now, you've stumped me again.

THEODORUS: Think of the maximum as the probability of propositions that are certain. Can you think of anything that would be even more probable than what's certain?

SOCRATES: Not at the moment. You could've justified their summing to the minimum in almost the same words. You'd've said, "think of the minimum as the probability of propositions that are the least uncertain, and the higher the probability the higher the uncertainty." What you did say and what you might've said instead are equally convincing, or unconvincing.

THEODORUS: Cantankerous old man . . .

SOCRATES: I didn't hear you.

THEODORUS: I'm hankering for a cold dram . . . of something. Your criticism's well taken. We could've taken the sums to equal a minimum rather than a maximum. Then all probabilities would be negative numbers. You may prefer that alternative; you may also prefer to look at the world in front of you by turning around, bending over, and looking at it through your spread legs.

SOCRATES: Positively not. What's the reason for summing probabilities?

THEODORUS: Here's an example. Let the mutually exclusive and jointly exhaustive propositions be just these two: lightning striking Mt. Olympus tomorrow and lightning not striking Mt. Olympus tomorrow. You're certain that one of these is true, but uncertain about which one. Suppose we added your degrees of partial conviction together, and their sum came to less than the maximum. What should we say of the difference between their sum and the maximum? It's positive and seems to represent a degree of partial conviction in some third alternative. But there's no third alternative to lightning striking and lightning not striking. You see that, don't you?

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[1]See Appendix, steps 17 and 54.

SOCRATES: I'm disappointed the best you can offer is only an example. If the sum came to more than the maximum, what then? The sum wouldn't be a probability, but so what? Your argument falls short in the forcefulness I've come to expect of geometers.

THEODORUS: I cower in shame before you.

5. *What's a convention? What's conventional about the axiom?*

SOCRATES: No. You're hiding your scorn of my slowness, and you're too impatient to explain at length. I know you. What number is the maximum probability?

THEODORUS: We take the maximum to be the number, one. But we could've adopted other numbers to represent the maximum. We could even have chosen a negative number, in which case we'd replace "maximum" in the axiom with "minimum." But positive one is the most convenient.

SOCRATES: My slowness shows again. I don't see why the maximum probability should be one either.

THEODORUS: It's natural to take the maximum as one if you think of lesser probabilities as parts of the maximum. For example, if the probability of lightning striking Mt. Olympus is one half of the maximum probability, we can say its probability is 1/2, provided the maximum is one. If the maximum were one and a half instead, then one half of the maximum would be 3/4. Taking the maximum probability to be one is just a convenience, that's all. Otherwise it's arbitrary.

SOCRATES: Arbitrary you say, but convenient. So it's a matter of convention.

THEODORUS: It's different with the axiom, though. The axiom is true. I feel quite confident of that.

6. *Can we justify an axiom, despite our inability to prove it without circular reasoning?*

SOCRATES: It may be true, but its truth is not all that obvious. Now the things you take as obvious in geometry, they're really obvious: "If equals are added to equals, the sums are equal too"; that's obvious.

THEODORUS: We're aware of the objection, Socrates, acutely aware of it. Nevertheless, as an axiom it's extraordinarily fertile. Everything we've wanted so far to prove has followed from it, and nothing we've wanted to reject has. This fertility is as important as obviousness, Socrates. So we had this predicament. One proposition served us well as axiom,[2] but seemed to be in need of proof itself. Now Theaetetus has hit on a clever way of persuading us of its truth without having to demote it from the status of axiom. Yes, indeed, quite clever.

SOCRATES: I wouldn't express my glee on a public highway by jingling my money bag, if I were you.

THEODORUS: Quite right. Thanks for calling it to my attention.

SOCRATES: You were saying Theaetetus has found a way of establishing the truth of his axiom without having to prove it. How can that be?

THEODORUS: We call it the Socratic method, Socrates, after you. When the geometer's

[2]Appendix, theorems 1-7.

method fails, there's always the Socratic method.

SOCRATES: That's very flattering, Theodorus. I'm pleased. What do you do?

THEODORUS: We play the horses.

SOCRATES: You what?

THEODORUS: We play the horses.

SOCRATES: I can't believe my ears. They're going to the Piraeus race track to bet on the horses, and they call it the Socratic method. I'm going back!

THEODORUS: Oh no. Come back here, Socrates! Now just one minute. You said you'd try to see it our way. You're not trying at all.

SOCRATES: Traitor.

THEODORUS: Will you listen! Don't you seek out people who think they know things they don't know, and make them pay for it? They put their prestige and reputation as authorities on the line, and you wipe them out. Well, what's so different about having people put money on the line? Answer me that.

SOCRATES: I don't see any analogy at all.

THEODORUS: Then you'll see for yourself. You're going to come with us to the track even if I have to drag you there. I'm not going to let you run off believing I'm some kind of traitor and debaucher of youth. First of all, we're placing bets, not so much on the horses, as on the truth of the axiom I told you about. Secondly, we go to the track because it's easiest to find people there who are willing to bet against it. Are we really gambling if we're betting on the truth of an axiom, which is certain? Of course not. There's not the slightest thing ignoble in our conduct, not the slightest departure from the highest standards of conduct! This is science!

THEAETETUS: Oh, hi, Socrates. What a surprise to see you here. Teach', Shark and Dutch are going to work the patsies at the odds board; we'll need lotsa drachmas tonight—large crowd. How much you got?

SOCRATES: Shark? Dutch?

THEAETETUS: Yeah. Socrates is Dutch. Not you, Sir. Little Socrates's alias is "Dutch." Mine's "Scarney." Remember if we meet, you don't know us.

THEODORUS: Here's the blasted money; beat it.

THEAETETUS: Wow! We can wipe out those suckers! S'long.

SOCRATES: Goodbye, Scarney. Give my regards to Dutch and Shark.

THEAETETUS: Remember you don't know us.

THEODORUS: I don't want to hear a word out of you, Socrates. Just follow me to the odds board, and I'll explain to you what's going on. First, recall Theaetetus's axiom of probability: The probabilities of mutually exclusive and jointly exhaustive propositions sum to one. A horse race gives us such a set of propositions. Suppose it's a three-horse race. Then we have these three propositions: horse A wins; horse B wins; and horse C wins. If a race is not run, and no horse wins, all bets are cancelled. On that condition at least one of the propositions must be true. So they're jointly exhaustive. Secondly, no referee has ever declared a race a tie, and if he did, there'd be such mayhem that all bets would be cancelled. So the three propositions are mutually exclusive. Are you following me?

SOCRATES: Yes.

THEODORUS: So of these three propositions: A wins; B wins; C wins; at least one must be true and at most one will be true under the conditions where bets hold. So their probabilities must sum to one. But a very curious thing happens at this race track, Socrates. Sometimes the probabilities sum to more than one; sometimes to less than one. You know what that means, Socrates?

SOCRATES: Someone thinks he knows something that he can't possibly know.

[Odds, Stakes, and Partial Beliefs]
[In Thirteen Parts]

1. *What are odds?*

HORSE	ODDS
ARISTOS	3 : 1
BUCEPHALOS	4 : 1
GERAX	5 : 1
DEIPYROS	8 : 1
ELIS	9 : 1
ZEPHYROS	11 : 1
HEPHAISTOS	20 : 1
IDOS	30 : 1

Table 1: For each horse, suggested odds against its winning.

THEODORUS: Either someone's claiming to know that our axiom of probability is false, or it's as if that's so. Here we are. Let's look at the odds board.

SOCRATES: I suppose this board belongs to some bookie, but I don't see one around. . . . Something about the style of writing reminds me of what I've seen in your school . . .

THEODORUS: Get your mind on business! Do you know what odds are used for?

SOCRATES: Not really.

THEODORUS: They're ratios associated with each horse that tell gamblers how much they can win by betting that a horse will win. Let's see what the odds are against the first horse—There: 3 to 1. That means that a bet of 1 drachma that the horse Aristos will win will net 3 drachmas for you if it's true. But if it's false, you lose your drachma. Try it on the next horse.

SOCRATES: Bucephalos is 4 to 1. So if I bet one drachma that Bucephalos will win, and he does, I don't lose my drachma. Instead someone gives me four more. And I suppose if I bet 10 drachmas that he'll win, and he does, someone gives me 40 drachmas.

THEODORUS: Exactly.

SOCRATES: And if Bucephalos loses, I'm out 10 drachmas.

2. *What are stakes?*

THEODORUS: You have the idea. Are you familiar with the idea of the bet's stakes?

SOCRATES: No. Sorry.

THEODORUS: A person's stake—here the word is singular—is the amount he risks and could lose. Add together the money everyone's risking in a single bet, and that total's the stakes—here the word is plural—in that bet. The stakes belong to the winner of the bet. The stakes are the prize, in effect.

SOCRATES: So the stakes in this bet on Bucephalos are . . . let's see. I risked losing 10 drachmas to the other party. He risked losing 40 to me. So 50 drachmas are the stakes. Or we might play for lower stakes, stakes of 5 drachmas: I risk losing just 1 to him, and he risks losing 4 to me.

3. *What is the PROBE of a bet?*

THEODORUS: Correct. Either way your share of the total risk is 1/5.

SOCRATES: 1/5? I wanted to blurt out that it was 1/4. But no. It is 1/5.

THEODORUS: The odds are 4 to 1 against Bucephalos winning. By putting up 1 drachma for each 4 he puts up, you put up 1/5 of the stakes.

SOCRATES: Does this fraction have a name too, as odds do? What's this 1/5 Portion Risked Of Bet monEy?

THEODORUS: Glad you asked. Let's make an acronym from what you just called it: P.R.O.B.E. The "Portion Risked" is that fraction of the stakes, "Of Bet monEy," which you must put up as your part of the bet.

SOCRATES: I guess a gambler'd want to know his PROBE in any bet. Is there a general way of figuring my PROBE in a bet, if I did bet?

THEODORUS: Yes; it's implied by the odds you accept. The denominator of your PROBE is the sum of the two numbers in the odds. Your numerator depends on which side of the bet you're on. If you bet against the horse winning, your numerator is the odds' first number. If you bet on the horse winning—bet in favor of its winning, that is—your numerator is the odds' second number.

SOCRATES: OK. Let me practice. Deipyros is 8 to 1. That means the odds are 8 to 1 against his winning. So if I bet that he'll win, I must put up 1/9 of the stakes; my PROBE's 1/9. If I bet against his winning, my PROBE would be 8/9. That's considerably more, but his losing is the more likely outcome.

4. *What is a partial belief? What is its connection to a PROBE?*

THEODORUS: True. How much more likely, in your opinion?

SOCRATES: A lot.

THEODORUS: Not a definite enough answer. You should know your own mind better than that.

SOCRATES: I don't see why I should. Either the horse'll win or it won't. I'm not of the opinion that it will, and I'm not of the opinion that it won't. What more can I say? I'm no expert on horses.

THEODORUS: I ask you for your degree of conviction, partial conviction, that the horse will win, whether or not your opinion's based on knowledge. To answer me, you need only know thyself, not thy horse. Anyway, the boys and I figure we can estimate more exactly how likely you think a win by Deipyros to be. Socrates, we've found a window into your mind!

SOCRATES: How fascinating.

THEODORUS: It's as if another's mind were occupying a mysterious house, dark and shuttered. We've learned how to unlock the shutters. We throw them open; the light within streams to the world outside. We see the shadows of what's within.

SOCRATES: Homeric similes from a geometer! What do you see shadows of?

THEODORUS: We see his mind's degrees of partial conviction in various propositions, as fractions of a total conviction, a full certitude. When I see how a man bets with my eyes, with my extra-penetrating insight I can discern the strength of his inner convictions.

SOCRATES: You have a crystal ball, have you?

THEODORUS: No, Sir. People show me their minds in the bets they take.

SOCRATES: Well, I don't take bets.

THEODORUS: You take chances; we all take chances everyday. Every time we do something that increases our risks, we're in effect gambling. Suppose a man trades a secure investment for a merchant ship, in the hope of increasing his wealth by exporting and importing goods. He's increased his risk, for the ship may sink or suffer damage, or mice may eat his cargo, or prices may fall below what he paid for his goods. Yet he accepts these risks because prospects of good outcomes counterbalance them. Prudent men calculate these matters carefully, and we admire them and benefit from their actions. Yet it's a form of gambling. Isn't that so, Socrates?

SOCRATES: Yes.

THEODORUS: So betting is just one more form of recreation imitating serious business, like little boys' war games.

SOCRATES: OK. In this broad sense I do gamble. But I'm not a gamester.

THEODORUS: I prefer to use the game of bets to see into people's minds because it's simpler. First, there are only two outcomes, unlike the serious business, where the outcomes may not even be countable. Secondly, the bet is decided quickly and without doubt. In serious business, the consequences of actions ramify and never come to an end. Do you still object, Socrates?

SOCRATES: If you use betting as an idealization of the serious business of living, then, Theodorus, use bets to show how you peer into people's minds. How do you see their degrees of conviction?

5. *What is a disadvantageous bet? The window-opener.*

THEODORUS: Here's my window-opening principle, which I believe is a law of psychology:

NO ONE TAKES A BET HE REGARDS AS DISADVANTAGEOUS.

A bet on a proposition's truth which he regards as disadvantageous is one in which his PROBE exceeds his degree of confidence in its truth. This definition's the window-opener.

SOCRATES: Aha!

THEODORUS: For example, if you're willing to put up 1/9 of the stakes in a bet that Deipyros will win, your degree of conviction that Deipyros will win cannot be less than 1/9 of the maximum of full conviction in his winning.

SOCRATES: Why not?

THEODORUS: Because then you'd think the bet disadvantageous, wouldn't you? Of course. If you put up 1/9 of the stakes, risking that much, you should have no less than a 1/9 chance of taking the stakes. Your portion of the risk, your PROBE, shouldn't exceed your chance of reward. Won't you admit that?

SOCRATES: I'm not sure. Let me figure it out for the other party to my bet. I bet the horse will win; he bets it'll lose. His portion of the risked money was 8/9. So he cannot believe Deipyros's chance of losing the race is less than 8/9. Or, to put it in terms of the horse's winning, he figures its chance at 1/9 at the very most.

THEODORUS: At the very most. Otherwise he'd find the bet disadvantageous to himself, and he'd refuse to bet you.

SOCRATES: So my PROBE determines a point between 0 and 1 of considerable significance. My degree of conviction that Deipyros will win the race cannot be less than my PROBE, and the other's degree of conviction that Deipyros will *win* the race cannot be greater than my PROBE.

THEODORUS: I think you have the hang of it.

SOCRATES: Isn't there another side to the definition? Let's say we bet and my PROBE is 1/9. If I think my chance of taking the stakes is greater than that, and he thinks it much less, we each think the bet's to our own advantage, don't we?

THEODORUS: Correct. Disagreements about chances grease agreements to bet. We can state the definition this way:

THE WINDOW-OPENING DEFINITION, BIFURCATED

(a) IF PROBE > DEGREE OF BELIEF, THEN THE BET IS THOUGHT DISADVANTAGEOUS.

(b) IF PROBE # DEGREE OF BELIEF, THEN IT IS NOT THOUGHT DISADVANTAGEOUS.

SOCRATES: I notice a coincidence. The PROBEs of the two parties to a bet will always sum to one. The definition of PROBE implies that.

THEODORUS: Yes.

SOCRATES: You also represent certitude, maximum conviction, by the same number as the sum of their PROBEs, although you admit it's only a convenience. Would it spoil your window-opening principle to represent certitude by a number other than one?

THEODORUS: No; it only complicates it slightly: A person regards a bet on a proposition as disadvantageous if and only if his PROBE exceeds the ratio that his degree of conviction in the proposition bears to the maximum degree of conviction, whatever you make it. That rules out infinity as the maximum, but any finite number is permissible.

6. *How should we treat money as a measure of desirability? The lesson from insurance.*

SOCRATES: I still feel diffident about assenting to your window opener, because it seems right for someone in a desperate situation to accept a PROBE greater than his degree of conviction

. . .

THEODORUS: Yes, I'm aware of that objection. The cases are like this one: A man is desperate for three drachmas, but has only two. With two he can do nothing; it's as if he had none. With three he can accomplish great things. Someone offers him a gamble. They will flip a coin; if it's heads, he wins one drachma, increasing his wealth to three drachmas; if it's tails, he loses his two worthless drachmas. His degree of conviction that the coin will come up heads is 1/2; his PROBE is 2/3; yet he quite correctly takes the bet.

SOCRATES: I don't see why that does not refute your idea of disadvantage.

THEODORUS: Well, it does refute it, if you want to get picky. But I could just repair it by distinguishing monetary value from subjectively real value. What we compare to the bettor's degree of belief is not his PROBE directly, but the ratio: the real value to him of the money he risks, divided by the real value to him of the stakes he stands to gain. In the problem I just gave you, that ratio would be less than 2/3. But look here, we don't have to get into the distinction between real value and monetary value. And, since we want to make observations of a person with as few assumptions about what's going on in his mind as we can get away with, we should find a way to evade the need for this distinction.

SOCRATES: I suppose you dismiss the case as too rare to constitute a serious limitation on your principle.

THEODORUS: Not at all; it's quite common, in fact.

SOCRATES: Oh?

THEODORUS: Yes, in purchasing insurance. Do you understand insurance?

SOCRATES: I think so. But I've never thought of it in the same context as gambling.

THEODORUS: Buying insurance is the reverse of gambling. The gambler buys into a risk; the insured buys out of one. The gambler pays for a chance at a desired outcome; the insured pays to remove the chance of a feared outcome. Each pays a fraction of the value of the outcome. So even the insured has his PROBE.

SOCRATES: Give me a concrete example, the man who purchased a merchant ship, let's say. Suppose he bought insurance.

THEODORUS: OK. But let's simplify insurance the way bets simplify serious business. Let there be only two outcomes: Either his ship's a total loss, say, a loss of 1000 drachmas, or it's no loss at all. There's a small chance of a total loss, a prospect he finds unbearable. So he buys insurance: If the ship's a total loss, the insurer will reimburse him fully. In return for the insurance, he pays the insurer a small sum. How much should he pay?

SOCRATES: That depends on how likely he thinks his ship will be a total loss.

THEODORUS: Exactly. Suppose he thinks there's one chance in a hundred his ship will be lost. Then, by my principle, his PROBE should not exceed 1/100. In other words, on 1000 of insurance he'd not pay a premium exceeding 10 drachmas.

SOCRATES: I see.

THEODORUS: The insured's PROBE is the fraction of the insured amount which constitutes his premium.

SOCRATES: But merchants commonly accept a PROBE higher than their degree of conviction that the disaster will occur. Is that your claim?

THEODORUS: Yes, about 50% higher, if they accept the insurer's estimate.

SOCRATES: Really! They—we—take PROBEs 50% higher than the insurers' estimates of our chances of loss!

THEODORUS: In apparent violation of my principle.

7. *Restriction on the window-opener, concerning money.*

SOCRATES: So your principle is of no value for seeing into people's minds.

THEODORUS: Not so fast. The fault's in using money as a measure of value over a great range of values. If I restrict stakes of bets and amounts of insurance to *small sums*, we can see that people do observe my principle. When only small sums are involved, they regard bets and insurance as disadvantageous when their PROBEs would exceed their degree of conviction in the proposition bet upon or insured against.

SOCRATES: "Small sums"?

THEODORUS: Small enough so that units *of money* approximate units *of real value*, but not so small that betting loses its attraction.

SOCRATES: For very small sums, the attraction might be to flout your principle altogether. Recklessness would have no penalty.

THEODORUS: True. By "small sums" I mean more than piddling amounts. Actually, we could do without money altogether, but it would take some explaining to distinguish units of money from units of real value and then to relate them.

SOCRATES: Ah! Impatient with your old student again.

THEODORUS: Just impatient with preliminaries. "Small sums" solves our problem well enough for us to get on with the observation of minds. Agreed?

SOCRATES: You yourself just gave an example in which a person with only two drachmas . . .

THEODORUS: Fiddlesticks. He was desperate for three, remember? What's small depends on a person's circumstances. In his, the stakes were large. A sum is small for a person, if the sure gain of it is not more of a gain than the sure loss of it is a loss.

SOCRATES: Does the restriction to small sums also rid us of the other type of counter-example? We've mentioned people who see an advantage to a bet even when their PROBEs exceeded their degrees of belief. Now I'm thinking of people who would still feel at a *dis*advantage, even if their degrees of belief exceeded their PROBEs.

DIODORUS: Yes. People who are risking a large portion of their wealth feel that way. They wouldn't feel that way when only small portions of their wealth are at risk. Here are the violations I hope to eliminate by the restriction to small sums.

Gamblers / insurance-buyers	Gambling	Insuring
violate part (a) of definition of disadvantage	when desperate for more than they have, or when their possible loss is piddling	when too much of what they have is at risk
violate part (b) of definition of disadvantage	when too much of what they have is at risk	when desperate to keep what they have, or when their possible loss is piddling

Table 2. Situations in which people do not act according to the definition of disadvantage.

8. *Other conditions affecting part (b) of the window-opener.*

SOCRATES: I might not believe you'll pay up if you lose to me. Wouldn't I be right to think a bet with you at a PROBE less than my degree of belief will still be disadvantageous to me?

THEODORUS: Yes, but I'll accept conditions for payment which reassure you completely.

SOCRATES: But what if we're not betting on horses, but something else, and I'm not sure we'll agree on what happened? Part (b) seems less reliable then. Maybe . . .

THEODORUS: OK! We only bet on horses or whatever conditions you are sure we can all ascertain. In other words, we bet under routine conditions in which you have no doubts about these matters, and we only bet for small sums. Can you now accept both parts of the principle?

SOCRATES: Well then, you do see into the minds of bettors, but not so clearly. If I were to bet on Deipyros at 8 to 1 odds, you'd discern that my degree of conviction that the horse'll win the race fell within a range from full conviction down to 1/9 of full conviction, and you'd discern that my partner's degree of conviction fell within the rest of the range, from 1/9 down to full conviction that the horse'll lose the race. And if I were not to bet, you'd know nothing. You accused me of being indefinite about my own degrees of conviction; I find you almost as indefinite.

THEODORUS: I plead guilty as charged. The PROBEs which a person accepts are only shadows of his inner degrees of conviction. My principle is a window onto the mind nevertheless, for all its shadowiness. And it's as much of a window as we need to prove Theaetetus's axiom. I, uhh, could give a less shadowy principle, but I'm less sure of it and you'd only drag me into argument that's irrelevant to our goal.

9. *Avoid circularity; don't derive the window-opener from the idea of "expected value."*

SOCRATES: Since you grant the lesser plausibility of this other principle from the outset, I'll be content to just hear it. Tell me.

THEODORUS: No, no. Well, OK. Suppose for simplicity that the small stakes in a bet equal unity. So either a bettor will win (1–PROBE), namely what the other bettor contributes to the stakes, or he'll lose an amount we represent negatively as (–PROBE), namely what he himself contributes to the stakes. We can represent his degree of confidence in his winning the bet by the shorthand, "prob," which is also a number between zero and one. I suggest the bettor's attitude, pro or con, toward his bet can be represented by the sign of this quantity:

$$(1–\text{PROBE})(\text{prob}) + (–\text{PROBE})(1–\text{prob}).$$

SOCRATES: Possible winnings times his confidence in winning, less possible losses times his lack of confidence in winning. Multiplying it out,

$$\text{prob}–\text{probPROBE}–\text{PROBE}+\text{probPROBE}.$$

But how's this an attitude?

THEODORUS: The principle reduces to this: The bettor's attitude to an unsettled gamble is just the sign, negative or not, of this: (prob–PROBE).

SOCRATES: I see the algebra. I don't see the attitude.

THEODORUS: Listen! The two parts of our window opener are consequences of this principle. When this quantity is negative the bet is seen as disadvantageous; when positive or zero, not disadvantageous.

SOCRATES: The derivation of the window-opener I grant, but I have many questions about what it's derived from.

THEODORUS: Of course you do. I do too. But forget them for now, because all we're interested in now is proving Theaetetus's axiom. We only need some of this principle's consequences, not the principle itself.

SOCRATES: But doesn't the formula assume Theaetetus's axiom? If tied outcomes are ruled out, and if there's no chance of the bet being called off, then this person's degrees of belief in the two alternatives are (prob) and (1–prob) if and only if he conforms to Theaetetus's axiom. So if you rely on Theaetetus's axiom to generate this formula, and you rely on this formula to prove the window-opener, and you rely on the window opener to prove Theaetetus's axiom . . .

THEODORUS: OK, OK, I don't need you lecturing me about begging the question. The consequences we need are weaker than the principle, and they have a plausibility independently of their derivation from the principle, and a greater plausibility at that. So forget the derivation. I know the window-opener's true in general from my observation of myself, and from the boys telling me it's true of them too.

SOCRATES: You're extrapolating from your own case to everyone?

THEODORUS: Just about, although we've identified four types of exceptional people.[3]

10. *What are probabilities? Subjective degrees of conviction distinguished from objective chances.*

SOCRATES: Say now, you haven't explained probabilities to me yet. Are they PROBEs?

THEODORUS: No, not PROBEs. The PROBEs are shadows of probability. Probabilities of propositions are persons' degrees of belief in them. Think of the mind as a house. Think of the light streaming out the window. The inner source of the pattern in the light is the degree of a person's conviction that something is so; its shadow on the ground outside is the PROBE of any bet that it's so. They're less than or equal to the probabilities they reflect.

SOCRATES: You've made probability a very personal thing then.

THEODORUS: Yes. Probabilities of the same proposition vary with the persons believing it, vary even with the times of their believing it.

SOCRATES: I'd thought the probability of, say, Gerax winning was his real chance of winning, which depended on the horses, and not on what a person like me thinks of the horses.

THEODORUS: Objective chance can be a probability too, Socrates. We needn't deny that, just because we think degrees of conviction are probabilities, personal ones. Tonight we only talk of personal probabilities, however, because Theaetetus's defense of his axiom makes sense only in terms of personal probabilities.

SOCRATES: Well, the one degree of conviction that matches the objective chance of Gerax winning is the true degree, and all the other degrees of conviction are just false, obviously.

THEODORUS: Not obviously. I tell you, put this idea of objective chance aside for now. We don't need to compare subjective strength of conviction with objective chance. Tonight we only study how a person is rational or irrational, depending on how his degrees of conviction relate to each other, not to anything else. So play along, will you please?

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[3]In Table 2.

SOCRATES: Do you mean to say, I'm to pretend there's no "the" probability of Gerax winning, but hundreds of probabilities of Gerax winning, all the different ones in different people's minds simultaneously?

THEODORUS: Exactly, even hundreds of them in you, not sumultaneously, but as you change your mind from one moment to the next for whatever reason, or no reason at all.

SOCRATES: I find it hard to keep things clear. Perhaps a picture . . .

THEODORUS: Alright. Your mind is a house; inside are your beliefs standing upright like people. Some are taller than others; their heights represent your degrees of belief.

11. *Others can discover your degrees of conviction by observing your behavior.*

SOCRATES: Ach! All you or anyone else can do is speculate, since you can't see in.

THEODORUS: Your behavior is a window into your mind. I cannot look directly at your beliefs through the window, but I can offer you bets at small stakes, which is like attaching a lantern to a pole and sticking it in the window to your mind. By placing the lantern behind a belief, I make it cast a shadow through the window onto the ground outside where I can observe it; that shadow is your PROBE in the bet.

SOCRATES: You can't tell a belief's height from the length of its shadow, however.

THEODORUS: The window-opener does let us compare the two. If, when I offer you a bet, I manage to place the lantern almost directly over a belief, slightly to its rear, the shadow the belief casts is shorter than the belief. But if I place the lantern too far above and behind the belief, the shadow it casts outside is longer than *it* is. Well, I poke around with my lantern a few times, that is, I probe your mind with offers to bet, and you accept some and reject others. When the shadow I've caused your belief to cast is longer than your belief is tall, you reject the bet. I discover that your PROBE exceeds your probability. When you do accept, the shadow, your PROBE, is shorter than your probability, or perhaps equal in length, although the latter leaves your motive for accepting rather weak. Does this picture help?

SOCRATES: Sure. I don't bet on the shadowed when it's shorter than its shadow. That's shimple.

THEODORUS: PROBEs are behavioral and public; probabilities are mental and private. PROBEs we observe; probabilities we infer.

SOCRATES: Ah, so that's it; you're inferring a cause from its effect. That which casts a shadow causes a shadow to be cast. Yes?

THEODORUS: No. Nothing causes itself to cast a shadow! *We* cause it to cast a shadow by probing it. It does nothing to determine which shadow it casts; it just is! To understand the ways of the mind, I prefer the metaphor I've been using: What we observe are the reflections of (prob) or its manifestation of itself. Our inference is based on empathy with its rationality. The bonds of (prob) to the preference, which the sign of (prob–PROBE) indicates, and to the accepting or rejecting of the PROBE in a bet are definitive of the sorts of things they each are. Conceiving of them bonded that way is unlike the way we'd have to conceive of them to capture the relation that a cause has to its effect. So I recommend silence on causality, OK? OK, enough of this. I see the boys are busy placing bets. So we probably have a violation of our axiom here.

12. *When do a bettor's PROBEs reflect that his degrees of belief violate the axiom?*

SOCRATES: How can you tell for sure?

THEODORUS: From the stated odds on each horse I calculate the PROBE of the person who bets the horse will win. Then I use my principle of PROBEs to calculate the degrees of conviction persons can have in the proposition that the horse will win, if they regard the PROBE as not disadvantageous to the one who bets. I've converted the odds to PROBEs and summed them up.

SOCRATES: In your head?

THEODORUS: Of course. Besides being a mathematician, I've had lots of practice. But I use my handy pocket abacus here to do the sum . . . The PROBEs are:

HORSE	PROBE OF A BET ON THE HORSE
ARISTOS	1/4
BUCEPHALOS	1/5
GERAX	1/6
DEIPYROS	1/9
ELIS	1/10
ZEPHYROS	1/12
HEPHAISTOS	1/21
IDOS	1/31

	SUM: 29,031/29,295 (ROUNDING UP: 0.991)

Table 3: The PROBEs of bets on horse.

As you see, Socrates, their sum is less than one.

13. *Proof that PROBEs can reflect degrees of belief.*

SOCRATES: But Theaetetus's axiom didn't say that PROBEs must sum to one, only that a person's probabilities must.

THEODORUS: Quite right. Imagine two people using these odds to bet each other on each horse. One bets *on* the horse each time; the other *against* the horse. The person who uses these odds to bet that a horse will win need not be irrational, for, although his probabilities can't be less than these PROBEs, they may exceed them and so sum to one. But someone who accepts all these odds for bets that the horses will lose is irrational. His degrees of conviction that a horse will lose may be equal or less than the *other* fellow's PROBEs, but must not exceed them. You made that point yourself.[4]

SOCRATES: I remember. A person who bets against Deipyros has a PROBE of 8/9. His

[4]See page 10.

degree of conviction that Deipyros will lose may not be less than 8/9, if the window opening principle of PROBEs describes his behavior. Therefore, by Theaetetus's axiom, his degree of conviction that the horse will win is at the very most equal to the other fellow's PROBE of 1/9. Similar reasoning applies to the other seven cases. Therefore, his probabilities of a win for each of the eight horses must sum to less than one, since the other fellow's PROBEs do.

THEODORUS: Bravissimo!

SOCRATES: That good, eh? I'm afraid I'd disappoint you if you asked me to repeat my wondrous performance.

THEODORUS: I'll write out the steps of your reasoning in my notebook here. As I write each step, you tell me what justifies it. Then, so what if you forget them; they're here for you to refresh your memory.

SOCRATES: I hate to have to rely on writing, but in this case . . .

THEODORUS: I'll refer to the PROBEs of bets on a proposition, A, or its denial, NOT-A, as PROBE(A) and PROBE(NOT-A). I'll refer to our bettor's degrees of belief in these propositions as prob(A) and prob(NOT-A). He will be the one who accepts or "covers" other people's bets on the horses at the odds stated on that odds board. So he's betting against the horses. And let's refer generally to the eight propositions stating that a horse wins as A_i, where the i is a number from 1 to 8. We say the A_is are mutually exclusive and jointly exhaustive. Are you following me, Socrates?

SOCRATES: You say you're just repeating what I said? Amazing.

THEODORUS: We're going to prove this bettor's degrees of belief violate the axiom.

SOCRATES: OK.

THEODORUS: Here's your first step. For each A_i, $\text{prob}(\text{NOT-}A_i) \$ \text{PROBE}(\text{NOT-}A_i)$. Do you recognize it?

SOCRATES: Yes. The window-opener. Anyone who bets against each horse has degrees of belief not less than his PROBEs. So our bettor does.

THEODORUS: Second step: $\text{prob}(\text{NOT-}A_i) \$ 1–\text{PROBE}(A_i)$. See what I did?

SOCRATES: A substitution in my first step, guaranteed by the definition of a PROBE.

THEODORUS: Third: $1–\text{prob}(\text{NOT-}A_i) \# \text{PROBE}(A_i)$.

SOCRATES: A bit of algebra applied to my second step. You subtracted a 1 from both sides of the inequality; then you multiplied both sides by a negative 1, which changed the direction of the inequality.

THEODORUS: Fourth: Either $1–\text{prob}(\text{NOT-}A_i) = \text{prob}(A_i)$, or it doesn't.

SOCRATES: Logical.

THEODORUS: Fifth, if it doesn't, our bettor's degrees of belief violate the axiom, since these are two mutually exclusive and jointly exhaustive propositions.

SOCRATES: And if the equality does hold?

THEODORUS: Sixth, if $1–\text{prob}(\text{NOT-}A_i) = \text{prob}(A_i)$, then $\text{prob}(A_i) \# \text{PROBE}(A_i)$.

SOCRATES: You used the equality to make a substitution in my third step.

THEODORUS: So seventh, $\text{SUM}(\text{prob}(A_i))\#\text{SUM}(\text{PROBE}(A_i))$.

SOCRATES: If each degree of belief is not greater than its corresponding PROBE, the sum of them is not greater than the sum of the PROBEs.

THEODORUS: Eighth: $\text{SUM}(\text{PROBE}(A_i)) < 1$.

SOCRATES: That's what the odds board tells us. A premise.

THEODORUS: Ninth: $\text{SUM}(\text{prob}(A_i)) < 1$, in violation of the axiom.

SOCRATES: A deduction from the seventh and eighth steps, by transitivity of the relations < and #.

THEODORUS: Tenth: So on either alternative given in your fourth step, our bettor violated the axiom.

SOCRATES: . . . which was the very thing to be demonstrated, or Q.E.D., *quod erat demonstrandum*.

THEODORUS: So we know that the person who accepts these odds for purposes of betting against each horse is violating Theaetetus's axiom.

SOCRATES: Clever.

THEODORUS: Or he might be thinking that there are at least 9 chances in a thousand that he'll win all his bets. For, as his thinking goes, there are 9 chances in a thousand, or more than nine, that no horse will win. But we've already determined that we can exclude that case from consideration. All bets are off if no horse wins.

SOCRATES: Just what mistake is he making? Is he obeying Theaetetus's axiom and not realizing the eight propositions about each horse winning are jointly exhaustive of all possibilities, or does he realize that and really is violating Theaetetus's axiom?

THEODORUS: Ask him. Let's suppose he says he knows all bets are off if no horse wins. He must be violating the axiom somehow. That's when our Socratic method comes into play.

[The Basic Biased Book: First Example, PROBEs too Low]

SOCRATES: So what next?

THEODORUS: Here's the general method. We convert odds to PROBEs and sum the PROBEs. If the sum equals one, we don't make any bets on the basis of those odds. If they sum to *less than one*, we bet on every horse *to win*, in amounts determined by the PROBEs. If they sum to *more than one*, we bet every horse *to lose*, again by the PROBEs. It's simple. And we always win. Can we lose if we're betting in favor of an axiom of rational belief?

SOCRATES: I don't understand. If I bet someone now that every horse will win, I'd lose for sure.

THEODORUS: True. That's not what I meant. Here's "Shark." I'll find out how the boys have set up the "book," and then I'll explain it to you. Boy!

"SHARK": You wanna bet, Mistuh?

THEODORUS: Tell me how you're betting.

"SHARK": I bet 300 drachmas on Aristos to win. I bet 133.33 on Deipyros to win. I bet 57.14 on Hephaistos to win. I bet 38.71 on Idos to win. Wanna bet, Mistuh?

THEODORUS: Perhaps my friend here will place a bet. How about it, Socrates?

SOCRATES: No.

"SHARK": You waste my time. Goodbye, wise guys.

THEODORUS: Notice, Socrates, "Shark"'s betting on four of the horses. So "Dutch" is betting the other four. Between the two of them, they're placing bets on all eight horses, individually. They're not betting that the proposition, "every horse will win," is true. That would be foolish. And did you notice the unusual amounts "Shark" was offering to bet?

SOCRATES: Yes. What's that all about?

THEODORUS: Each bet is 1200 times the PROBE of a bet on the horse's winning. "Shark" knew that's what I was really trying to find out. Sometimes I have to correct his arithmetic.

SOCRATES: Why 1200 times?

THEODORUS: The multiplier is an arbitrary number as far as testing our theory is concerned. The boys set it to fit the amount of money they have to bet and the amount other people are willing to bet. It's just the sum of the two, the stakes. Here. Let me show you how it's working tonight.

HORSE	PROBE OF BET HORSE WINS x	STAKES	= AMOUNT TO BET
ARISTOS	1/4	1200	300 DRACHMAS
BUCEPHALOS	1/5	1200	240
GERAX	1/6	1200	200
DEIPYROS	1/9	1200	133.33
ELIS	1/10	1200	120
ZEPHYROS	1/12	1200	100
HEPHAISTOS	1/21	1200	57.14
IDOS	1/31	1200	38.71

		TOTAL:	1189.18 DRACHMAS

Table 4: The boys' calculations of their bets

SOCRATES: You can't be serious. The boys are out there risking over a thousand drachmas on the horses?

THEODORUS: More like six thousand, I'd say. We count as one complete "book" a collection of eight bets, one on each horse to win at the amounts you see here. Once the boys have completed the collection, they start a second collection. My guess is that they're going for five books tonight.

SOCRATES: Explain to me why you think you can't lose.

THEODORUS: It's nothing but simple arithmetic. Every "book," I mean, collection of eight such bets, is certain to win around 10.82 drachmas. Here. I'll show you.

WINNER	ODDS	COST OF BET	NET GAIN ON THIS BET	–	LOSSES ON 7 OTHERS	=	NET GAIN FROM ALL 8 BETS
ARISTOS	3:1	300	900	-	889.18	=	10.82
BUCEPHALOS	4:1	240	960	-	949.18	=	10.82
GERAX	5:1	200	1,000	-	989.18	=	10.82
DEIPYROS	8:1	133.33	1,066.64	-	1,055.85	=	10.79
ELIS	9:1	120	1,080	-	1,069.18	=	10.82
ZEPHYROS	11:1	100	1,100	-	1,089.18	=	10.82
HEPHAISTOS	20:1	57.14	1,142.80	-	1,132.04	=	10.76
IDOS	30:1	38.71	1,161.30	-	1,150.47	=	10.83

Table 5: The boys' calculations of their net winnings.

Do you understand this table? Imagine you've placed all eight bets.

SOCRATES: The first row shows me what I gain by betting 300 drachmas on Aristos, if Aristos wins. That means, of course, that I lose the other seven bets. So I subtract from my winnings on Aristos what I lost on the other seven bets. What's left is my net gain. The other rows repeat those calculations for the other seven possible outcomes of the race, Bucephalos winning, Gerax winning, and so on. I see that I always come out ahead. Always.

THEODORUS: Always, no matter which horse wins, and always by the same amount, except that whenever we have to round off our bet, we get a little variation in our net gain. Another way of looking at the matter is to add your bet to your gain on that bet if you win, the stakes in that bet. That's the amount you walk away with if you win exactly one of the bets, 1,200 drachmas, near enough, for each bet. You placed a total of 1,189.18 drachmas in bets. That's what you walked in with. So you're richer by 10.82 drachmas whichever horse wins.

SOCRATES: Your total contribution to all the eight stakes combined is less than the amount of any one of the stakes.

THEODORUS: Right.

SOCRATES: And you're sure to take one of the stakes.

THEODORUS: Now, you've got it.

SOCRATES: Amazing. You're betting with no chance of losing! Except what do you do to protect yourself against welchers?

THEODORUS: Good point. Sometimes we use professional bookies to hold our bets. But they charge six and a half percent of the winner's bet. Tonight our net gain is not very large. If Aristos wins, a bookie's charge could wipe out our gain. They'd charge 6.5% on our 300 drachma bet. That's 19.50 drachmas. We'd be out 8.68 drachmas instead of ahead by 10.82. I think the boys are trying to avoid the professionals tonight. They find some old respectable types who'll hold money for nothing.

SOCRATES: But the bookie who owns the odds board—where is he? You could ruin him by placing all your bets with just him. Could it be there isn't . . .

[YOUNG SOCRATES COMES BY WITH A STRANGER.]

"DUTCH": Look here, Stranger. Here are two old respectable gentlemen who look like they can be trusted with our bets.

STRANGER: Sirs, this boy and I have just made a bet on the horse, Elis. Would you be kind enough to hold it for us?

SOCRATES: No.

THEODORUS: That's not very gentlemanly of you, Socrates, [FORCIBLY OPENING SOCRATES'S HAND]. Each of you just put your money right there. Who gets what?

STRANGER: If Elis wins, the boy gets the whole 1,200 drachmas. If Elis loses, I get it all. But no matter who wins, I already feel repaid if I am making the acquaintance of the Socrates, Athenian philosopher.

SOCRATES: You are, Sir. And this is Theodorus, the geometer.

STRANGER: Greetings. I too am a philosopher.

SOCRATES: You are? Then you can tell me why a philosopher should play the horses. My good friend here has been trying to persuade me of that, but I seem to be a bit thick headed tonight. You, however, are a philosopher who has just risked 980 drachmas, betting a boy that Elis will lose.

STRANGER: I'm not in favor of betting any more than you. But that boy needs to be taught a lesson. He told me mine was the fifth bet he'd placed on Elis. How can he expect a 9 to 1 shot to win? He must learn to respect the odds. That's why I placed the bet, to ensure that he loses his shirt. He won't be so foolish with his money in the future. The race is beginning. Let's watch.

SOCRATES: What's to watch? Just horses running around in circles.

THEODORUS: Look at that horse breaking out of the pack! What a finish! What horse was that? It was Elis!

"DUTCH": Hi, Mr. Socrates. I'm here to collect my bet. Thank you. Tough luck, Stranger. Got to collect the rest of my bets now. Goodbye.

THEODORUS: Perhaps the boy's more of a philosopher than you realize, Stranger. He's a student of mine. Come to Athens tomorrow and put him to the test yourself at my school. Perhaps that'll make up for your loss of 980 drachmas.

STRANGER: I'll do that. Thank you.

[The Basic Biased Book: Second Example, PROBEs too High]

SOCRATES: Where's Theaetetus been all this time? I haven't seen hide nor hair of him.

THEODORUS: Oh, he works the most lucrative spot by himself. He's back at the stables.

SOCRATES: I don't understand.

THEODORUS: The owners of the horses are back there before the race, and they have the most distorted sense of the chances of their horses' winning. Theaetetus gets each to cite odds on his horse. Invariably the PROBEs sum to more than one. So Theaetetus bets each owner that his horse will lose. I see him coming now. He'll explain it to you.

THEAETETUS: You won't believe the kind of odds I got tonight. Even the guy who named his horse after a cripple gave me even odds. Here are tables showing the results. The first one shows my strategy in betting:

HORSE	ODDS	PROBE	PROBE FOR HORSE TO LOSE	x	STAKES	=	BET THAT HORSE LOSES	GAIN IF BET IS WON
ARISTOS	1:2	2/3	[1/3	x	1200	=	400]	800
BUCEPHALOS	3:1	1/4	[3/4	x	1200	=	900]	300
GERAX	7:1	1/8	[7/8	x	1200	=	1,050]	150
DEIPYROS	7:1	1/8	[7/8	x	1200	=	1,050]	150
ELIS	7:1	1/8	[7/8	x	1200	=	1,050]	150
ZEPHYROS	7:1	1/8	[7/8	x	1200	=	1,050]	150
HEPHAISTOS	1:1	1/2	[1/2	x	1200	=	600]	600
IDOS	7:1	1/8	[7/8	x	1200	=	1,050]	150
TOTALS		2 1/12					7,150	

Table 6: Theaetetus's calculations of his bets.

The first line shows you that the owner of Aristos was so confident of winning that I was able to badger him into paying me double my bet if he lost. Only one was sucker enough to fall for the even bet ploy. That's the seventh line. I was able to get a little more out of the Bucephalos guy. The stable manager helped me a lot there. And the rest we shamed into betting that their horses had just as good a chance of winning as any other horse. The PROBEs summed to 2 1/12, well in excess of 1, as we expected. So the probabilities of their collective minds summed to at least that.

SOCRATES: Did you risk 7,150 drachmas altogether?

THEAETETUS: Yes, Socrates. But I knew I'd get it all back. Here are the results in the next table. My net gain from all bets is over and above the 7,150.

WINNING HORSE	GAIN FROM 7 BETS	–	LOSS OF ONE BET	=	NET GAIN ON ALL BETS
ARISTOS	1,650	–	400	=	1,250
BUCEPHALOS	2,150	–	900	=	1,250
GERAX	2,300	–	1,050	=	1,250
DEIPYROS	2,300	–	1,050	=	1,250
ELIS	2,300	–	1,050	=	1,250**
ZEPHYROS	2,300	–	1,050	=	1,250
HEPHAISTOS	1,850	–	600	=	1,250
IDOS	2,300	–	1,050	=	1,250

Table 7: Theaetetus's calculations of his net winnings. The seven bets that would be winning bets, if the horse listed were the winner, are the bets made with each owner of the other horses, that his horse will lose.

The second table demonstrates that I was certain to win 1,250 drachmas, no matter which horse won. And of course I did. I put up 7,150 drachmas. All was returned to me except the 1,050 I lost on Elis. But I netted 2,300 on the other seven bets, which left me ahead by 1,250. The stable manager got his half, and here's my 625 drachmas! And here's the 7,150 you loaned me, Theodorus. God! Am I excited!

SOCRATES: Why does the stable manager get half?

THEAETETUS: Because he'll kick me out of the stable if he doesn't. Actually, he's proved quite useful since he learned I never lose. He's learned that the higher the stakes are, the more I win, and the more he gets. So, in a way, he earns it with his scornful looks whenever an owner tries to avoid a bet or keep it cheap. He also holds the bets to make sure no one welches.

SOCRATES: How generous of him. Let me make sure I understand your second table. The fifth line refers to what actually happened, namely, Elis won; the other seven lines refer to what might have happened instead.

THEAETETUS: Right.

SOCRATES: Since the PROBEs summed to more than 1, you bet against each horse. So you won seven bets instead of just one. The last column of your first table shows me what you won. I just have to add up the column, leaving out the 150 you did not win in your bet against Elis . . . 2300 drachmas.

THEAETETUS: Right. That's what my second table shows on the fifth line.

SOCRATES: Yes. I see: "Elis; 2,300." How much did you lose on Elis? Your first table, second to last column, shows what you risked on Elis: 1,050 drachmas.

THEAETETUS: Right. My second table shows that on the fifth line too.

SOCRATES: Combining your one loss of 1,050 with your seven wins totalling 2,300 gave you a net gain of 1,250 drachmas, as your table says.

THEAETETUS: And the neat thing, Socrates, is that it didn't matter one bit which horse won. I'd get my 1,250 . . .

SOCRATES: 625 after taxes.

THEAETETUS: . . . no matter what. That's what the other seven lines on my second table show.

[The Basic Biased Book: Third Example, Applied to One Person]
[In Three Parts]

1. *What do you think of a person who takes all the bets in a book biased against him?*

SOCRATES: Something's puzzling me. If anyone had taken all the bets against the horses according to the odds listed over there,[5] we'd've said his degrees of belief violated the axiom. Theodorus demonstrated that to me. Now you are taking all the bets against the horses, and you are not violating the axiom, or are you?

THEAETETUS: No, I'm not. According to our window-opening principle—have you heard of that?

SOCRATES: Yes; Theodorus told me all about it.

THEAETETUS: Did he introduce you to our symbols for representing a person's degrees of conviction in a proposition A, "prob(A)"?

SOCRATES: Yes, and for a person's PROBEs too.

THEAETETUS: Good. The window-opening principle implies this: For each proposition, call it A, that I bet against, my PROBE(NOT-A) # my prob(NOT-A). But that's consistent with my degrees of belief in all these propositions summing to 1.

SOCRATES: Would you mind showing me how?

THEAETETUS: Well, since

PROBE(NOT-A) = 1–PROBE(A),

it follows from my taking the bets that

for each A, 1–PROBE(A) # my prob(NOT-A).

By algebra, that's equivalent to

for each A, PROBE(A) $ 1–prob(NOT-A).

Now, trust me, I believe the right side is equal to prob(A). So

for each A, PROBE(A) $ prob(A),

and the sum of all my prob(A) is equal to 1, less than the 2 1/12 the PROBEs summed to.

SOCRATES: But that's just what I'm wondering about.

THEAETETUS: Well, at least there's nothing in my betting behavior that should cause you to wonder. The sum of all the PROBE(A) is well over 1, as you saw. My betting behavior committed me to the truth of all the inequalities I derived. And they are completely consistent with my degrees of belief in all the A's not violating the axiom.

SOCRATES: OK. I guess it would be strange if you had to violate the axiom in order to

[5] See Table 1 on page 7.

be certain of winning a bundle.

THEAETETUS: That would be strange. The odds I got weren't my pals' odds. So my betting strategy was different from theirs. The person on the winning side in our books is not forced to adopt irrational degrees of belief. But the person on the losing side in these books, if he takes all the bets himself, is on that side because of his irrational degrees of belief.

2. *When someone takes all the bets on the losing side of a biased book, is he irrational?*

"DUTCH": How's your luck been, gentlemen? Shark and I put together five complete bets and have won 54.10 drachmas in all. Theodorus, here's the money you lent us, 5,945.90 drachmas.

SOCRATES: I'm glad to see, Theodorus, that you haven't become a usurer too.

"DUTCH": Oh yeah, I forgot. Here's the two drachmas interest I owe you for the day's loan.

THEAETETUS: Sorry, I forgot too. Here's the three drachmas interest I owe you.

THEODORUS: Thanks, fellows. You know I didn't mean you had to . . . See here, Socrates . . .

"DUTCH": Chariot race is next. Care to bet, anyone?

LAMPROCLES: Hi, Dad.

SOCRATES: Son! What are you doing here?

LAMPROCLES: I was just going to ask you the same thing, Dad.

SOCRATES: Don't be impertinent, Son. Are you betting on the horses?

LAMPROCLES: Yeah, Dad. I really get my kicks from the horses, Dad.

SOCRATES: Then we have something to thank the horses for.

LAMPROCLES: Yeah, Dad. Lotsa kicks. Can you lend me some change, Dad? I'm low, and I haven't won a bet all night.

SOCRATES: Why? Are you going to bet on the chariot race?

LAMPROCLES: Yeah, Dad.

SOCRATES: Good. I'll watch how you do it, and you can explain to me whatever I don't understand. I'll lend you whatever you need to cover your bet.

LAMPROCLES: Hey, Dad. You know what? You're a cool dude, Dad. It's a deal. What d'ya want to know?

SOCRATES: First, tell me what the chances of each of the four chariots are.

LAMPROCLES: Oh, yeah, Dad. Well, I think it's going to be a close race. Pretty even, I'd say.

SOCRATES: Do we need to know "odds," whatever that means?

LAMPROCLES: Oh, yeah, Dad. You should know the odds when you make a bet.

SOCRATES: Well, what are the odds in this race?

LAMPROCLES: I'd say the red chariot's a "2 to 1 shot." That's track lingo, Dad.

SOCRATES: What's it mean, Son?

LAMPROCLES: It means he has a good chance of winning, kind of close, like I said. Now if it was a "20 to 1 shot," that's what we call a "long shot." Long shots don't stand much chance of winning. But they pay well when they do.

SOCRATES: You know this stuff pretty well, don't you?

LAMPROCLES: Yeah, Dad. I study pretty hard.

SOCRATES: Well, what are the odds on the other charioteers, Son?

LAMPROCLES: Well, since they're all pretty even, they'd all have the same odds, Dad.

SOCRATES: That sounds reasonable enough. So they all are "2 to 1 shots"?

LAMPROCLES: Hey, you're catching on real quick, Dad.

SOCRATES: Then we're ready to place our bets. Sir! Would you care to wager on the next race with me and my son?

"DUTCH": Sure thing, Mistuh. What's your pleasure?

SOCRATES: OK, Son. We know that each of the four chariots is a 2 to 1 shot. What do we do next?

"DUTCH": I'm gonna make life simple for you gents. At those odds I'm willing to bet no charioteer wins.

LAMPROCLES: We got it made, Dad.

"DUTCH": Here's the deal. We make four separate bets, one on each chariot. I bet that the chariot will lose. The odds are 2 to 1 on each bet. If the chariot wins I pay you 200 drachmas; if he loses you only pay me 100 drachmas. Is it a deal?

LAMPROCLES: It's a deal. Hey, Dad, you sure have a talent for meeting up with chumps. This guy just bet us that every chariot would lose.

SOCRATES: It sounds like a winner, Son. But I'm new to this sort of thing. Do we have enough money to cover our bets?

LAMPROCLES: Oh, yeah, Dad. Don't worry. We're going to win. See? I have 300 drachmas here, if that'll make you feel better.

SOCRATES: And look at those chariots go! My, that was over fast. The Bronze Chariot's the winner.

"DUTCH": Time to settle up, gents. The first chariot lost; you pay me 100 drachmas. The second chariot lost; you pay me 100 drachmas. The third chariot won; I pay you 200 drachmas. The fourth chariot lost; you pay me 100 drachmas. Thank you very much.

LAMPROCLES: Did you see that, Dad? We won. Just like taking candy from a baby.

SOCRATES: How much do you have?

LAMPROCLES: Let's see. I have 200 drachmas.

SOCRATES: You had 300 drachmas. What happened to the other 100?

LAMPROCLES: Well, you see, Dad, we didn't win all our bets. That's life, you know. You win some, and you lose some.

SOCRATES: I see. We made four bets, but only won one, which wasn't enough to cover our losses.

LAMPROCLES: Don't take it bad, Dad. You can't win all the time. That's life.

SOCRATES: You know that for a fact, do you?

LAMPROCLES: Oh yeah, Dad. I learn a lot about life down here.

SOCRATES: I'm glad to hear that; we could use an expert on life in the family.

LAMPROCLES: Thanks, Dad.

SOCRATES: But that guy must have been on to something we didn't know.

LAMPROCLES: No, Dad. He didn't know anything. I can tell.

SOCRATES: How can you tell?

LAMPROCLES: If he knew something, he wouldn't've bet that the third chariot would lose. That was his big mistake.

SOCRATES: He had to bet the third would lose. What if he only bet the first, second, and fourth would lose, and then the first chariot won? The guy wouldn't have made anything off of his bets with us.

LAMPROCLES: There's just one thing wrong with your thinking, Dad.

SOCRATES: What's that?

LAMPROCLES: The first chariot didn't win; the third did. That guy didn't know anything, Dad.

SOCRATES: Well, I believe you're right. That guy didn't know anything that anybody and everybody else doesn't already know.

LAMPROCLES: Yeah, I could spot that real easily.

SOCRATES: So how come this know-nothing walked off with a hundred of our drachmas?

LAMPROCLES: Well, you see, Dad, this is where the lessons in life come in. You got to learn that things don't always go your way. Life is a gamble. Well, I'm going to run along now. There's my gang over there. See you around, Dad.

3. *How explain the irrationality of taking all the bets on the losing side of the book?*

YOUNG SOCRATES, "DUTCH": Excuse me, Socrates. Socrates?

SOCRATES: Yes, Socrates? Or must I still call you Dutch?

YOUNG SOCRATES: No. The charades are over for the night. Here are your son's hundred drachmas. I didn't want you to think we'd keep his money. But I couldn't resist. You don't often run into types like that with money to bet.

SOCRATES: I can understand, for a fool and his money are soon parted.

YOUNG SOCRATES: You have to be real lucky to find a fool who still has his money.

SOCRATES: By the window-opening principle, Lamprocles's degrees of belief in each of the propositions were not less than his PROBE in each bet. So the sum of them was not less than the sum of the PROBES. But the sum of the PROBES was greater than 1; so the sum of his degrees of belief was greater than 1 too. My oh my; boy oh boy.

YOUNG SOCRATES: I'm sorry. Here's Lamprocles's money.

SOCRATES: Thank you. In return for this fine gesture, Lamprocles will pay for refreshments for you, "Shark," Theaetetus, Theodorus, and me.

THEODORUS: I know a tavern round the corner, not too disreputable. I mean, no riffraff; just ship owners, foreign merchants, insurers. Maybe some gaming tables.

THEAETETUS: It's nice, Socrates. See this abacus? Each of us bought one there.

THEODORUS: Here we are, and there is the merchant who sold us them.

FOREIGNER: Herro, gentermen. You rike your new carcurators?

THEODORUS: We can't afford to be without them, Sir. In Greek the name for a calculator is abacus.

FOREIGNER: Ah, so: abacus. Maybe this genterman rike one, too? I show you. Four function keys. See? Only 7.95 drachma. You buy?

SOCRATES: Yes, I'll take two; one for me; one for my son. It'll make a useful gift for Lamprocles.

FOREIGNER: Thank you. Good evening, gentermen.

THEODORUS: Now you too can make tables like the ones the boys and I were making.

It'll add a whole new dimension to your Socratic method.

SOCRATES: I'll make a table for my son's bets. Since his odds were 2 to 1 against each of the four chariots, the PROBEs summed to 4/3, greater than 1. That's when you, Socrates, decided to bet on each to lose.

YOUNG SOCRATES: Right.

SOCRATES: You risked 200 drachmas; my son 100. The sum of these gives me the multiplier you used, the stakes. So here's the table:

CHARIOT.	ODDS,	PROBE	[PROBE CH. LOSES	x STAKES	=BET TO LOSE]	NET GAIN
GREEN	2:1	1/3	2/3	300	200	100
WHITE	2:1	1/3	2/3	300	200	100
BRONZE	2:1	1/3	2/3	300	200	100
RED	2:1	1/3	2/3	300	200	100
		4/3				

Table 8: Socrates's calculations of the bets with Lamprocles.

And here are all the possible alternative outcomes:

WINNER;	GAIN ON 3 WINNING BETS	–THE LOSING BET	=NET GAIN,ALL BETS COMBINED
GREEN	300	-200	100
WHITE	300	-200	100
BRONZE	300	-200	100
RED	300	-200	100

Table 9: Socrates's calculation of net winnings from Lamprocles.

THEAETETUS: Excellent.

SOCRATES: So hows about it, boys? Can I join the gang?

THEAETETUS: Sure thing. But what shall we call you?

SOCRATES: Call me Jimmy.

TAVERN OWNER: Here's your usual, gentlemen; wine, bread, dice. Who's the new face?

SOCRATES: I'm Jimmy.

TAVERN OWNER: Oh, another foreigner?

SOCRATES: No. I'm Jimmy the Greek.

[Is the Axiom of Probability a Description or a Norm?]
[In Two Parts]

1. *Do Lamprocles's bets disprove the axiom?*

YOUNG SOCRATES: I'm Dutch.

TAVERN OWNER: Hallo Meneer. Als u echt een Nederlander bent, zeg dan iets in het Nederlands.

YOUNG SOCRATES: [Coughs]

TAVERN OWNER: Wij worden allemaal verkouden.

SOCRATES: And you, "Shark." Where are you from?

"SHARK": [No answer.]

SOCRATES: Well, what's your real name?

"SHARK": No one knows that, Sir.

THEODORUS: It's so.

SOCRATES: Dear, dear. But "Shark"'s as good as any other misnomer; so Shark you are. Theodorus told me we were to prove Theaetetus's axiom of probability true. But instead we've proved it false, haven't we, Theaetetus?

SHARK: What's the matter, Theaetetus? You've gone all pale!

SOCRATES: Any truth worth the name of "axiom" describes probabilities as they have to be; probabilities are people's degrees of conviction. So the axiom, if true, would describe everybody's degrees of belief. But it doesn't describe Lamprocles's, if his PROBEs were any indication of them. Shark, you're the only one left with a voice. Do you agree we've misnamed it axiom?

SHARK: No, Sir. The axiom may or may not be false as a description of a person's actual beliefs, but it's true *as a norm.*

SOCRATES: A norm you say. Not a necessary description. A norm of what?

SHARK: A norm of . . . of rationality, that's what. We did prove that the axiom should, that is, ought to describe the degrees of belief of anyone motivated by advantages.

SOCRATES: How?

SHARK: Whenever the axiom doesn't describe a person's degrees of belief, there will be bets with PROBEs he would regard as not to his disadvantage, but which ensure a net loss for him whatever happens.

SOCRATES: How do you know whether the axiom describes a person's mind?

SHARK: How? You did say, didn't you, that you knew our "window-opener" principle?

SOCRATES: Theodorus mentioned it: If someone is offered a bet with small stakes on a proposition's truth, but his PROBE exceeds his degree of conviction in the proposition, then he regards the bet as disadvantageous.

SHARK: By that principle, we know that Lamprocles's degrees of conviction in the four propositions he bet on were not less than his PROBEs in the bets.

SOCRATES: They were equal or greater than his PROBEs, for he did take the bets. So his degrees of belief violated the axiom.

SHARK: Yes. Suppose his degrees of belief had conformed to the axiom. His maximum acceptable PROBEs would not have exceeded them, and then he wouldn't have fallen for sucker bets. At least one of the bets he was offered would've had a PROBE in excess of his degree of belief, and he'd've rejected the bet. That's why everyone's degrees of belief ought to conform to the axiom.

SOCRATES: Nice.

SHARK: If the window-opener principle describes degrees of belief, then the axiom is a norm of rational degrees of belief.

2. *How the window-opener is both descriptive and normative.*

SOCRATES: The rule for a PROBE is that it ought not exceed a person's degree of belief. Desperate people are exceptions to the window-opener rule, but you're not saying they're irrational, are you?

SHARK: No. But even desperate people are no exceptions to the axiom as a norm.

SOCRATES: So the PROBEs of bets on each proposition in a set of mutually exclusive and jointly exhaustive propositions ought not to sum to more than the maximum probability, but may sum to less. Right?

SHARK: That's correct, and whenever you can finagle it, they should sum to less. That's what we did tonight. The less it is, the more incentive you have to place the bets.

THEAETETUS: Yes! My PROBEs summed to less than 3/4. It was the cat's meow!

SOCRATES: So, no longer the sheet-faced fugitive from Hades, Theaetetus?

THEAETETUS: We didn't prove my axiom false; we proved it true, true as a norm.

SOCRATES: I stand corrected. Shark's hit on a theory of incentives to bet: The greater the difference between a person's degree of belief in a proposition, minus the PROBE of a bet on it, the more attractive the bet to him.

YOUNG SOCRATES: Get those suckers.

SOCRATES: So you've recovered from your coughing fit?

YOUNG SOCRATES: Yes, Sir.

SOCRATES: Then tell me, my dear Socrates, whether your principle, which you call "the window-opener," is a description of minds, or a norm for minds to follow.

YOUNG SOCRATES: Norm.

SHARK: No. Description.

THEAETETUS: It's both.

THEODORUS: It's neither; it's a definition of "disadvantageous."[6]

SOCRATES: Taverner! More wine!

[EVERYONE TALKS AT ONCE.]

THEODORUS: Fine. We've reached a consensus. We're all somewhat correct. The window-opener's in part factual, part definitional as in a theory of the mind, and part normative thus:

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[6]See pages 9-10.

THE PROBE PRINCIPLE, alias THE WINDOW-OPENER
FOR PROBES WITH SMALL STAKES AND ROUTINE CONDITIONS

DEFINITIONAL: IF THE MAXIMUM DEGREE OF BELIEF IS SET AT 1,

(a) IF PROBE ON X > DEGREE OF BELIEF IN X, THEN THE BET ON X IS DISADVANTAGEOUS.

(b) IF PROBE ON X # DEGREE OF BELIEF IN X, THEN IT IS NOT DISADVANTAGEOUS.

FACTUAL: WE REFUSE DISADVANTAGEOUS BETS.

NORMATIVE: WE OUGHT TO REFUSE THEM; WE MAY ACCEPT BETS NOT DISADVANTAGEOUS.

The normative part is a corollary of the axiom of probability taken as a norm, not part of the proof of the norm. But the other two parts we need as premises in the proof. When we argue that our axiom's the right norm, we're treating our window-opener as a descriptive premise, telling how minds do work in routine conditions, which occur when stakes are small and the circumstances of the bet reassure the bettor that the bet will be settled and prizes delivered.

THEAETETUS: If we can't blame Lamprocles's degrees of belief for his foolish behavior, then his foolish behavior implies nothing about his probabilities.

SHARK: So we're committed to his wrong degrees of belief accounting for his acceptance of those bets. To prove he disobeyed our axiom, we assume he did obey the PROBE principle.

YOUNG SOCRATES: We prove that a person ought to do something by showing that, when he fails to do it, it causes something bad.

SOCRATES: And that truism is your normative premise, not the normative parts of the PROBE principle. You can't get a normative conclusion from an argument without some normative premise, you know.

YOUNG SOCRATES: Right. But the normative premise we use is just the obvious one that you shouldn't adopt a procedure that guarantees that your purposes will be defeated.

SOCRATES: I wonder if those "routine conditions," which Theodorus mentioned, don't conceal some question-begging. You all insist we have a proof. But it's in hiding still. Show me it.

[Recapitulation of the Argument for the Axiom as a Norm]
[In Nine Parts]

1. *If your degrees of belief conform to the axiom, you won't succumb to a biased book.*

THEODORUS: The argument for the axiom as norm, gentlemen, must be stated precisely, for we are in the presence of a philosopher. [BOWS CEREMONIOUSLY TO SOCRATES.] The argument has two parts. First we prove that the conformity of a bettor's degrees of belief to the axiom is *sufficient* for his regarding as disadvantageous a set of bets that assures him a loss no matter what happens. Second we prove that conformity is *indispensable* for his regarding such a set of bets as disadvantageous. Are we in agreement on these things to be proved?

THEAETETUS: Yes, teacher. You see, Socrates, we prove two conditionals. The first is, if he conforms to the axiom, he regards all such sets of bets as collectively disadvantageous. That's the same as saying, if he accepts such bets, he doesn't conform to the axiom. The second conditional is, if he doesn't conform to the axiom, there is a set of such bets he doesn't think disadvantageous. And that's the same as saying, if no such set of bets seems OK, that is, seems to be not disadvantageous, he does conform to the axiom. See?

SOCRATES: My word! I'll have to take notes. Speak slowly.

THEODORUS: To the sufficiency lemma, then . . .

SOCRATES: "Lemon"?

THEODORUS: No lemons; "lemma." Each of the two proofs is called a lemma; together they constitute the whole proof. So a lemma is a proof within a proof. Now for the sufficiency lemma. Lamprocles didn't refuse the bets offered. Therefore, by the PROBE principle, factual part, he didn't regard them as disadvantageous. Therefore, by the PROBE principle, definitional part (b), the PROBEs didn't exceed his degrees of belief. But the PROBEs did exceed the maximum probability. Therefore, his degrees of belief did also, in violation of the axiom . . .

THEAETETUS: That's it. You see, Socrates? To prove the sufficiency conditional, we assume its antecedent, and deduce its consequent: "If the bettor accepts such bets, he doesn't conform to the axiom."

THEODORUS: But, by the sad outcome resulting from accepting all the bets, he ought to have refused at least some of the bets. And he would've, if his degrees of belief had conformed to the axiom of probability, for then one or more of the PROBEs would've exceeded his degree of belief, making at least one of the bets appear disadvantageous by the PROBE principle, definitional (a) and factual parts. Therefore. . .

THEAETETUS: There! Teacher proved the sufficiency conditional in its alternate form: "If the bettor conforms to the axiom, he regards all such sets of bets as collectively disadvantageous." Once is enough; twice is nice.

THEODORUS: . . . his degrees of belief ought to have conformed to the axiom of probability. Shark, you continue the proof. Theaetetus, sit down.

2. *If your degrees of belief are excessive, you are open to accepting a biased book.*

SHARK: So far we've proved that if Lamprocles had conformed his degrees of belief to the axiom, he would've found at least one bet disadvantageous. Now we must prove that if he, or anyone, doesn't conform his degrees of belief to the axiom, there will exist a set of bets, no one of which will seem disadvantageous to him, and yet the whole set of bets assures that he will suffer a net loss. Am I on the right track?

THEODORUS: Exactly the right track.

THEAETETUS: That's the indispensability conditional, Socrates. Watch how Shark assumes its antecedent and deduces its consequent.

SHARK: So we suppose Lamprocles, or anyone, to have degrees of belief for mutually exclusive and jointly exhaustive propositions that fail to sum to 1. Therefore, they sum to more, or to less, than 1. In either case, we know a recipe for constructing a set of bets, no one of which will he find disadvantageous.

THEAETETUS: Oh! The second lemma's a dilemma.

THEODORUS: Stop interrupting! Shark, continue.

SHARK: Suppose the first possibility: His degrees of belief in mutually exclusive, jointly exhaustive propositions sum to greater than the maximum of 1. We devise bets in which he bets in favor of each of the propositions so that his PROBE in each of the bets does not exceed his degree of belief in the proposition to be bet on. So he does *not* regard any of the bets as disadvantageous, by the PROBE principle, definitional (b) and factual parts. So he accepts them, (as he may by the principle's normative part).

SOCRATES: Just for small stakes though.

SHARK: For small stakes. Yet, because his degrees of belief sum to more than 1, we can always contrive that his PROBEs in these bets also sum to greater than 1 without this sum of PROBEs exceeding the sum of his degrees of belief. That fact assures that the total amount he risks in all the bets exceeds the stakes in any one bet. For we've made the total stakes the same in each bet, and the parties' PROBEs in each bet must sum to 1. Since he must win exactly one bet, he takes the stakes of only one bet. But his contribution to the stakes of the remaining bets exceeds his gain from the one bet he wins. So he's sure to suffer a net loss on all the bets combined.

SOCRATES: That's what happened to my Lamprocles.

THEODORUS: Good, Shark. Let little Socrates continue.

3. *If your degrees of belief are deficient, are you open to a biased book?*

YOUNG SOCRATES: Shark divided his part of the proof into two cases. He considered the case where a person's degrees of belief in mutually exclusive, jointly exhaustive propositions summed to more than 1. Now I must consider the alternative, where, if his degrees of belief sum to less than 1, there will exist a set of bets, no one of which will he regard as disadvantageous, and yet the set of them assures him a net loss. Is that right?

THEODORUS: Exactly right.

YOUNG SOCRATES: This person's degrees of belief in the propositions sum to less than the maximum of 1. So we consider bets in which he bets against the proposition, betting that it's false. But we can't say what his degrees of belief in the denials of the propositions actually are, since we're presupposing they don't conform to the axiom. Hmmm . . .

THEAETETUS: [WHISPERS TO SOCRATES.] Socrates doesn't know how to deduce the indispensability conditional's consequent from its antecedent.

SOCRATES: What's the problem?

THEAETETUS: In this case the vulnerability of the violator of the axiom follows from our PROBE principle in an indirect way.

SOCRATES: Theodorus has a proof written up for just this case.[7] Can't we make use of it here?

THEODORUS: There we assumed the person did take all the bets and we proved he had violated the axiom; here we're assuming that he violates the axiom, and so we cannot assume he will deem the bets permissible; we must prove he will.

YOUNG SOCRATES: So let's have him evaluate the advantage of, not his own, but his

[7]See pages 17-18.

co-bettor's bets on the truth of the propositions, since we do know something about our subject's degrees of belief in their truth: They sum to less than the maximum. We can always contrive that in each bet, his co-bettor's PROBE exceeds his own degree of belief that the proposition is true.

THEODORUS: Let's say it more formally. If our axiom-violator uses his degrees of probability in mutually exclusive, jointly exhaustive propositions, A_i, to assess the advantage of the other guy's PROBEs, the guy who'll bet on the truth of each, we contrive bets so that

for each A_i, our subject's prob(A_i) < the other guy's PROBE(A_i).

YOUNG SOCRATES: Therefore, by the PROBE principle, definitional and factual parts, he regards his co-bettor as at a disadvantage in every one of the bets. For we can read the principle as defining "disadvantage to a bettor in the opinion of someone": The PROBE of the one betting on truth exceeds the degree of belief in truth of the one making the judgment of comparative disadvantage.

THEAETETUS: That's it! Now go for it!

YOUNG SOCRATES: Yet we can also contrive that his co-bettor's PROBEs in all the bets still sum to less than 1, even though they exceed the sum of our subject's degrees of belief, also less than 1. So the total amount his co-bettor risks in all bets is less than the stakes of a single one of the bets. For we've made the total stakes the same for each bet. His co-bettor wins one bet exactly, gaining on this bet more from our patsy than he loses to our patsy on all the other bets combined. So it's absurd for our patsy to think his co-bettor at a disadvantage.

THEAETETUS: Yes, yes! But where's the blood? Give me blood!!

YOUNG SOCRATES: Here's my problem. I can show that

his own PROBE(NOT-A_i) against each A_i < 1–prob(A_i),

but I *can't* show

1–his prob(A_i) = his prob(NOT-A_i).

We just said he violates the axiom. So I *don't* know that, for him

PROBE(NOT-A) < prob(NOT-A),

which would make his side of the bet seem advantageous. What if for this guy,

1–prob(A_i) > prob(NOT-A_i)

by enough so that he finds both sides of the bets disadvantageous?

THEAETETUS: Hot pursuit! Chase him down the dead-end!

YOUNG SOCRATES: Well, either he does or he doesn't find both sides of the bets disadvantageous. If he doesn't, he likes his side of the book; we've got blood. And . . ., yeah. If he finds both sides of the bets disadvantageous, then it must be that his

prob(A_i)+prob(NOT-A_i) < 1.

So, do we let him give us the slip? No! We give up our original book and go for a simple two-bet book; we offer to bet on A and on not-A. It's not clear this person thinks a bet *against* A is a bet *on* not-A. The principle nevertheless holds true that:

PROBE(on A) = 1 – PROBE(against A), and
PROBE(on NOT-A) = 1 – PROBE(against NOT-A),

independently of his degrees of belief. So we arrange that

his prob(A) < our PROBE(on A), and
his prob(NOT-A) < our PROBE(on NOT-A), and still
our PROBE(on A) + our PROBE(on NOT-A) < 1.

He judges our side of the bets to be disadvantageous, even though the sum of our risks is less than the total stakes. What does he think of his side? Hmmm . . .

THEAETETUS: It's obvious that what one side of a bet loses, the other side wins. The stakes are sitting there to be taken by one or the other bettor. If one bettor is at a disadvantage in taking it, the other bettor must be the beneficiary of his disadvantage, so that empirically if one side is disadvantageous, it must be that the other side is advantageous.

YOUNG SOCRATES: Yeah. The PROBE principle should say that.

THEAETETUS: That observation forces this guy to compute this way: He can't compare his prob(NOT-A) to his PROBE(against-A) in applying the PROBE principle. But rather, since he compared his prob(A) to PROBE(on A), he must compare [1–prob(A)] to PROBE(against A). Look at the inference:

his prob(A) < PROBE(on A); so
his prob(A) < 1– PROBE(against A); so
1–his prob(A) > PROBE(against A).

If the quantity [1–prob(A)] exceeds his PROBE(against A), then he finds the "against" side advantageous, even if the quantity [1–prob(A)] is not a degree of belief in anything. Only thus will he get the empirically verifiable result that a disadvantage on one side is an advantage on the other side. This use of a function of a degree of belief, rather than the degree pure and simple, is an extension of the PROBE principle as we stated it, I admit. But obvious features of bets are forcing the constraint on the workings of the mind, not anything question-begging.

YOUNG SOCRATES: This comparison of PROBEs "against" A to [1–prob(A)] applies even if he decides to assess his side of the book for advantage first. He may see that

his prob(A) < his PROBE(against NOT-A), and
his prob(NOT-A) < his PROBE(against A),

but what can it mean to him, if he doesn't accept the equivalence of "against A" with "on NOT-A"? But he might! Yikes! Is he getting away? No! In this case he does accept

prob(A)+prob(NOT-A) = 1

after all, and we go back to the original case, where we now see he will accept his side of the original book as advantageous. Do I have to prove this? Blood comes with sweat and tears; here goes. Premises: We are supposing we're dealing with a guy who believes

for any x, PROBE(on x) = PROBE(against not-x)

and

for any x, PROBE(on not-x) = PROBE(against x)

By the new part of the PROBE principle, the on-PROBEs are compared straightforwardly to his prob(x) or his prob(not-x), but his against-PROBEs are compared to [1–prob(not-x)] or [1–prob(x)]. And, in order for him not to violate simple logical consistency that a side of a bet cannot be both advantageous and disadvantageous, he must assure that prob(x) and [1–prob(not-x)] are always on the same side of, that is, greater than, less than, or equal to, both PROBE(on x) and the quantity [1 – PROBE(against x)]. Similarly, he must assure that prob(not-x) and [1–prob(x)] are always on the same side of both PROBE(on not-x) and the quantity [1 – PROBE(against not-x)]. No other relation will save him from self-contradiction than

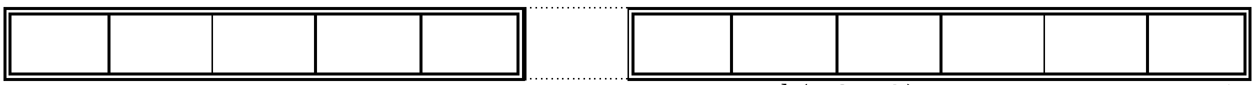
prob(x)+prob(not-x) = 1. Q.E.D.

SOCRATES: Could I have that again, preferably in pictures?

YOUNG SOCRATES: Draw you a picture? Of course. [UNDER HIS BREATH: Y'wannapicha; y'gettapicha.] We make three assumptions about this person:

i. his prob(A) and his prob(NOT-A) sum to less than 1,
ii. he accepts the equivalence of bets against A with bets on NOT-A,
iii. he evaluates bets for and against so that sides are opposite in advantage.

We're going to demonstrate that he is in self-contradiction. Here's the picture:

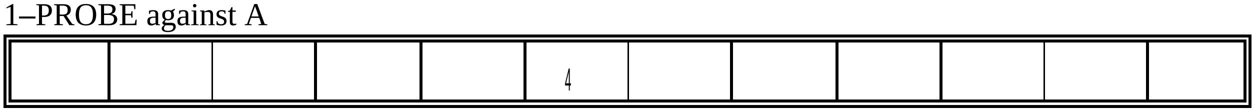


0 prob(A) prob(NOT-A) 1
[. . . . 1–prob(NOT-A) . . . .]
[. . . . . 1–prob(A) . . . .]

Diagram 1 (a).

Suppose his prob(A) is 5/12 and that his prob(NOT-A) is 1/2. Mark them off on the picture, starting from opposite ends. Note the gap in the middle between the two. But 1–prob(A) and 1–prob(NOT-A) overlap this gap. Now set a bet so that the PROBEs divide in that gap. For instance, let the PROBE on A be 11/24. He's committed to the four expressions on the left side of this second picture being the same and the four on the right being the same:

PROBE on A
PROBE against NOT-A
1–PROBE on NOT-A
1–PROBE against A



PROBE against A
PROBE on NOT-A
1–PROBE against NOT-A
1–PROBE on A

Diagram 1 (b).

We see that

prob(A) < PROBE on A.

So the left side is disadvantageous, whereas

1–prob(A) > PROBE against A.

So the right side is advantageous. But also

prob(NOT-A) < PROBE on NOT-A.

So the right side is disadvantageous. Contradiction. It's easy to show that, if we assume ii and iii, then a sum of prob(A) and prob(NOT-A) exceeding 1 also leads to contradiction. So, he accepts this minimal instance of our axiom, that prob(A) + prob(NOT-A) = 1, and we bleed him with our original biased book. If he doesn't accept even this, we bleed him with a biased book of two bets, based on this violation. Any which way, the blood flows. I'm done, Teacher, whew.

THEODORUS: Good, except for the lack of pity. Is the proof complete, or are there any loose ends to be tied up? Socrates?

4. *Does Young Socrates's lemma assume the axiom and beg the question?*

SOCRATES: You've had to strengthen the PROBE principle to complete your proof. Theodorus, do you recall deriving the principle for me from another principle?[8] It seemed to me at the time that, if your PROBE principle depended on that derivation, it could not be used to prove the axiom of probability, since it's derivation assumed the axiom. Now I'm wondering if your strengthening of the PROBE principle has introduced some question begging.

THEAETETUS: No, Sir. We better state the clarifications to the PROBE Principle:

CLARIFICATIONS OF THE PROBE PRINCIPLE

1. The PROBEs that a person evaluates for advantage in relation to his degrees of belief need not be his own PROBEs, nor need he be evaluating his advantage.

2. If one PROBE is assessed as advantageous or disadvantageous, the other side must be assessed oppositely. To that end, it is required that:

3. If comparisons are made to PROBEs "against" X, they must be compared to the quantity [1–prob(X)], even if that quantity does not represent the comparer's degree of belief in NOT-X.

SOCRATES: Notice the 1–prob(X) is compared to the PROBE against X, while the prob(X) is compared to the PROBE on X. Sounds real close to assuming the axiom.

THEAETETUS: Notice that the last conditional clause in 3 explicitly denies exactly what you'd need to get from "real close" to really.

SOCRATES: I agree; the "even if" clause of 3 shows that the axiom's not assumed.

THEAETETUS: The person whom my friend Socrates here was concerned with did identify PROBE on NOT-A with PROBE against A, and he was forced to agree to the axiom in one instance at least. But Soc also considered the person who was not forced.

SOCRATES: I'm satisfied.

THEODORUS: Anything else you want to add, Theaetetus?

THEAETETUS: Let me think. If a person's degrees of belief sum to more than 1, he regards each bet as not to his disadvantage, yet suffers a net loss from all combined, if he does as he may and takes them. Shark proved that. If his degrees of belief sum to less than 1, he regards his co-bettor's bets with him as disadvantageous to his co-bettor, even though the co-bettor has a net gain from them all, which gain is his own loss. Friend Socrates proved that, and also proved we could get him to agree to one or another book of bets. In either case, we've proved that an incongruity exists between the person's judgment of each of the bets and the results from all of them combined, if his degrees of belief violate the axiom. He judges the bad to be not bad; the good to be not good. That's the indispensability lemma. But I see a loose end.

THEODORUS: Well, tie it up.

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[8]See page 13.

5. *Reasoning from members of a collection to the collection as a whole.*

THEAETETUS: The loose end's a premise that Shark and Soc both leave tacit: If a person views each of a set of bets as being to someone's disadvantage, or not to his disadvantage, then he views similarly the set of bets as a whole. For example, Shark must exclude the possibility that a person may regard each bet as not to his disadvantage, but the set of all the bets as disadvantageous. Socrates must exclude the possibility that a person may think each bet to the disadvantage of someone, but the set of all the bets not to his disadvantage.

THEODORUS: I sympathize with the person who'd accept any one of eight bets, but not all together, on the grounds that the combined risk's more than he can afford. We remove that objection by assuring that not only the stakes in any one bet are small, but also that the combined stakes of all eight bets are still small. Here, let's show the premise as a principle:

PRINCIPLE OF ASSESSING COLLECTIVE DISADVANTAGE,
WITH COLLECTIVE STAKES SMALL

FACTUAL: ASSESSING ALL BETS RELATIVELY TO THE SAME INFORMATION,
(a) IF WE THINK NONE OF A SET OF BETS DISADVANTAGEOUS INDIVIDUALLY, THEN WE DON'T THINK THE SET DISADVANTAGEOUS COLLECTIVELY;
(b) IF WE THINK EVERY BET IN A SET DISADVANTAGEOUS INDIVIDUALLY, THEN WE THINK THE SET DISADVANTAGEOUS COLLECTIVELY.

NORMATIVE: PROCEDURES (a) AND (b) ARE PERMISSIBLE.

Can you think of any other objection to these premises?

THEAETETUS: No.

6. *Does the argument commit the fallacy of hasty generalization?*

SOCRATES: Well I can. Or rather, some thoroughly unsympathetic fellow might say to us, "You restrict the conditions under which you can place your books, yet you draw universal conclusions anyway. Do we have here the fallacy of hasty generalization?"

THEODORUS: What!

SOCRATES: I mean, he might say to us defenders of the argument, "If you demonstrate the rationality of conforming to your axiom only when you can use your window opener, or PROBE principle, and you can use that principle only when stakes are small, haven't you failed to demonstrate any universal requirement to conform to your axiom?" What shall we say to him?

THEODORUS: Consternations! Would you have us say this: Just insofar as anyone's degrees of belief happen to be discernible from the bets he'd take, because we're betting small sums, he ought to conform to the axiom. But whenever his degrees of belief happen to be obscured by the way he values large sums of money, it's rational for him to jolly well be a jackass in his degrees of belief! Is that your logic?

SOCRATES: Not mine; we're on the same side, Theo, my old friend. It's that unsympathetic fellow we're both trying to answer.

THEODORUS: None of your tricks! I'm losing my patience with you! "Hasty generalization"! You better beat a hasty retreat!

[EVERYONE TALKS AT ONCE.]

THEODORUS: OK! He's right; we have one more step, justifying an extrapolation. See if you can reason with him.

YOUNG SOCRATES: Suppose we proved that it's not smart of you to jump into a lake over your head with a lot of lead tied around your waist. We'd've had to suppose special conditions to ensure your drowning, like there's no lifequard nearby, and the shore is further than you could get to while holding your breath. Otherwise you wouldn't see the danger of tying lead to your waist. With all due respect, Sir, wouldn't it be silly of you to counter our proof with the claim that we only demonstrated that your jumping in over your head burdened with lead was foolish in situations where you might not've had any way of saving yourself from your foolishness?

SOCRATES: But, if I knew I could be foolish with impunity, there might be some sport in it.

SHARK: That's not the way we make the extrapolation. We can defend our generalizing, Sir. The considerations that force us to use only small sums in our bets are irrelevant to the general truth we wish to establish . . .

YOUNG SOCRATES: That's right. We're trying to establish something about people's degrees of belief. How they happen to value sums of money is irrelevant to their beliefs about outcomes . . .

THEAETETUS: The way they value money interferes with our ability to detect what their degrees of belief happen to be. When we get around that interference, we reveal the one phenomenon with relevance, namely a person's degrees of belief . . .

SHARK: Right, the camouflage is penetrable, and nothing in the procedures for our getting around the interference from a person's values interferes with our generalizing about his degrees of belief, even when we bookies aren't in a position to tell what they are. They must conform to the axiom.

SOCRATES: A person may have a policy, concerning any set of mutually exclusive and jointly exhaustive propositions, never to have more than one bet on any proposition in the set in force at any one time, and never to have bets in force against all of them in force at any one time. That policy immunizes anyone who violates your axiom from your type of punishment.

SHARK: Such a person is choosing an action, namely refraining from a bet, while by his own lights the bet would be more advantageous to him than refraining. His policy is to prefer that act which he believes to be less advantageous; that in itself is a form of irrationality. We should've added a clause to that effect to our PROBE principle: We ought to accept bets when they are more advantageous than any alternative to the betting. Don't you agree?

SOCRATES: Hmmm . . .

SHARK: Why would he adopt such an irrational policy when he could see from its *ad hoc* nature that its only appeal is its ability to protect him from the consequences of his other irrationality, namely, violating the axiom of probability? One irrationality begets another.

YOUNG SOCRATES: Shark, what you recommend is plausible. But let's not strenghen the PROBE principle any more. Let me try a different tack. Socrates, if all we were interested in was how a person could avoid succumbing to our book of bets, then, yes, there are several ways to avoid them: Don't make bets, value money oddly, rob all bookies rather than place bets with them, lots of ways, none relevant to our interest. We establish one source of vulnerability, namely, having degrees of belief that don't conform to the axiom; we establish a way to remove that vulnerability: Conform to the axiom.

THEODORUS: You see, Socrates, the situation's similar to a scientist's generalizing to a universal law from the peculiar conditions of the experiment that confirms the law. Suppose a scientist confirms that the difference between the weight of a body in air and its lesser weight while submerged in a liquid is equal to the weight of the liquid it displaces. The scientist confirms this by using a stone and some water. He weighs the stone in air and again when it's submerged. He weighs the water that the stone displaced. How foolish it would be to insist that the scientist has only discovered a law that holds for the water in his experiment, or only a law for fresh water, not sea water, or only for water, not for any other liquid, or only for granite and not for basalt, or only . . .

SOCRATES: I get the point. The restrictions on the conditions under which you have a window into a person's mind do not invalidate your generalizing from what's rational for that mind in those conditions to what's rational for all minds in any conditions.

THEODORUS: Yes. Put the generalization this way: We've shown that anyone who violates the axiom of probability thereby becomes more liable to exploitation than he would've been if he hadn't violated it, and so becomes more in need of auxiliary protections to prevent others from exploiting him than he should've needed.

YOUNG SOCRATES: That's just what I said to begin with.

7. *Definition of a sense of coherence of beliefs.*

THEAETETUS: Our axiom is sufficient and indispensable for a strict standard of right betting, but I'd like to highlight what we've proved by introducing a new concept. Anyone's degrees of belief are "coherent," let's say, if and only if they lead him to view a set of bets on those beliefs as disadvantageous when his PROBEs would ensure a net loss.

THEODORUS: Ah, yes. This is better than such a blanket term as "rationality." Coherence is one part of rationality. We have a norm of coherent belief, don't we?

THEAETETUS: We've proven, then, that a person's degrees of belief are "coherent," if and only if they conform to my axiom.

THEODORUS: So, gentlemen, our proof is complete. Q.E.D., and bottoms up.

ALL: Q.E.D., and bottoms up!

8. *The concepts of indifference and the fair bet are not used in the argument, and are now introduced as corollary norms.*

THEODORUS: Our axiom of coherence has a corollary norm. It's this: There must be a PROBE in any bet at which the bettor *ought to* see neither advantage nor disadvantage, either to himself or to the other side. In other words, you can find a PROBE in any bet for a coherent person to bet at, that will *equal* his degree of conviction that he'll win.

SHARK: And we may get him to tell us what it is. Make him set the odds in a bet before he knows which side of the bet we're willing to take. If he thinks he may end up on either side of the bet, he'll set the odds so that he doesn't sense disadvantage for himself on either side.

SOCRATES: A bit coercive or deceptive, wouldn't you say? I doubt whether these conditions would allow you to generalize your results to cases that involve no coercion or deception.

YOUNG SOCRATES: Who'd bother betting anyway, if he didn't sense advantage?

THEAETETUS: Teacher's right, though. There must be such a point at which a coherent person's degree of conviction equals his PROBE. A person ought to feel *indifference* to the bet at that point. He might not take the bet, but he'd have to agree that the bet would be *fair* to both the parties to the bet at those PROBEs.

SHARK: And he wouldn't be irrational to bet at odds he deemed not advantageous, just fair.

THEODORUS: You two have just stated the Norm for Indifference.

COROLLARY NORM FOR INDIFFERENCE

THERE EXISTS A PROBE FOR ANY BET AT WHICH prob(A) = PROBE(A)
A PERSON OUGHT TO FEEL INDIFFERENCE BETWEEN TAKING AND NOT TAKING A BET WITH A PROBE AT THAT POINT

SHARK: Still, someone who bets because he deems it advantageous differs from someone who bets since he deems it merely fair. We can tell if a bettor is of the first or second type without being coercive or deceptive. We entice with an ever-so-small bonus to the side of the bet he does not take. Anyone who took his side only because he considered it fair will now switch sides, no matter how small the bonus may be. That's not the case for the one who bets from a sense of advantage to his side. A prize of *any* degree of smallness should provide an indifferent person with an incentive to change sides.

SOCRATES: A rough indicator of indifference at best, since you can't offer prizes of any degree of smallness. Face up to the fact that indifference is a dimensionless point, not easy to locate and distinguish from its neighbors.

YOUNG SOCRATES: We're not committed to there being any behavioral tests for a person's sense of indifference or fairness, although Shark may find one. If he fails, then there's no window-opener for indifference. If a person just tells us he's indifferent, fine, and even better if we have behavioral tests to corroborate his words. Are you going to suspect us of question-begging again, if we talk of indifference points, Socrates Sir? Because, if you are, I just want to say, we never used these notions in our arguments. They come in now, after the arguments, in corollaries.

9. *Why be rigorous in stating all our argument's assumptions?*

SOCRATES: What patient attention to logic, and at your ages! Youngsters typically shirk careful reasoning, I've found. Even the smart ones do, who enjoy ideas, preferring instead the big, breathtaking insights, provided they're quick and easy. And they so often are, especially when they're false! Do you think many of your age-mates would've stayed to listen to your proof? Even considering only the keenest ones; would they?

THEAETETUS: Are you kidding?

SHARK: No way.

YOUNG SOCRATES: Forget it!

THEODORUS: A few would, but by now they'd be strongly tempted to wander off, as if some law of diminishing returns applied to arguments.

SOCRATES: You are the precocious ones. The rest of us feel our mistakes knock the wind out of us before we appreciate the step-by-step plodding of good argument. It makes our falling into error less frequent or less long-lasting, and it alerts us to the less obvious truths.

THEAETETUS: Oh, we do it for fun. It's a challenge to argue convincingly. It's, well, it's winning; it's a triumph.

YOUNG SOCRATES: Well, that's you, Theaetetus. For me a finished argument is a thing so beautiful I fall in love with it. I say it over and over 'til I have it by heart. Poetry doesn't make my heart burst the way a beautiful proof does.

SHARK: Well, that's you!

SOCRATES: And what of you?

SHARK: Liquid is more constant than vapor, and solid more constant than liquid. In this unbearably crazy, topsy-turvy world, the only thing like a rock, firm enough to hold me, making sanity preferable to insanity, is logic. It's my bitter comfort. With it I condense vaporous confusion into liquid, and then I chill what's still mushy in it to ice.

SOCRATES: Brrr. My utilitarian view must be for old men, hey Theodorus?

THEODORUS: The practical benefits come regardless of impractical motives. What do you think of our argument?

SOCRATES: Your explicit premises, particularly the definition in your window-opener, helped me appreciate the proof's soundness and non-circularity.

THEODORUS: Bartender! Another round!

THEAETETUS: I've made some notes for you, Socrates.

SOCRATES: Very thoughtful of you. I'll look them over right now.

1st Appendix to Dialogue I: *Fifteen theorems derived.*

Dear Socrates,
I prepared this crib sheet just for you, to demonstrate the sufficiency of my axiom.

Respectfully yours,
Theaetetus

KEY TO SYMBOLISM:

Understand A, A_i, B, etc. to be place holders for propositions.
After step 68, integers, i, supplant indexed letters, A_i.
Read "p(A)" as "the probability of A." It stands for a number, expressing a particular person's degree of conviction at a particular time in A's truth.
Read "max" as "the maximum probability." No probability exceeds this number.
"T" is the symbol for the necessary proposition, expressed by all tautologies.
"F" is the symbol for the impossible proposition, expressed by all denials of tautologies. F can be in a set of mutually exclusive, jointly exhaustive propositions. See steps 17 and 54.

PARTIAL DEFINITIONS:

"NotA" means: it is not the case that A.
"A or B" means: A and/or B.

If A implies B, then the pair: A, not B, are mutually exclusive.

If A is equivalent to B, then A implies B and B implies A.

AXIOM:

If A_1, A_2, . . . A_n are jointly exhaustive of possibilities and mutually exclusive of each other, then $S_{i=1}^{n} p(A_i) = max$.

CONVENTION:

max = 1.

NOTE ON PROOFS OF THEOREMS: Individual steps are either
(1) instances of the axiom or theorems already proved,
(2) tautologies or truths of mathematics,
(3) hypotheses of dilemma, conditionalizing, or reductio ad absurdum arguments, initiating the vertical lines to the left of the step numbers,
(4) conclusions from arguments, terminating the vertical lines to the left of step numbers,
(5) steps of algebra, which are equivalences
(6) steps of logic, which are not equivalences
(7) consequences of definitions.

DEDUCTION OF THEOREMS	JUSTIFICATION OF STEPS
1. A, notA are mutually exclusive and jointly exhaustive	from logic
2. p(A)+p(notA) = max	1 allows instance of the axiom schema
Theorem I: 3. p(notA)=max-p(A)	algebra
4. | Suppose p(A)<O	hypothesis of reductio ad absurdum
5. | Either max$0 or max<0	logic
6. | |Suppose first case: max$0	hypothesis as premise for dilemma
7. | |max-p(A)>max	from steps 4 and 6 (remember, subtracting a negative number is same as adding a positive number)
8. | |p(notA)>max, which is self-contradictory	from 7 and Theorem I
9. | |Suppose second case: max<0	2nd hypothesis of dilemma
10. | |either p(A)>max (which is self- contradictory) or p(A)#max	tautology
11. | |p(A)#max	from 10 by logic
12. | |max-p(A)>max	from 4, 9, 11 (compare step 7)
13. | |p(notA)>max (which is self- contradictory	12, Theorem I
14. | hypothesis, p(A)<0, implies self-contradiction	5, 6-8, 9-13
Theorem II: 15. p(A)$0	from 14 by reductio
16. | Suppose all A_i's, 1<i#n, are mutually exclusive	hypothesis of conditionalizing
17. | the A_i's, (not any of the A_i's), which is the prop denying each, are mutually exclusive and *jointly exhaustive*	from 16, by logic
18. | S(p(A_i))+p(not any of A_i)=max	17 allows instance of axiom schema
19. | p(not any of A_i's) = max-p(one or other A_i)	instance of Theorem I
20. | S(p(A_i))+max-p(one or other A_i)=max.	19, substitution into 18
Theorem III: 21. S(p(A_i))=p(one or other A_i) if all A_i's are mutually exclusive	algebra, conditionalizing
Theorem IV: 22. If A, B are mutually exclusive, then p(A)+p(B)=p(either A or B)	from 21 for 2 A_i's

	23. | Suppose A implies B	hypothesis of conditionalizing
	24. | A, notB are mutually exclusive	partial definition of implication
	25. | p(A)+p(notB)=p(either A or notB)	instance of Theorem IV
	26. | p(notB)=max-p(B)	instance of Theorem I
	27. | p(A)+max-p(B)=p(either A or notB)	26 substitution in 25
	28. | p(A)-p(B)=p(either A or notB)-max	27 algebra
	29. | either p(either A or notB)=max, or p(either A or notB)<max	by logic
	30. ||Suppose 1st case: || p(either A or notB)=max	1st hypothesis of dilemma
	31. || p(A)-p(B)=0	28, 30 substitution & algebra
	32. || p(A)=p(B)	31 algebra
	33. || p(A)#p(B)	32
	34. || Suppose 2nd case: ||p(either A or notB)<max	2nd hypothesis of dilemma
	35. || p(A)-p(B)<0	28, 34 substitution & algebra
	36. || p(A)<p(B)	35 algebra
	37. || p(A)#p(B)	36
Theorem V:	38. If A implies B, p(A)#p(B)	23-37by dilemma & conditionalizing
	39. | Suppose A is equivalent to B	hypothesis of conditionalizing
	40. | A implies B	39 by definition
	41. | p(A)#p(B)	41, Theorem V
	42. | B implies A	39 by definition
	43. | p(A)$p(B)	42, Theorem V
	44. | p(A)=p(B)	41 & 43
Theorem VI:	45. If A is equivalent to B, then p(A)=p(B)	39-44 by conditionalizing
	46. A, not A are mutually exclusive	by logic
	47. p(A)+p(notA)=p(either A or notA)	instance of Theorem IV
	48. A, not A are jointly exhaustive	by logic
	49. p(A)+p(not A)=max	46, 48 allow instance of axiom
	50. p(either A or notA)=max	47, 49 substitution
	51. all tautologies are equivalent to (either A or notA)	by logic
	52. T is equivalent to (either A or notA)	51 & by definition of T
Theorem VII:	53. p(T)=max	50, 52 substitution

	54. T, F are mutually exclusive and jointly exhaustive	logic
	55. p(T)+p(F)=max	54 allows instance of axiom
	56. max+p(F)=max	53 substitution in 55
Theorem VIII:	57. p(F)=0	56 algebra
	58. (either A or B) is equivalent to (either A or both notA and B)	logic
	59. p(either A or B) = p(either A or both notA and B)	58 by Theorem VI
	60. A, (both notA and B) are mutually exclusive	logic
	61. p(A)+p(both notA and B)= p(either A or both notA and B)	60, Theorem IV
	62. B is equivalent to (either both A and B or both notA and B)	logic
	63. p(B)=p(either both A and B or both notA and B)	62, Theorem VI
	64. (both A and B), (both notA and B) are mutually exclusive	by logic
	65. p(both A and B)+p(both notA and B) =p(either both A and B or both notA and B)	64, Theorem IV
	66. p(B)=p(both A and B)+p(both notA and B)	65 substitution 63
	67. p(B)–p(both A and B)= p(both notA and B)	66 algebra
Theorem IX :	68. p(A)+p(B)–p(both A and B)= p(either A or B)	60, 62, 67 substitutions
	69. p(not1)+p(not2)–p(both not1 and not2)=p(either not1 or not2)	substitution in 68 "not1" for A "not2" for B (see key to symbolism)
	70. either p(both not1 and not2)=0 or p(both not1 or not2)>0	from Theorem II
	71. |if p(both not1 and not2)=0, |then p(not1)+p(not2)=p(either |not1 or not2)	1st hypothesis of dilemma & 69 by math
	72. |if p(both not1 and not2)>0, then |p(not1)+p(not2)>p(either not1 |or not2)	2nd hypothesis of dilemma & 69 by math
Theorem X:	73. p(not1)+p(not2)$p(either not1 or not2)	70-72

74. if 1 implies C, then p(1)#p(C)	substitution in Theorem V
75. if 1 implies C, then –p(1)$–p(C)	multiplying by –1 changes # to $
76. if 1 implies C, max–p(1)$max–p(C)	algebra
77. if 1 implies C, p(not1)$p(notC)	76 substitution by Theorem I
78. if (1,and2) implies C, then p(not(1,and2))$p(notC)	77 substitution of one compound sentence for one simple sentence
79. if 1 and 2 imply (note plural) C, then p(not(1,and2))$p(notC)	intuition of equivalence *of 78* (where one compound premise implies C) *with 79* where two premises imply C.
80. if 1 and 2 imply C, then p(not1)+p(not2)$p(notC)	79, Theorem X by transitivity of $; and Theorem VI, for (either not1 or not2) is equivalent to not(1,and2)

Five rules for generalization of 80 for 1, 2, ..., i,... for any integer, i.
All rules preserve validity.

Rule I: Replace 1 with (1,and2), 2 with 3, and in general every integer by the next integer, i with i+1, e.g.,

81. if (1and2) and 3 imply C, then p(not(1,and2))+p(not3)$p(notc)	from 80 by Rule I

Rule II: Decompose the antecedent of the condition replacing the single sentence (1,and2) with two separate sentences 1, 2 (note no parentheses).

82. if 1, 2, and 3 imply C, then p(not(1,and2))+p(not3)$p(notC)	from 81, intuition of equivalence of 81 and 82

Rule III: Subtract all the terms on the left, except the first one from both sides of the inequality.

83. if 1, 2, and 3 imply C, then
p(not(1,and2))$p(notC)-p(not3)

Rule IV: Use theorem IX and transitivity of $ to replace left hand term, p(not(1,and2)), by p(not1)+p(not2)

84. if 1, 2, and 3 imply C, then
p(not1)+p(not2)$p(notC)-p(not3)

Rule V: All the terms that were subtracted in accordance with Rule III are now to be added back.

85. if 1, 2, and 3 imply C, then
p(not1)+p(not2)+p(not3)$p(notC)

Rules I-V can now be reused on step 85 to produce a theorem about four premises implying C.

86. if (1,and2), 3, and 4 imply C,
then p(not(1,and2))+p(not3)+
p(not4)$ p(not C) from 85 by Rule I
87. if 1, 2, 3, and 4 imply C, then
p(not(1,and2))+p(not3)+p(not4)
$p(notC) from 86 by Rule II
88. if 1, 2, 3, and 4 imply C, then
p(not(1,and2))$p(notC)–p(not3)
–p(not4) from 87 by Rule III
89. if 1, 2, 3, and 4 imply C, then
p(not1)+(not2)$p(notC')–
p(not3)—p(not4) from 88 by rule IV
90. if 1, 2, 3, an 4 imply C, then
p(not1)+p(not2)+p(not3)+p(not4)
$p(not C) from 89 by Rule V

Rules I-V can now be applied to 90 to produce a theorem about five premises implying C. More generally, if you have a theorem about i premises implying C, you can produce from it another theorem about i+1 premises implying C, by applying Rules I-V, along with Theorem IX.

Theorem XI: For any N, if 1, 2, ..., N imply C, then p(not1)+p(not2)+...+p(notN) $ p(notC)

Steps 77, 80, 85, and 90 are instances of the Theorem X.

Theorem XII: Shark's theorem, proved in the text of another dialogue.
If the probability of the disjunction of A_1, A_2, . . . A_n is 1, & the probability of the conjunction of any two of them is 0,
then $S_{i = 1}^{n} p(A_i) = 1$.

91. |Suppose p(A)=0
92. |p(A)=p(A&B)+p(A&NOT-B) consequence of step 63
93. | 0=p(A&B)+p(A&NOT-B) from 91 and 92
94. |Either p(A&B)>0, or not.
95. |If p(A&B)>0, then p(A&NOT-B)<0 from 93

96. |Consequent of 95 contradicts Theorem II	
97. |p(A&B)=0	from 91, 95 and 96
Theorem XIII: 98 If p(A)=0, then for any B, p(A&B)=0.	conditionalizing from 91 to 97.
99. |Suppose p(A)=1	
100. |p(NOT-A&B)=0	from 98 and theorem XIII
101. |p(B)=p(A&B)	from 99 and 92: A,B switched
Theorem XIV: 102 If p(A)=1, then for any B, p(A&B)=p(B)	conditionalizing from 98 to 100
Theorem XV: 103 If p(A)=0, then for any B, p(AorB)=p(B)	from theorem IV

Suppose the probability of (B&C) is 0. Does that mean B and C are mutually exclusive, the way B and NOT-B are? No. There's a distinction to be drawn between the two. Strong mutual exclusion, like that between B and NOT-B, derives from their meaning. Just to understand them is to know their conjunction's probability is 0. But weak mutual exclusion can exist between two propositions when we're sure for any reason their conjunction's probability is 0. Some propositions may be weakly, but not strongly, mutually exclusive. For example, if someone says of some animal, "an elephant is its father" and "a mouse is its mother," I wouldn't give you the least credence; they're weakly mutually exclusive.

We stated the third and fourth theorems in terms of strong mutual exclusion. Yet they seem to hold true even for weakly exclusive propositions. We can prove they do. We can prove a counterpart to the axiom for the weakly exclusive.

There's a similar distinction between joint exhaustion of all possibility and joint exhaustion of all probability. It turns out we can prove that the probabilities of weakly mutually exclusive, weakly jointly exhaustive propositions sum to the maximum probability.

> If A_1, A_2, . . . A_n are jointly exhaustive of probabilities
> and at least weakly mutually exclusive of each other, then $S_{i=1}^{n}$ p(A_i) = max.

That's the axiom's counterpart. The axiom implies that its counterpart holds true for a set of weakly exclusive and exhaustive propositions.

First consider the matter of exhaustion. Consider a set of strongly mutually exclusive and jointly exhaustive propositions, and delete any propositions with 0 probability from the set. If there are any, (the only case of interest), the reduced set's no longer exhaustive of all possibility, but it remains exhaustive of all probability, and that suffices for the truth of the axiom's counterpart. For nothing's been deleted that can have any effect on the sum of the probabilities of the propositions in the set. They still sum to the maximum.

Now consider the matter of mutual exclusion. Take two of the strongly mutually exclusive propositions remaining in the set and disjoin each to one of the 0-probability propositions that were deleted from the set. Let these two disjunctions replace their strongly mutually exclusive disjuncts in the set. The set is now weakly mutually exclusive, for the conjunction of these disjunctions is possible, but its probability is 0. I can demonstrate that. Take a pair of strongly mutually exclusive propositions: A and NOT A. Disjoin B to each: (A OR B) AND (NOT-A OR B). That conjunction's equivalent to B, as calculating the areas shows. So its disjuncts are not strongly mutually exclusive. If the probability of B's 0, however, so is this conjunction's. So its two disjuncts are weakly mutually exclusive.

But the probabilities of the two weakly exclusive disjunctions are the same as the probabilities of their two strongly exclusive disjuncts. So, again, no change was made that could affect the sum of the probabilities. Therefore, if the axiom holds true for strongly mutually exclusive, jointly exhaustive propositions, its counterpart must hold true for weakly mutually exclusive, jointly exhaustive propositions also.

Consider that wilted geranium flower you've tucked in your ear. I can't see it clearly; either it has 0 or 1 or 2 or more than 2 petals. Let that be our set of four strongly exclusive and exhaustive propositions. Their probabilities sum to 1. A goddess tells me it has at least 1 petal, so the probability of 0 petals is 0. I delete that possibility from the set, creating a weakly exhaustive set of three propositions. The axiom's counterpart holds true of the set, for deleting a summand of 0 cannot affect the sum. Now replace "it has 1 petal" with the disjunction "it has 1 petal or none" and replace "it has 2 petals" with "it has two petals or none," and "it has more than 2 petals" with "it has more than 2 or none." The resulting set of three propositions is only weakly exclusive as well as weakly exhaustive, for "it has 1 petal or none" is compossible with "it has 2 petals or none," although the conjunction of the two disjunctions has a probability of 0, since it's equivalent to "it has none." The probabilities of these disjunctions are the same as those of their disjuncts whose place they take in the set.

Therefore the axiom's counterpart holds true of the set, for our replacing summands with identical summands cannot affect their sum.

Can we prove the axiom about strongly mutually exclusive, strongly jointly exhaustive sets from the axiom's counterpart about weakly mutually exclusive, weakly jointly exhaustive sets? Yes, for despite the contrastive language, strong versus weak, the axiom's initial proviso is an instance of the more general proviso of the theorem. Joint exhaustion of all possibilities is one way of having joint exhaustion of all probabilities, and strong mutual exclusion is one way of having weak mutual exclusion. Thus, we can derive the axiom from the theorem.

Why then, choose one as axiom rather than the other? The truth of the one chosen as axiom is more obvious.

2nd Appendix to Dialogue I: *Probability through Venn Diagrams*

Dear Professor Socrates the Elder,

Theaetetus's boring proofs are putting me to sleep. Here's probability in eight steps with pictures. I hope I finish these notes before I drop off, because you'll need the techniques before the night is out.

1. How to represent propositions spatially with diagrams of two overlapping circles.
2. Three rules for telling from the spaces in the diagram which proposition they represent, and telling from the proposition which spaces represent it.
3. How to keep track of the structure of a very complex expression of a proposition.
4. How to prove the equivalence of two equivalent expressions of a proposition.
5. How to calculate the probability of a proposition from a diagram.
6. Expanding all the above to three expressions and three-circle diagrams.
7. How we can at least prove that an expression is not a theorem with our diagrams.
8. How, from 4 above, we can tell when two propositions have the same probability even when we don't know the probability of either of them.

Sincerely,
Socrates the Younger (I almost wrote the Yawner)

1. *How to represent propositions spatially*, so that you'll be able to calculate equivalences between the different expressions of them, and also calculate the relations of their probabilities. First I find a square and call it the "field." The table top will do:

Diagram 2: The spatial field.

All our sentences, expressions of propositions, compound as well as simple, can be reinterpreted as signifying regions of this field rather than propositions, all, that is, except the expressions for the impossible proposition. They signify no region at all, or the non-region, as I will say.

Any part of the field is a region. See the rings left by our glasses? Pretend they're boundaries of regions. Make more rings with your glass. One can include another, and any two can overlap. Let your visual imagination draw boundaries every which way you please. Each possible region could have an expression signifying it.

If you imagine regions any which way, there'll surely be more regions than you have expressions to signify them with, especially if single points are regions and any set of points is too. A region can consist of a single point, but we will always work with extended regions so we can see them. The regions of our field are inexhaustible. Through our ideas we gain only limited access to them. For example, when the idea or sentence, A, awakens in me, it gives me access to two mutually exclusive regions, the region signified by A, and all the rest of the field, signified by NOT-A, in addition to the field itself and the non-region:

Diagram 3: One circle defines four regions, 0, 1, 2, and 1-2. (No space is 0.)

The area within the circle is called A; the rest, NOT-A.

2. *Rules for telling which regions are in the signification of compound expressions*. Three suffice; they're for negations, disjunctions, and conjunctions.

(i) The regions in a negation's signification are all those not in the signification of the expression it negates. Since region 2 is in the signification of A, region 1 is in the signification of its negation, NOT-A.

(ii) A disjunction's signification is all the regions in the signification of *any* one or other of its disjuncts. So (A OR NOT-A) signifies the field, 1-2, since one disjunct, A, signifies region 2, and the other, NOT-A, signifies 1.

(iii) A conjunction's signification is whatever regions are signified by *every* one of its conjuncts. So (A&NOT-A) signifies no space at all, since no region is signified by all its conjuncts. We can say it signifies 0.

If you have a second idea, compatible with the first and so overlapping it, you gain access to four mutually exclusive and jointly exhaustive regions and to 15 regions over all, and to 0, no space at all.

Diagram 4: The two-circled diagram.

The letter A stands for the left circle, which now is subdivided into two regions, 3 and 4. So A signifies 3-4. The letter B signifies the right circle, also subdivided into two regions, 2 and 4. So B signifies 2-4, without 3.

Introducing the idea, B, hasn't lost me access to A. But it has shown me that A may be true in two ways, namely, with B, region 4, or without B, region 3. (Regions are spaces, not propositions, and aren't true or false. But a region's having two parts is analogous to a proposition's having two ways of being true.)

We use the three rules to find expressions for all the fifteen regions that two ideas give access to. NOT-A still signifies all the region outside A, but it's now subdivided into two regions, 1 and 2. Since A signifies 3-4, NOT-A signifies 1-2. NOT-B signifies 1-3, since B is 2-4. The numbered sub-regions are signified by conjunctions. Since a conjunction signifies a region which every one of its conjuncts signifies, (A&B) signifies region 4, for A signifies 3-4 and B 2-4. (A&NOT-B) signifies region 3, because A signifies 3-4 and NOT-B 1-3. (Understand "1-3" as the union of regions 1 and 3; 2 is not included.) (NOT-A,&B) signifies 2, and (NOT-A,&NOT-B) signifies 1. These four regions are the most specific our two ideas give us access to. Were they propositions instead of regions, we'd say they come true in one way only.

We've named 8 regions so far. Disjunctions signify the others. Since a disjunction signifies all the regions which are in the signification of at least one of its disjuncts, (A OR B) signifies 2-3-4; (A OR NOT-B) signifies 1-3-4; (NOT-A OR B) signifies 1-2-4; (NOT-A OR NOT-B) signifies 1-2-3; the field, 1-2-3-4 is signified as before by (A OR NOT-A) or by (B OR NOT-B). What's left? "Combination" counting tells us. There's one way to pick no region, four ways to pick one, six ways to pick two, four ways to pick three, and one way to pick four, when we pick from four mutually exclusive and jointly exhaustive regions. We've only listed four of the six ways of picking two of the four regions. What signifies 1-4, and what signifies 2-3?

3. Disjunctions of conjunctions will do it. *Here's how to keep track of the order in which the parts of the compound expression were put together.* Assess the formula: (A&B)OR(NOT-A,&NOT-B). It was assembled from A and B ultimately. And A signifies 3-4; B signifies 2-4. The significations of all other expressions are computable from these. It might help to put the numbers in place of the letters and, following the order in which the expression was constructed from its components, replace the intermediate expressions with the numbers they signify thus:

(A&B)	OR(NOT-A,&NOT-B)
(3-4&2-4)	OR(NOT3-4,&NOT2-4)
(3-4&2-4)	OR(1-2&1-3)
(4)	OR(1)
4-1	

First you put the significations of A and B in place of them. You applied the rule for negations next, replacing them with their significations. Next you applied the rule for conjunctions, replacing them by their significations. Finally you replaced the disjunction by its signification.

(A&NOT-B)OR(NOT-A,&B) will signify 2-3.

(A OR B)&(NOT-A OR NOT-B) also signifies 2-3:
(3-4 OR 2-4)&(NOT3-4 OR NOT2-4)
(3-4 OR 2-4)&(1-2 OR 1-3)
(2-3-4)&(1-2-3)
2-3

4. You can use the diagrams to *prove the equivalence of expressions*. They're equivalent if they signify the same regions. For example, the two just mentioned each signify 2-3. Can you prove A equivalent to (A&B)OR(A&NOT-B)?

A's signification is 3-4. So I'll show (A&B)OR(A&NOT-B) has the same:
(3-4&2-4)OR(3-4&NOT2-4)
(3-4&2-4)OR(3-4&1-3)
(4)OR(3)
3-4

Prove (A OR NOT-A) is equivalent to (B OR NOT-B).

(3-4 OR NOT3-4)	(2-4 OR NOT2-4)
(3-4 OR 1-2)	(2-4 OR 1-3)
(1-2-3-4)	(1-2-3-4)

Prove NOT(BOTH A&NOT-A) is equivalent to them.
NOT(BOTH 3-4&NOT3-4)
NOT(BOTH 3-4&1-2)
NOT(0)
1-2-3-4

Prove A equivalent to A&(A OR B).

3-4	3-4&(3-4 OR 2-4)
	3-4&(2-3-4)
	3-4

Prove NOT-A&NOT-B equivalent to NOT EITHER A OR B.

5. How to *compute the probabilities of propositions* using your diagrams. We think of the probabilities connected to expressions as giving the area of the field, measured as a percent of the whole field. So if a region has a probability of .5, that means its area is one half the field. Except that I don't draw circles to scale when I give their probabilities. You'll have to imagine circles with probabilities close to 1 as taking up most of the field, and those with low probabilities as taking up little.

The circles in our diagrams correspond to our ideas; our four mutually exclusive and jointly exhaustive regions are analogous to a generating set of propositions, those that come true in *one* way only. So we cannot construct their areas from more basic areas. We start by supposing theirs given, although if only three of the four areas were given, we'd know the fourth, since the four must total 1.

Diagram 5: Two-circle diagram, with relative areas (=probabilities) of the four mutually exclusive, jointly exhaustive regions. (Note the decimal points.)

How did I arrive at those numbers? Pretend, Sir. Each represents *the area* of the region it's in as a portion of the whole field. I chose the numbers for the mutually exclusive, jointly exhaustive regions arbitrarily, subject to only two requirements: All numbers are non-negative, and they sum to the maximum probability, 1.

The probability of a region can be 0. But a 0 probability in a region doesn't mean it's a self-contradictory thing, like the non-region we've designated 0. It's a region with an area equal to 0. A point. A point and a non-region have both no area, but one exists and the other doesn't! We know the impossible proposition to be false a priori; other propositions that we are certain are false a posteriori have a probability of 0 too. We treat them as regions with points in them but no area. The situation is similar with the necessary proposition; it is like the total field. Other propositions we are certain of also have areas equal to the total field, but we may think of these regions as missing some of the points in the total field. That won't make their area any less! Now, if all this seems to border on self-contradiction, think of what it would sound like if I said the area of a point were infinitesimally small rather than zero!

Once the mutually exclusive and jointly exhaustive regions are assigned their probabilities, all the regions we have access to have theirs. For example, what's the probability of (A OR B) using diagram 5? (A OR B) signifies 2-3-4. I sum the probabilities of the constituent regions, for they are mutually exclusive: .2+.3+.4=.9. So prob(A OR B)=.9.

What's the probability of (A&NOT-A)? First I calculate its region:

3-4&NOT3-4
3-4&1-2
0

Second, I calculate the size of the region. The size of no space at all is 0. So prob(A&NOT-A)=0.

What's the probability of NOT(BOTH A AND NOT(EITHER B OR A))? First, calculate the region:

NOT(BOTH 3-4 AND NOT(EITHER 2-4 OR 3-4))
NOT(BOTH 3-4 AND NOT(2-3-4))
NOT(BOTH 3-4 AND 1)
NOT(0)
1-2-3-4

Second, calculate the size of the region. The region is the field. So prob(NOT(BOTH A AND NOT(EITHER B OR A)))=.1+.2+.3+.4=1.

You see the utility of having the probabilities of the four mutually exclusive regions given; all other probabilities follow. In contrast, if I gave you the probabilities of A and of B, you couldn't compute the probabilities of all other regions. You'd infer the probabilities of NOT-A and of NOT-B. The probabilities of the necessary and impossible propositions you already know. Just six regions. But the other ten you don't know. To be given the probabilities of A and of B is to be given very incomplete information about the field.

But the probabilities of the four mutually exclusive, jointly exhaustive regions would've been complete information. Let A be .5, and B be .5 too. If I added to the information by telling you that the probability of (A OR B) is .8, you could calculate more. Area 1 must be .2. If A, area 3-4, is .5, but area 2-3-4 is .8, then area 2 must be .3. Reversing the reasoning tells me area 3 is .3 also. So area 4 is .2, if all sum to 1. So your information is now complete.

Diagram 6: Given prob(A)=.5; prob(B)=.5; prob(A OR B)=.8. The area of the region, A OR NOT B, is .7.

You may wonder why propositions that we are certain are true or are certain are false, but which are not the necessary or impossible propositions, are assigned regions different from those two propositions. Imagine the actual world is a point in the total field, but we don't know exactly where in the field to locate it. Only an omniscient Apollo knows that. But since we think of our degrees of conviction as well calibrated with the world, we believe a proposition just to the extent it has a chance of being true. Then, when we assign it a region, we are suggesting that the chance of the actual-world point lurking in that region is equal to the area of the region. If the point is in the region, the proposition is true; if it's outside, the proposition is false.

Now when we say a proposition other than the impossible one has a region containing points but no area, we are able to say that the chance of any of *those* points being the actual-world point is 0. But it's still possible! And when we say a proposition other than the necessary one has a region of area 1, and yet lacks some points in the field, we are saying that the chance that any of those missing points is the actual-world point is 0. But it's still possible! For example, in the spatial field of a diagram imagine that dimensionless points are regions of the field. Each of them has a name consisting of two numbers. The first number's the point's distance from the left boundary of the field; the second number's its distance from the bottom boundary. If we make each of the boundaries' lengths equal to 1, the point in the center of the field has the name: +1/2, 1/2,. If I had a perfectly arbitrary way of picking a single point from the field, so that any one point was just as likely to be selected as any other, what's the probability of picking +1/2, 1/2,? I'll tell you. It's 0. So I'm certain it won't be picked, although it's possible.

Alternatively, I might say the point has an infinitesimally small chance of being picked.

That's greater than 0, but so close to 0 that any positive decimal number is greater than it! These are difficult matters, Socrates, dealing with counting infinities of points, which Zeno called our attention to, and I'd rather not get mired down in Zeno's paradoxes now.

6. If we have three ideas, we should've *used three circles.* I'm thinking of a process taking time. First I have two ideas; after a while, a third comes to me. I want us to describe that change. Let's do the "kinematics" of belief, in which a believer moves from a two-circled diagram to a three-circled one. Kinematics is the description of a change, without entering into a discussion of its causes. So let me begin by describing the end of the change, when we have three ideas. Here's the three-circle diagram, with eight mutually exclusive, jointly exhaustive regions, and 256 regions over all:

Diagram 7: Three circle diagram & two circle diagram (= diag. 4) for comparison.

Each of the four basic regions of the two circle diagram are subdivided in the three circle diagram:

Basic regions of *two circles* become divided in three circles
1 is succeeded by 1-2
2 is succeeded by 3-4
3 is succeeded by 5-6
4 is succeeded by 7-8

Table 10: How the regions of a two circle diagram are divided by a third circle.

What degrees of belief should I associate with the eight new basic propositions? Or, to stay with the analogy, what are the eight new areas as fractions of the field? A new idea makes our old information about probabilities incomplete, even if it was complete before we had the new idea. There's no rule telling us how to distribute the probability of old region 1 between its two successors in the three-circle diagram, regions 1 and 2, for example.

We've divided "kinematics of belief" into two distinct aspects: having new ideas and forming degrees of belief in the propositions the ideas give us access to. The problem is telling if the aspects are independent. There is, however, a consideration that does give us a constraint on succession of degrees of belief, namely, getting an idea does not in and of itself cause us to change our degree of belief in any proposition we already had access to. Thus our first rule:

i. The probabilities of the pairs of successor regions must sum to the probability of the old region that the pair succeeds.

Having a new idea renders our old degrees of belief incomplete. We're at liberty, however, to complete them in any way at all, subject to just two more rules:

ii. All degrees of belief are non-negative.

iii. The probabilities of the new *basic* regions, which our new idea gives us access to, must sum to 1.

If I have two ideas and probabilities as shown on diagram 5, then having a third idea renders those probabilities incomplete. I'm at liberty to construct any complete set, however, which preserves those old probabilities. For example,

Diagram 8: Probabilities of eight regions that are arbitrary except for being non-negative, summing to 1, and the probabilities of pairs of successor regions sum to the probabilities of the regions of two circle diagrams which they succeed.

The probabilities of all other areas follow from these. So we can use it for practice. What's the probability of B&C? Diagram 7 gives me the regions to use: 3-4-7-8 & 2-4-6-8. So it signifies region 4-8. Diagram 8 tells me the probabilities to sum: 0 for region 4 + 0 for region 8 = 0 for region 4-8. The probability of (B&C) is 0. That does not mean B and C are mutually exclusive, the way B and NOT-B are. There's a distinction to be drawn between the two. Strong mutual exclusion, like that between B and NOT-B, derives from their meaning. Just to understand them is to know their conjunction's probability is 0. But weak mutual exclusion can exist between two propositions when we're sure for any reason their conjunction's probability is 0. Some propositions may be weakly, but not strongly, mutually exclusive. (That distinction is covered in the 1st Appendix to Dialogue I.)

What's the probability of (A&NOT-C) OR NOT(B&NOT-C) in diagrams 7 and 8?

7. *How the diagrams relate to the theorems of probability*. It's impossible for any assignment of non-negative numbers to all the mutually exclusive regions of the field, summing to 1, to make a theorem false of the field. On the other hand, if it is possible to assign numbers to regions so that a formula comes out false of the field, the formula's not a theorem. That means *we have a technique for rejecting expressions as theorems*.

For example, is the formula, prob(A)<prob(A OR B), true of the field in our diagram 5? A is region 3-4; (A OR B) is region 2-3-4. So we're asking whether prob(3-4)<prob(2-3-4). Prob(3-4)=.3+.4=.7; prob(2-3-4)=.2+.3+.4=.9. Yes. It's true in our diagram 5; also in diagram 8. Is it true in all possible diagrams? In other words, is it a theorem? Find numbers for the regions of a diagram that would make the formula false in it. Remember you can put 0 in a region.

Diagram 9: Two-circle diagram in which prob(A)<prob(A OR B) is false.
We can reject it as a theorem.

But I think prob(A) # prob(A OR B) must be a theorem. I can't see how to assign non-negative numbers to all the mutually exclusive regions, summing to 1, that would make it false. It is a theorem (implied by V in the 1st Appendix to Dialogue I). But our inability to find a set of falsifying numbers doesn't prove that. Theaetetus has to use algebra to construct conclusive proofs of theorems. With my diagrams, however, I can prove conclusively that certain formulas are not theorems. So our procedures are complementary.

8. If P and Q imply each other, prob(P)=prob(Q). This theorem is most useful, because expressions you prove equivalent by the diagram method imply each other. This theorem lets you *infer that their probabilities are equal, even when no probabilities are given*. If, for example, I'm to say whether NOT BOTH A AND B has the same probability as EITHER NOT-A OR NOT-B, I number the basic regions of a two circle diagram:

Diagram 10: Two circles only, because the expressions refer to only A and B.

NOT BOTH A AND B	EITHER NOT-A OR NOT-B
NOT BOTH 3-4 AND 2-4	EITHER NOT 3-4 OR NOT 2-4
NOT 4	EITHER 1-2 OR 1-3
1-2-3	1-2-3

Therefore, by this theorem, prob(NOT BOTH A AND B)=prob(EITHER NOT-A OR NOT-B).

I can't stay awake any longer. I'll add more techniques for conditional probability when you learn it.[9]

[9]See the Appendix to Dialogue II.

Notes to Dialogue I

Some historical background:

Despite the setting of this dialogue, the ancient Greeks did not develop the mathematical calculus of probability. The participants in this dialogue bet on horse races and on dice. Do not think these games are the same as their modern counterparts. Odds at a modern race track will not permit a biased book against the track, as happens in the dialogue. In fact, odds at a modern track are variable, as determined by the "parimutuel" system. They begin as biased books against bettors, however, and always remain biased books. The open craps game, to be played by the participants in the second dialogue, is also not the same as casino craps, in that our participants can make bets favorable to themselves, if they can find suckers to take them. In the modern game, all the players are "suckers" if we ignore non-monetary rewards to the players, for all bets are disadvantageous in monetary payoffs, except to the casino and to the state, which taxes every bet.

The mathematical theory of probability came to prominence in the work of Blaise Pascal (who died in 1662) and Christiaan Huygens, who died in 1695. The Dutch were among the first to apply it to practical problems. See Ian Hacking, *The Emergence of Probability* (Cambridge: Cambridge University Press, 1975), chapters 12 and 13. One of the dialogue's participants, the Younger Socrates, takes the monicker, "Dutch," for a while.

The name "Dutch book" is American slang dating back to at least the beginning of the 20th century. See the entry for "Dutch book" in the Random House *Historical Dictionary of American Slang* (1994). It means "a botched method of accounting or odds-making that allows no percentage in favor of the bookmaker." *Webster's Third New International Dictionary* defines the transitive verb, "dutch," similarly. A Dutch book might also be called a dutched book. See also John Scarne, *Scarne's New Complete Guide to Gambling* (New York: Simon and Schuster, 1974), p. 48. (In the dialogue, Theaetetus takes "Scarney" as an alias, which is the way Scarne pronounces his name.)

Philosophers learned the phrase "Dutch book" from R. Sherman Lehman, "On Confirmation and Rational Betting," *Journal of Symbolic Logic* 20 (1955) 251-262. He characterized the argument set forth in the first dialogue as a proof of the necessary and sufficient conditions for avoiding a Dutch book. At first sight, this does not seem to be accurate, for consider the three ways a book of bets on a race might be constructed. The sum of the probabilities of each horse winning may sum to less than 1, or they may sum exactly to 1, or they may sum to more than 1. The first book, and perhaps the second also, is a Dutch book as gamblers use the term. But the second is not a Dutch book as philosophers use the term. The third book is a Dutch book as we philosophers have come to use the term, since we are also stating the necessary and sufficient conditions for avoiding this book. Yet this is not a Dutch book as gamblers use the term. Logicians will not be concerned with this last difference, since they see the symmetry between a bookie taking a book of the first type and losing, and his finding a sucker who takes all the bets in a book of the third type, biased in the bookie's favor. In effect the sucker is exactly analogous to a bookie succumbing to a Dutch book. Nevertheless, not only does one have to stretch the slang term's meaning to fit the philosopher's concept, but also overlook a possible pejorative connotation in the slang, and overcome people's unfamiliarity with the meaning. Therefore I have avoided the term, preferring "biased book" instead to mean books of either the first or third type, and the second type of book is unbiased.

First in the use of biased books to justify the axioms of probability were a youngster of 23, Frank Ramsey (who died in 1929 at age 26), and Bruno de Finetti. See p. 182 of Ramsey's "Truth and Probability" in his *The Foundations of Mathematics* (London: Routledge & Kegan Paul, 1931), and de Finetti's "Sul significato soggestive della probabilità" *Fundamenta Mathematicae* 17 (1931) 298-329, translated in part in Klaus Heilig, "Carnap and de Finetti on Bets and the Probability of Singular Events: The Dutch Book Argument Reconsidered," *British Journal for the Philosophy of Science*, 29 (1978) 325-46. Ramsey's paper and a later one by de Finetti, "Foresight: Its Logical Laws, Its Subjective Sources," are reprinted in Henry E. Kyburg, Jr. and Howard Smokler, eds., *Studies in Subjective Probability*, 2nd ed. (Huntington, NY: Robert Krieger, 1980). Ramsey's is pages 25-52; the biased book is mentioned on page 41. He and de Finetti (in 1931) state only half the argument: If you violate the axioms, a book can be made biased against you. In 1937, de Finetti completed the argument: If you do not violate the axioms, no book can be made biased against you. De Finetti's is pages 57-118 of Kyburg and Smokler; one biased book is on pages 62f.

The interpretation of probability as a degree of belief is assumed in the dialogue. The biased book argument neither proves nor was ever intended to prove that a person's beliefs have degrees. But if they do and if they are compared to "betting quotients" (=PROBEs) to judge advantage in bets, then the argument becomes available for justifying probability theory as normative for degrees of belief. Several sections of the dialogues broach some of the issues here. And an argument that beliefs should come in degrees is presented in the last dialogue in the only section in it numbered 13. Other interpretations of probability theory have their own ways of justifying the axioms of the theory. See, for example, J. R. Lucas, *The Concept of Probability* (Oxford: Clarendon Press, 1970), ch. 3. I pick no quarrel with alternative interpretations of probability; there is room for all.

Those who accept that probabilities are degrees of belief disagree among themselves about the extent to which those degrees ought to be determined by considerations of evidence and logic. I opt for the "personalist" view of probabilities, in which logic is less determinative of rational degrees of belief than is deemed desirable by those who opt for the "logical" or "epistemological" view of probabilities.

AXIOMS AND CONVENTIONS:

2.

I call a set of mutually exclusive, jointly exhaustive propositions, which does *not* have the impossible proposition as a member, a "partition" of the "space" of possibilities. Theaetetus's axiom is not restricted to partitions, as I use the term. See the Appendix to Dialogue I, and in the second part of Dialogue V for further explanations.

ODDS, STAKES, AND PARTIAL BELIEFS:

3.

The more common name for a PROBE is "betting quotient," since the fraction is the quotient of the money bet as numerator and the total stakes as denominator.

4.

That probabilities are degrees of belief is a view that can be behavioristic in methods but perhaps not in metaphysics, if it must acknowledge that mental events, such as people's desires and convictions, are real causes of their behavior. The alternative instrumentalist account of theories of the mind is also consistent with the view. So are views that actions are simply expressive of the intensity of desires and degrees of conviction. The issue is addressed more explicitly in the next dialogue.

5.

Behaviorist method restricts what we learn about a person's mind to what can be learned from observing his behavior. The method ignores the person's reports, or lack of them. See behaviorist method in Theodorus's claim that he has a window onto Socrates's mind, even when Socrates doesn't know his own mind. Even if Socrates did, through introspection, Theodorus would still prefer his window. It is open to question, however, whether the personalist can really avoid appealing to introspection.

See Émile Borel, *Probabilities and Life*, 4th ed., (N.Y.: Dover, 1962), ch. 3, sect. 8, "The Wagering Method," for something like a PROBE principle. Borel specifies large stakes, however, and then, to make the potential for losses even larger, he considers the wager repeated many times. Theodorus avoids justifying the PROBE principle by appeal to the idea that, if violating the principle had been repeated over and over again, it would have led to ruination. Further development begins in the part titled "Is the Axiom of Probability a Description or a Norm"

6.

See Mark R. Greene's article on Insurance in vol. 21 of *The Encyclopedia Britannica*, 15th ed. (1985) p. 687, for pricing insurance at about 50% above expected value of loss—a clear argument that when small sums are at risk we are better off to "insure ourselves."

7.

Theodorus remarks that the subterfuge of using small stakes can be avoided by not using money at all to describe the stakes. For a simple account of the alternative, see Howard Raiffa, *Decision Analysis* (Reading, Mass.: Addison-Wesley, 1968), chapter 4.

I designate a degree of belief "a probability" whether or not it is part of a coherent set of degrees of belief, that is, whether or not all the person's degrees of belief at a particular time constitute an additive function. That means the *probabilities* of a person at a particular time may not constitute an additive function or even a function. Later, in the last part of Dialogue II, when the restriction can be fully motivated, I restrict the symbol "prob" to degrees of belief that are part of an additive function. So, while all degrees of belief are probabilities, not all probabilities are "prob's." Many mathematicians would not allow an incoherent set of degrees of belief to be called a set of probabilities.

9.

Theodorus is using the standard formula for calculating expected value to derive the window opener. The derivation would be valid, but question begging in the context of his argument.

10.

Theodorus says that one need not choose between a subjective interpretation of probability and an objective one (the view that probabilities are either propensities in things or the limits of frequencies of random events). For a defense of that opinion, see D. H. Mellor, *The Matter of Chance,* (Cambridge, University P, 1971), chs. 1-2, Richard Jeffrey, "De Finetti's Probabilism," *Synthese* 60 (1984), 73-90, Brian Skyrms, *Choice and Chance* 3rd ed. (Belmont: Wadsworth, 1986), ch. VII.4, and David Lewis, "A Subjectivist's Guide to Objective Chance," reprinted in W. L. Harper, R. Stalnaker, and G. Pearce, eds., *Ifs* (Dordrecht, Holland: D. Reidel, 1981).

12.

Uncritical use of the division operation on calculators can cause its user to subject himself to a biased book or miss a book he might otherwise have exploited, as Steven Hudson demonstrated to me. Theodorus never uses the division operation. First he multiplies all the denominators together to get the denominator of the sum. Then he revises each fraction's numerator by multiplying it by the product of the denominators of all the other fractions. Then he adds up these corrected numerators to get the numerator of the sum.

IS THE AXIOM OF PROBABILITY A DESCRIPTION OR A NORM?

1.

Most who interpret probability as degree of belief concede that people's degrees of belief often violate the laws of probability, and so the theory is normative of rationality rather than descriptive of people's actual beliefs. But the PROBE principle must be generally descriptive of most people's behavior most of the time, if the biased book argument is to succeed. Furthermore, a principle that describes behavior is needed if we are to reject such alternative theories as Glenn Shafer, *A Mathematical Theory of Evidence* (Princeton: Princeton University P, 1976). His theory predicts that, if the PROBE principle describes, then the less grounds one has for one's belief, the more willing one will be to regard PROBEs exceeding one's degrees of belief as not disadvantageous. Shafer rejects the view that his theory is descriptive, however, believing it to be normative instead, in a way that people can actually follow, and not yielding impossible standards. See his "Jeffrey's Rule of Conditioning" *Philosophy of Science* 48 (1981) 337-362.

The issue of norms impossible for us to follow can be settled by distinguishing theories of competence from theories of performance, as Chomsky distinguishes them. Our theory is a theory of competence.

2.

In my biased book argument I use the PROBE principle in place of the concepts of expected value and fair bet. Thus I show that the argument is independent of these concepts and evades criticisms directed at them.

RECAPITULATION OF THE ARGUMENT FOR THE AXIOM:

1.

One sense of "dilemma" is an argument consisting of two lemmas.

Several philosophers besides Shafer (*A Mathematical Theory of Evidence*, p. 22) claim the biased book argument is not persuasive, e.g., Henry E. Kyburg, Jr., "Subjective Probability: Criticisms, Reflections, and Problems" in his *Epistemology and Inferenc*e (Minneapolis: U of Minnesota, 1983). The references in that article and in Barbara Davidson and Robert Pargetter, "In Defence of the Dutch Book Argument" *Canadian Journal of Philosophy* 15 (1985) 405-424, will lead you to most of the other early critics. Thus the need to state the argument with care.

5.

One criticism I do not evade is that of Frederic Schick, "Dutch Bookies and Money Pumps" *The Journal of Philosophy* 83 (1986) 112-118. Schick denies the principle of assessing collective advantage.

6.

A recent critic is Patrick Maher, *Betting On Theories* (Cambridge University Press, 1993), section 4.6. I address the substance of his charge of fallacy here.

Shark's attempt to strengthen the PROBE principle to include the norm (sc., that we should accept bets that are more advantageous than doing something else instead) would make assuming the principle equivalent to assuming the axiom of valuing actions by their "expected value." See part 9 of the earlier section "Odds. Stakes, and Partial Beliefs" for an earlier appearance of this idea. We will justify the idea in a later dialogue. In the course of justifying it, however, we will assume the axiom of probability. To have now that very principle assumed in the course of justifying the axiom of probability would make a rather tight circle of our reasoning. The two principles would be locked together in a holistic system, whose only justification then would be a pragmatic vindication in terms of the system's guiding us to successful prediction and control.

Theodorus's appeal to Archimedes's law of buoyancy is anachronistic by 200 years.

1st APPENDIX.

Richard Jeffrey says the biased book argument justifies "the special addition law" for probabilities (in his *The Logic of Decision*, pp. 60f.) That's his name for the axiom, although he does not identify it as a sufficient axiom for probability. But it justifies the whole set of axioms customarily assumed for probability theory. The standard set of axioms for elementary probability theory are those of Andrei Kolmogorov, *Foundations of the Theory of Probability* (N.Y.: Chelsea Publishing, 1950). They are Theaetetus's theorems 2, 3, and 7 in addition to two postulating a field of probability. The proofs demonstrate that the axiomatization presented in the dialogue, together with the convention about the maximum probability, is as complete as Kolmogorov's is. An elementary presentation of the usual development of probability theory, well integrated into a logic course, is Richard Jeffrey, *Formal Logic: Its Scope and Limits*, 2nd edition (NY: McGraw-Hill, Inc., 1981), but regrettably not the 3rd edition.

2nd APPENDIX

The area interpretation of probability is spelled out in D.V.Lindley's *Making Decisions* (John Wiley, 1971).