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II. THEAETETUS DEFENDS HIS DEFINITION OF CONDITIONAL PROBABILITY

IT'S LATER THAT EVENING IN THE TAVERN. THEAETETUS HAS BEEN ENTERTAINING EVERYONE BY DEDUCING THEOREMS FROM HIS AXIOM. ALL THE OTHERS, BUT FOR SOCRATES, HAVE FALLEN ASLEEP.]

SOCRATES: But is it a proposition that you're believing?

THEAETETUS: Well, it may not be a proposition that I believe, when I believe I exist now. But whatever it is, it's an object of my belief whose truth I'm certain of. And you're certain you exist now. . . . Socrates?

SOCRATES: Certain am I? Theaetetus, if I become extinct, I hope it's while listening to your proofs.

THEAETETUS: Why so?

SOCRATES: Because the transition would be so subtle.

LAMPROCLES: Hi, Dad.

SOCRATES: Son! What are you doing in a place like this at such an ungodly hour?

LAMPROCLES: Gee, Dad, I was just going to ask you the same question.

SOCRATES: Don't be impertinent.

LAMPROCLES: I don't have any more change, Dad. Can you stand me a drink?

SOCRATES: What do you mean? Of course I can stand.

LAMPROCLES: Dad, you misunderstand me.

SOCRATES: Oh, stand you? Of course; I can stand practically anything short of another proof.

LAMPROCLES: Thanks, Dad. I could stand something high proof too.

SOCRATES: This is Theaetetus, Son. He'll give you real high proof stuff.

[The Coherence of a Community of Believers]
[In Three Parts]

1. *What error does Lamprocles commit that subjects him to this biased book?*

THEAETETUS: Hi, Lamprocles. See these three walnut shells? Bet you can't guess which one has the nut underneath. [SWITCHES THE SHELLS QUICKLY.]

LAMPROCLES: Wanna bet?

SOCRATES: I thought you said you had no money left.

LAMPROCLES: I keep a little aside . . . for emergencies.

SOCRATES: Like the present. Well, I've been watching Theaetetus shuffling those shells; four times out of five it ends up in the shell on the left.

LAMPROCLES: No kidding?

THEAETETUS: What do you think? Is it in one of the outside shells?

SOCRATES: Lamprocles, how are you with probabilities? If there's a .8 chance that it's in the shell on the left, what's the probability that it's in either the shell on the left or on the right?

LAMPROCLES: Well, I ain't no wizzkid, but, you see, when you say it's in the left or the right, you're saying something longer than it's just in the left, and so you're saying more, and so it won't be as likely and that's . . . uh, I think you square it, yeah.

SOCRATES: Square what?

LAMPROCLES: Square your .8.

SOCRATES: So the probability that it's in an outside shell is .64?

LAMPROCLES: You got it, Dad.

THEAETETUS: That's 1:4 odds that it's in the leftmost shell, 9:16 odds that it's in an outside shell.

SOCRATES: Son, how much you have?

LAMPROCLES: Let's see. Sixteen drachmas.

SOCRATES: I bet 2.4 against it's being in the left shell, at 1:4 odds.

LAMPROCLES: You're on, Dad.

SOCRATES: Oh, and Son, 7.68 on it's being in an outside shell, at 9:16 odds. Will you cover that one too?

LAMPROCLES: It's your money, Dad. [THEAETETUS REVEALS NUT UNDER INSIDE SHELL.]

SOCRATES: I lost my second bet; here's your 7.68. I won my first; give me 9.6. Thanks. Shall we bet the same again? Theaetetus, let him do the shuffling. [LAMPROCLES SHUFFLES, REVEALS NUT UNDER THE LEFT SHELL.]

SOCRATES: I lost my first bet; here's your 2.4. I won my second; give me 4.32. Thanks. Shall we try the same again? Good. [LAMPROCLES SHUFFLES, REVEALS NUT UNDER THE RIGHT SHELL.]

SOCRATES: Dear me, I won both my bets; 9.6 plus 4.32; give me 13.92 drachmas. You only have 12.16 left? That's good enough. Thanks.

LAMPROCLES: Daa-aaaad?

SOCRATES: What, Son?

LAMPROCLES: Only three outcomes are possible in this game. Either the nut's under the left shell, or under the middle shell, or under the right shell. How come your pair of bets nets a gain of 1.92 on the first outcome, a gain of 1.92 on the second, and a whopping 13.92 on the third? Something's crooked here.

SOCRATES: But you shuffled the nutshells, Son. Got to go now. Give my regards to Theodorus when he wakes. Goodbye.

2. *A biased book argument for coherence within a community of believers.*

FIRST SCOUNDREL: One look at you, Sir, and I know there's famine in the land.

SECOND SCOUNDREL: One look at you, Sir, and I know who caused it.

FIRST SCOUNDREL: 'Sblood, you starveling, you elf skin, you dried neat's tongue, you bull's pizzle, you stockfish! Oh, for breath to utter what is like thee! You tailor's yard, you sheath, you bow case, you vile standing tuck . . .

SECOND SCOUNDREL: Well, breathe a while, and then to it again, you bed-presser, you horseback-breaker, huge hill of flesh . . .

THEAETETUS: What a racket out there.

SECOND SCOUNDREL: . . . You trunk of humors, you bolting hutch of beastliness, you swollen parcel of dropsies, huge bombard of sack, stuffed cloakbag of guts . . .

THEAETETUS: Should we see to your father's safety?

LAMPROCLES: He'll be alright. I saw those two as I came in. They're jiving.

THEAETETUS: They reciprocate affection by imaginatively vilifying each other.

LAMPROCLES: Like I said. It passes the time, you know. "One can always be kind to people about whom one cares nothing." The nut's under the middle one.

THEAETETUS: Right. You're good at this game. Try again. You want to bet?

LAMPROCLES: I'm no fool, any more than you're a bore. Why are your friends dozing, Theaetetus?

FIRST SCOUNDREL: A crumb like you should've stayed in bread.

SECOND SCOUNDREL: You cowardly rascal, Nature disclaims in thee; a tailor made thee.

FIRST SCOUNDREL: Sir! Your wife, under pretext of keeping a bawdy house, is a receiver of stolen goods.

THEAETETUS: If you're no fool, Lamprocles, why did you think that each of four chariots had a one third chance of winning the race?

LAMPROCLES: I never believed such a thing, never.

THEAETETUS: If not, why'd you take the four bets at 2 to 1 odds? You should've foreseen you'd lose 100 drachmas no matter who won the race.

LAMPROCLES: I just didn't care how much I lost. Easy come; easy go. Dad said you were buying, remember?

THEAETETUS: But surely you don't forfeit 100 drachmas gratuitously. So why?

LAMPROCLES: Are you listening? A hundred makes no difference to me. If I have it, I buy; if I don't, others buy. Hint. Hint.

THEAETETUS: Taverner!

THIRD VOICE: Help, ho! Murder! Help!

THEAETETUS: That sounds like trouble.

LAMPROCLES: That sounds like my old man. We'd better go out and see.

THEAETETUS: Socrates! Let him be, you scoundrels; let him be! What's the quarrel about?

FIRST SCOUNDREL: This ancient ruffian, sir, whose life I have spared at suit of his gray beard . . .

SECOND SCOUNDREL: Villainous abominable misleader of youth, old white-bearded Satan . . .

SOCRATES: You whoreson zed! You unnecessary letter! Give me leave, and I'll tread these unbolted villains into mortar, and daub the walls of a jakes with them. Spare my gray beard, you wagtails?

LAMPROCLES: Dad! I didn't know you had it in you!

THEAETETUS: Please explain what caused this ruckus.

SOCRATES: It was to be a friendly bet, and then they turn on me.

FIRST SCOUNDREL: Friendly, my eye. He tried to set us up for a biased book.

THEAETETUS: "Biased book"?

SECOND SCOUNDREL: Like the bets that guy called "Dutch" and his sidekick make at the track all the time. We're on to their scam.

SOCRATES: It happened like this. I hear them arguing whether a certain ship will arrive tomorrow. One says probably it will; the other, probably it won't. I question them, as is my wont

. . .

FIRST SCOUNDREL: Oh, get this: "as is my wont."

SOCRATES: The first agrees to 1 to 2 odds that it will come in; the second to 1 to 2 odds that it won't. So I offered to bet the first man that the ship *won't* come in, I to pay him 100 drachmas if it does, he to pay me 200 if it doesn't. You admitted you considered the bet fair, didn't you?

FIRST SCOUNDREL: Yes. The ship has a 2/3 chance of arriving.

SOCRATES: Then I offered to bet the second man that the ship *will* come in, I to pay him 100 drachmas if it doesn't, he to pay me 200 if it does. You admitted you considered that bet fair, didn't you?

SECOND SCOUNDREL: Sure. The ship has a 2/3 chance of not arriving.

SOCRATES: I then sweetened the offer to make the bets seem more than fair to each of them, advantageous as they saw it; I offered to pay 150 drachmas if I lost. I only required that both bets be made, or I'd withdraw the offer.

FIRST SCOUNDREL: And we saw through you at once. Either the ship comes in or it doesn't. Whichever it is, you first collect the 200 drachmas from the one of us who loses. Then you give 150 of that to the other of us who wins, and you keep 50 for yourself.

SECOND SCOUNDREL: And we're out 50 drachmas of good times.

FIRST SCOUNDREL: We don't take kindly to geezers who try to exploit our disagreements like that.

SECOND SCOUNDREL: Especially just when we're finally working them out.

SOCRATES: Well, my offer made you serious about agreeing with each other so you could make me a counter-offer. Before that you were both bluster.

LAMPROCLES: Dad, come back into the tavern with us. [THEY GO IN.]

THEAETETUS: Please sit here with us. Let the others stay asleep over there.

SOCRATES: What went wrong, Theaetetus? Lamprocles?

LAMPROCLES: Don't you see, Dad? Each of them depends on the other's money for good times. So naturally, neither wants to see the other lose it foolishly.

THEAETETUS: I think you're right, Lamprocles. I was able to set up the—what did they call it?—the biased book in the stables because none of the horse owners had any interest in any of the others maintaining their fortunes. The same was true of all the people little Socrates and Shark were betting. If they'd all had a sense of communal dependence on one another's wealth, and were aware of our attempts to exploit their disagreements, they'd've foreseen that the losers in their group would lose more to us than the winners among them could recoup from us, so that their group must inevitably sustain a net loss.

SOCRATES: There's only one sure way for a group to protect itself from a biased book being made against it unawares. Its members must strive for agreement in their degrees of belief.

THEAETETUS: So my axiom's a norm for some types of groups, as well as for individuals. Any group whose members depend on each other's wealth, limited in amount, must strive for consensus on degrees of belief. For otherwise the group can be made to suffer a net decrease in its wealth, because any disagreements within the group imply combinations of members' degrees of belief that violate the axiom. They can be exploited to make up a biased book against their community.

SOCRATES: Very good, Theaetetus. You've managed to derive a second norm: Groups ought to reach agreement on what's probable and what's not.

3. *Are people motivated to conform their degrees of belief to other people's?*

THEAETETUS: I think we've answered an objection made against interpreting probabilities as personal.

SOCRATES: What objection?

THEAETETUS: One of those scoundrels had one probability for the ship coming in; the other had another. The objection is that they weren't disagreeing at all. One was announcing a bit of autobiography, his degree of belief in the ship coming in; and the other was announcing a bit of another autobiography, quite independent of the first and consistent with it.

SOCRATES: But that's true. You're not going to deny that two persons' differing degrees of belief in one proposition are logically consistent, are you?

THEAETETUS: No. I now see that they do have a motive to make these points of autobiography less dissimilar, though.

SOCRATES: What would that be?

THEAETETUS: A motive everyone in the group ought to share. Each has, or should have, the motive of reducing one's group's vulnerability to a biased book.

SOCRATES: That would give their expressions of opinion more than the force of autobiographies, wouldn't it?

THEAETETUS: Yes. In addition to being autobiographical, one's expressions of opinion to other members of one's group are claims on them to agree or to persuade one otherwise.

SOCRATES: So expressions of one's degrees of belief are not autobiography merely. They have a persuasive force. On the other hand, if there's a group with wealth available to all without limit, then no harm can come to it or its members from disagreements within the group, or even from the irrationality of a single member.

LAMPROCLES: Here's our drinks, finally. Hey, who's paying?

SOCRATES: You realize, Theaetetus, that the whole argument for your axiom depends on the PROBE principle. That principle's refuted if societies allow violators of the principle to replenish their wealth from its stock.

LAMPROCLES: The taverner wants payment. I don't have it. Someone pay him.

THEAETETUS: Lamprocles, did you say you can't pay for your drink?

LAMPROCLES: Yeah, clean broke.

THEAETETUS: Then you can't have a drink.

LAMPROCLES: Hey, Dad?

SOCRATES: I'm afraid he's right, Son. That's life.

[When a Biased Book Is The Lesser Evil]
[In Five Parts]

1. *What are the rules of the game of shooting craps?*

LAMPROCLES: Dice! [PICKS THEM UP AND THROWS THEM.] Merdsicles! I crapped.

SOCRATES: You what?

LAMPROCLES: I threw boxcars, Dad, two 6's. That's craps. It means I lost. [THROWS THE DICE AGAIN.] Craps again. Snake eyes. That's two 1's, Dad. I lose again.

SOCRATES: Aren't you glad you're broke already, Son?

THEAETETUS: See this money, Lamprocles? [SHOWS 625 DRACHMAS.] I won it all. I'll extend you credit for some crap shooting at small stakes, but only if you'll first learn how to bet it wisely.

LAMPROCLES: Hey, that's more like it. [WHISPERS TO DICE.] Be good to me. A 3. Craps again. Oh, well.

SOCRATES: What would be a win, Son?

LAMPROCLES: If I threw a 7 or an 11 on the first roll, I'd win. Winning's called "passing," Dad.

SOCRATES: So far, you've accounted for 2, 3, 7, 11, and 12. What if you roll one of the other six numbers?

LAMPROCLES: That's called "establishing my point," Dad. I continue throwing the dice until I roll that number a second time or I roll a 7. Rolling it a second time before rolling a 7 is a pass, I mean it wins. It's called "making my point." What's to learn, Theaetetus? I know this game.

SOCRATES: I know what counts as winning, and what as losing, Son. But the fun of the game, I gather, is in the betting. We bet that the dice shooter will win or that he won't; is that it?

LAMPROCLES: Not just that, Dad. You can bet on so many things.

SOCRATES: Let's say I want to make a bet on your winning, "passing." What should by my probability of your winning?

LAMPROCLES: I'd say about even, Dad.

THEAETETUS: What's "about even," Lamprocles?

LAMPROCLES: An even money bet. You know, what you risk is what you win.

THEAETETUS: *If* you win. Your father asked for probabilities, you answered with odds. The chance of winning is 488/990, which is less than 495/990, which is ½.

LAMPROCLES: That's so close to even as to make no difference. Let's play!

2. *Wise bettors know the dice's equiprobable mutually exclusive and jointly exhaustive states.*

THEAETETUS: Are you satisfied with the dice?

LAMPROCLES: [LOOKS AT DICE.] Each cube has two 1's on opposite faces, two 2's on opposite faces, and two 6's on opposite faces. I never looked at dice this closely before. I just pick them up, and off they go.

THEAETETUS: Well, let this be your first lesson, Lamprocles. The spots on opposite faces of a die should sum to 7. Each face has from one to six spots on it. The face with 1 is opposite the face with 6, the 2 opposite the 5, the 3 opposite the 4. The dice you threw are crooked, with only three numbers. Here, look at this other pair.

LAMPROCLES: Oh, yeah. Another bum pair. One's a translucent pink and the other a translucent yellow. You thought I wouldn't spot that?

THEAETETUS: No. These are fair dice. I never said fair dice had to be white. Do you know how to test whether a die's fair?

LAMPROCLES: [NO ANSWER.]

THEAETETUS: Rub the dice together so that you can tell the sides are flat—there's no rocking; compare their edges by feeling them to check they're the same length all around; hold the dice loosely with your fingers at diagonally opposite corners to ensure there's no tendency to tumble. The purpose of checking the symmetry of shape and weight distribution is to assure yourself that, after a die's tossed in a fair manner, each side's as likely to be face up as any other when it comes to rest. So each face will face upwards at a certain frequency.

LAMPROCLES: I know that. I'd say 1/6th of the tosses.

SOCRATES: The probabilities of six mutually exclusive, jointly exhaustive propositions summing to 1 again.

LAMPROCLES: How's that again, Dad?

SOCRATES: Son, probabilities are subjective degrees of belief, influenced by evidence, thank goodness. The six propositions about which face of the die will turn up are mutually exclusive and jointly exhaustive of possibilities, and you're certain just one is true.

THEAETETUS: But more than that, each proposition has the same probability as any other. The dice most people play with aren't perfectly balanced, though. I know a guy named Longcor who threw that type over a million times. Higher numbers were more likely to come up than lower ones:

NUMBER COMING UP:	1	2	3	4	5	6
FREQUENCY:	.155	.159	.164	.169	.174	.179

Table 11: Frequency of number showing up, using a cheap die. If the die had been perfect, the frequency of any number would be .1666---.

LAMPROCLES: Oh.

THEAETETUS: But the colored dice I gave you are perfect; the frequency of each face of a die coming up is 1/6 exactly, provided none of us can control the dice as we throw them. What's your probability, then, of losing on the first throw with fair dice in a fair shake?

LAMPROCLES: That's hard to say. The two dice can sum from 2 to 12, eleven alternatives. You lose on the first throw if the sum is 2, 3, or 12; that's three of the eleven possibilities. Is it 3/11?

THEAETETUS: That depends on whether the eleven possible sums are all equally probable. Are they, in your opinion?

LAMPROCLES: Mmmm . . .

THEAETETUS: The evidence of physical symmetries and observed frequencies guides a *wise* person's degrees of belief. We've considered the evidence. Now what has it taught you?

LAMPROCLES: So I learned the sums are equally probable?

THEAETETUS: No. And 3/11 is not what our evidence says. We can figure it out. First, since each die has six faces, there are thirty six ways to combine the six faces of the pink die with the six faces of the yellow die:

<1,6>	<2,6>	<3,6>	<4,6>	<5,6>	<6,6>
<1,5>	<2,5>	<3,5>	<4,5>	<5,5>	<6,5>
<1,4>	<2,4>	<3,4>	<4,4>	<5,4>	<6,4>
<1,3>	<2,3>	<3,3>	<4,3>	<5,3>	<6,3>
<1,2>	<2,2>	<3,2>	<4,2>	<5,2>	<6,2>
<1,1>	<2,1>	<3,1>	<4,1>	<5,1>	<6,1>

Table 12: All the possible ways the numbers can show on the top face of each die; the numbers are given in this order: <top of yellow die, top of pink die>.

LAMPROCLES: In the lower left corner, why do you distinguish between <1,2> and <2,1>?

THEAETETUS: <1,2> stands for 1 on the yellow die, 2 on the pink one; <2,1> stands for 2 on the yellow die, 1 on the pink one. They're distinct possibilities, all the 36 outcomes, mutually exclusive and jointly exhaustive of possibility.

SOCRATES: So my probabilities for the outcomes sum to 1. But what are my probabilities that sum to 1?

THEAETETUS: The dice are perfect. Each of these outcomes is just as likely as any other, since that's so for each die singly. So your degree of belief in each of the outcomes is 1/36 of the maximum probability. Is that clear, Lamprocles?

LAMPROCLES: Yeah. The evidence of symmetry leads me to equal degrees of belief in useless propositions about pink and yellow dice.

SOCRATES: You're moving closer to wisdom, wisecracking smart aleck. Theaetetus, do you realize you're the first one all evening to mention wisdom? Theodorus refused to consider whether subjective degrees of belief should conform to objective chances.[0] I see it was irrelevant at the time to justifying the axiom. I'm glad we're not supposed to ignore it all the time.

LAMPROCLES: Wise about <2,1>; what's the use of that?

3. *What are the important probabilities in the game of shooting craps?*

THEAETETUS: I'd rather discuss objective chance another time, Socrates; we need to answer the wisenheimer’s question. Lamprocles, let's add together the pair of numbers in each of the 36 outcomes. After all, the sum's what matters in the game. The sums of the numbers on the two upturned faces of the dice range from 2 to 12.

LAMPROCLES: Yes. Only there are more ways to get a 7 than to get a 2 or a 12.

THEAETETUS: Yes. Here's a table showing that. It's derived from our last one by adding the numbers in each pair together.

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[0] See the first dialogue, page 15.

	7		8	9	10	11	12
6		7		8	9	10	11
5	6		7		8	9	10
4	5	6		7		8	9
3	4	5	6		7		8
2	3	4	5	6		7	

Table 13: Sums computed from table 12. You can now count the number of ways of getting different sums of faces of two dice, and their probabilities.

Since each of these ways of getting a sum is just as likely to occur as any other way, the more ways there are of your getting a sum, the more likely is your getting it.

LAMPROCLES: With fair dice, [COUNTS LOWER LEFT AND UPPER RIGHT CORNERS] there's four chances in thirty six of missing on the first throw, rolling a 2 or a 3 or a 12.

THEAETETUS: Or 1/9. But with the crooked dice you threw, your chances were 4/9.

SOCRATES: Throwing a 7 or 11 with fair dice is a 2/9 probability, the same as with the crooked dice.

THEAETETUS: I said the chance of winning this game with fair dice is 488/990. Just figure out the probability for each way of winning and sum them. That's the way I got that probability.

4. *Bet to maximize your advantage.*

LAMPROCLES: Oh, be good to me! [THROWS THE FAIR DICE.] A 4. Now I have to throw another 4 before I throw a 7. I know I can do it.

THEAETETUS: Look at the table, Lamprocles. What's the chances of each?

LAMPROCLES: Six chances in thirty six of a 7, but only three chances in thirty six of throwing a 4. A 7 twice as likely as a 4?

THEAETETUS: On any one throw, yes. So you don't "know" you can throw another 4 before you throw a 7, do you? That's your next lesson. Always bet on the side that has the advantage. At even money, 1:1 odds, the bet against 4 is far better than the bet on 4.

LAMPROCLES: You mean, bet that I won't do it?

THEAETETUS: That's the gist of it. Socrates, if you bet on Lamprocles to win at even odds, 1:1, you're accepting a disadvantageous bet. His chance of winning just dropped to 1/3.

LAMPROCLES: You can't tell Dad to bet against his own son.

SOCRATES: If I took a bet where my PROBE exceeded my degree of belief by, let's see: 1/2 minus 1/3 equals 1/6, the havoc it would play with the PROBE principle! Son, I could not love you half so well, loved I not Theaetetus's axiom more. I bet you don't pass.

LAMPROCLES: Daa-ad!

SOCRATES: OK. I bet you pass. Two drachmas on the front line.

THEAETETUS: You're on at even odds, chump. I mean no disrespect, Sir.

LAMPROCLES: [TALKS TO DICE.] Be Daddy's good little babies! [THROWS DICE.] A 10. You didn't lose, Dad.

5. *Not another biased book!*

SOCRATES: Didn't win either. What should be my probability of your winning now?

THEAETETUS: It's still 1/3. Of the thirty six ways the dice may come up, only nine count; six are ways the 7 may come up and you lose; three are ways the 10 may come up and you win.

SOCRATES: Theaetetus, in a moment of fatherly weakness, I betrayed your axiom. Well, I regret it. Can I change my bet now?

THEAETETUS: A bet's a bet, chump, Sir.

SOCRATES: But you yourself admit that the odds against his passing are 2:1. You, betting he won't pass, should be willing to pay me 4 drachmas if he passes, against my paying you 2 if he doesn't.

THEAETETUS: We mustn't blur the distinction between the earlier time when we established our bet, assumed risks, made commitments, and the later time when we'll settle the bet. Our assessment of the risk we incurred may change in the meantime, but the terms of the bet don't. If, however, you've come to dislike your original bet, you can make a hedge bet. Do you understand hedging?

SOCRATES: How would you define hedging?

THEAETETUS: By "hedging a bet" we mean placing second bets, where a net loss from both bets will be less of a loss than losing the first bet alone or where a net gain from both bets will be less of a gain than winning the first bet alone.

SOCRATES: The point of hedging is what?

THEAETETUS: The trade-off's that you've a greater chance of sustaining the lesser loss. If the hedge bet sets you up for a biased book, the lesser loss is certain. But it is a lesser loss, and that's the point. In the other kind of hedge, you settle for less of a gain, in return for a greater assurance of gaining.

SOCRATES: OK. Let's return to our game. How do I hedge my bet?

THEAETETUS: Bet the 7 *will* come up before the 4. I'll take it at 1:2 odds against the 7 before the 4.

SOCRATES: I pay you 2 drachmas if 4 comes before 7; you pay me 1 drachma if 7 comes before 4?

THEAETETUS: It's a fair bet. But let's make the stakes the same as before, 4 drachmas. If you lose, you pay me 2 2/3; if you win, I pay you 1 1/3.

SOCRATES: I can take a fair bet?

THEAETETUS: By all means. The axiom is safe.

SOCRATES: OK, you're on. Let's see. Either 4 comes up before 7 or it doesn't. Suppose it does. My pass bet wins me 2. My hedge bet loses me 2/3 of the stakes of 4, that is, 2 2/3, giving me a net loss from both bets of 2/3. Suppose it doesn't. My pass bet loses me 2. My hedge bet wins me 1/3 of the stakes of 4, that is, 1 1/3, giving me a net loss from both bets of 2/3. I lose 2/3, no matter what. Theaetetus! How could you do this to an old man, your friend, your confederate! You've sold me—me!—a biased book!

LAMPROCLES: What's happening?

SOCRATES: Is there no honor among hustlers?

[Conditional Bets and Combination Bets]
[In Ten Parts]

LAMPROCLES: . . . bored, bored stiff. I thought we were going to shoot craps.
SOCRATES: Why don't you go bore a hole in yourself and let the sap run out.
LAMPROCLES: What was that, Dad?
SOCRATES: Let's get on with it.
THEAETETUS: Give me the dice; I'll shoot. Place your bets, gentlemen.
LAMPROCLES: If my credit's good, I put 20 on "you don't pass." Even money.
THEAETETUS: Good. You remembered. [PUTS 20 ON "DON'T PASS" FOR LAMPROCLES]
LAMPROCLES: I have to bet he'll lose, Dad. I'm not allowed to make disadvantageous bets. My PROBE's 7/990 less than my degree of belief that he'll lose.
SOCRATES: I can hardly believe my ears.
LAMPROCLES: I've been listening real close, Dad.
THEAETETUS: And you, Sir? Shall I have no supporters, poor orphan that I am?
SOCRATES: Orphan! You're a bookie balancing his book. I bet 20 that you pass.
THEAETETUS: Thanks for the perfect balance. Balancing a book's just hedging . . .
LAMPROCLES: Throw!
THEAETETUS: [THROWS DICE.] A 12. Lamprocles, you win 40, minus the loan from me, leaving you 20. Socrates, you lose 20. Thank you. Here's your 20, Lamprocles.
LAMPROCLES: Taverner!

1. *What are conditional bets?*

SOCRATES: Hedging is hindsight; is there any betting procedure that's foresight?
THEAETETUS: What do you mean?
SOCRATES: There should be a way of betting that circumvents the possibility of future disappointments.
LAMPROCLES: Dad, this is craps, not a debate!
SOCRATES: OK. Lamprocles, I bet you 10 that Theaetetus passes.
LAMPROCLES: You're on.
SOCRATES: Theaetetus, here's 20 in "if" money. If I don't win my bet with Lamprocles, I bet you 20 that you don't pass.
THEAETETUS: That's not . . .
LAMPROCLES: No debates! I'll take that bet too. Roll'em! [THEAETETUS ROLLS 5, ROLLS AGAIN: A 7; GIVES DICE TO SOCRATES.]
LAMPROCLES: I win again.
SOCRATES: Only your first bet. Here's the 10. Since I lost my first bet with you, my second bet comes into force. I've bet you 20 he wouldn't pass. I won that.
LAMPROCLES: Here's 20. Hey, Dad. Did you just do it to me?
SOCRATES: To my own son, even.
THEAETETUS: Watch your credit rating, Lamprocles. You just accepted another biased book.

LAMPROCLES: Aw gee, Dad. You confused me with that stuff about "if" money.

THEAETETUS: Work the two bets out.

LAMPROCLES: Either you pass or you don't. If you pass, Dad wins 10 and his second bet doesn't come into force. So his "if" money's returned to him. His net gain in this case is 10. If you don't pass, Dad loses 10, but his second bet comes into force, and he wins 20. Again his net's 10. So he gains 10 no matter what. How could you do that to me, Dad?

SOCRATES: How could you do it to yourself, Son? You should've called for a debate on the shrewdness of accepting the second bet. [HANDS DICE TO LAMPROCLES.]

THEAETETUS: What kind of bet was that: "If I don't win my first bet, I bet the opposite side at twice the stake of my first bet"?

SOCRATES: A conditional bet.

THEAETETUS: I know that, but the condition amounts to, "If I'm sure to win the bet, then I make the bet." Who but a fool would accept that condition at even money?

SOCRATES: Well, not all conditional bets are so blatant. Now I want to bet, if Lamprocles establishes a point, that he'll make it the hard way.

LAMPROCLES: "Hard way"? You know dice lingo, Dad?

SOCRATES: Is it, now! Such a coincidence.

LAMPROCLES: Only thing is, I might establish a 5 or a 9. You see from the tables I can throw a 6 in five ways, only one of which has both dice the same. The hard way of throwing any even number besides 2 and 12 is having both dice show the same number. So you're betting if I throw a 4, I'll throw a <2,2> before a 7, or if I throw a 6, I'll throw a <3,3> before a 7, or if I throw an 8, I'll throw a <4,4> before a 7, or if I throw a 10, I'll throw a <5,5> before a 7. But you're also betting my point will not be a 5 or 9, since they can't be made the hard way, right?

THEAETETUS: Is that what you're betting, or does your bet only come into play, if he establishes an even point?

SOCRATES: That's it. If he establishes an even point.

LAMPROCLES: What's going on now?

THEAETETUS: Your father's introducing a different kind of bet, one conditional on your establishing an even point, that is, on your not throwing a 2, 3, 5, 7, 9, 11, or 12. If you do establish an even point, his bet will come into force, otherwise not. The difficulty is to set the correct odds for a conditional bet.

2. *Why make conditional bets?*

LAMPROCLES: What's the point of making the bet now? Why not wait until after I make my first throw?

THEAETETUS: Good point. Why make conditional bets? Two reasons, it seems: We sometimes might make them in order to take into account rare conditions, when, if they should occur, it's understood by all parties that the bet's called off. For example, all bets are conditional on no one having cheated. We don't wait to ascertain that there's been no cheating. That'd be too late. We make our bets with the understanding that they're off if cheating's discovered. The bet's conditional on things being as they appear, but present realities may not be as they appear. Secondly and less commonly, some of the alternative ways things might be, consistent with what we know currently, our subsequent learning will eventually rule out, and the bet's made conditional on what's discovered. Usually, a bet's made before the discovery occurs, as a convenience to one of the bettors, who may be too busy to make the bet at the time the bet would come into force.

LAMPROCLES: Well, why make a bet conditional in the first place? As for cheating, let the suckers beware. That's what I say.

THEAETETUS: There's something both purposes of conditional bets have in common, namely, the odds become more calculable under the condition for the bet holding than they'd be if it didn't hold. By calculable I mean that we can go by frequencies, rather than by our inchoate estimates. Making conditional bets is part of not being a sucker. If you had made bets using those gaffed dice, wouldn't you want to declare the bets off after you learned they were gaffed?

LAMPROCLES: Yes. But Dad's bet's not the sort where the odds are more easily determined for the case when the condition holds—"if I establish my point"—nor will he be out of town when it's determined whether the condition holds. So why's he slowing up the action?

SOCRATES: One condition might be "if betting's legal here." Unless that condition holds, a win's not even collectible. Another reason for a conditional bet is that it can serve to reduce the stakes. And the condition might be something on which my acquiring my stake depends.

LAMPROCLES: Dad, it's legal; you're not waiting for your ship to come in; your money's in your lap. Forget conditional bets. Wait to bet until after I throw.

SOCRATES: My dear Theaetetus, you love examining all possible cases, but this time I suspect you missed one. Might it not be that I have to visit the gents, and I don't want to hold you up or miss the action?

LAMPROCLES: Dad, we'll wait for you.

3. *Conditional bets are equivalent to a combination of unconditional bets.*

THEAETETUS: Socrates, you don't have to wait until after Lamprocles throws to make unconditional bets. There are unconditional bets equivalent to conditional bets. Instead of making bets now that might come into force later, you can make equivalent bets now that come into force now.

LAMPROCLES: This is hopeless. [PUTS DICE DOWN.]

THEAETETUS: Not that anyone ought to be indifferent between the two procedures. But the differences are irrelevant to comparisons of the PROBEs, the way the difference between a loaf of bread and a pile of crumbs is irrelevant to their equality as nutrition, although you'd prefer the first to the second, wouldn't you, Lamprocles? For example, the conditional bet your father wants to make is equivalent to the combination of these two bets: an unconditional bet on your establishing an even point *and* making it the hard way, and another unconditional bet that you'll *not* establish an even point. In general, a bet on A, conditional on B, is equivalent to the combination of two unconditional bets, one on A&B, and the other on NOT-B. But before I prove that to you, . . .

LAMPROCLES: Oh, please don't!

THEAETETUS: . . .I must state two further conditions on this combination of bets. An example will motivate them; what do you think of a bet on A&B at 2:1 together with a bet on NOT-B at 3:1 odds against B?

LAMPROCLES: My PROBE in the first bet is 1/3, and my PROBE in the second bet is 3/4. Yes?

THEAETETUS: That's correct.

LAMPROCLES: I suppose there's some A and B those odds are correct for.

THEAETETUS: Wrong. The bets are incoherent. It's impossible to win.

LAMPROCLES: I see it's impossible to win both; they're bets on mutually exclusive propositions.

THEAETETUS: Yes, but you misunderstand me. I mean there's no way the combination will permit you to come out ahead. Suppose the stakes for the first bet are 6; you risk 2 to gain 4. Say you win it. But a win on your first bet is a loss on your second. To keep from suffering a net loss from both bets, your risk on the second bet cannot exceed 4. The stakes of the second bet we compute thus: Your PROBE in the bet is 3/4. Your maximum allowable risk is 4, and $4=(3/4)x$, where x is the total stakes. So the total stakes cannot exceed 5 1/3. So, in order not to have a net loss if you win the first bet and lose the second, your net win on the second bet must be set at 1 1/3. But if you win that, you've lost 2 on the first bet, for a net loss of 2/3. If you lose the second bet, you lose what you won on the first, for a net loss of 0. You cannot come out ahead.

LAMPROCLES: Is this another book biased against me?

THEAETETUS: Almost. Just set the stakes of the two bets equal. What happens?

LAMPROCLES: Let the stakes of both bets be 12. Either I win the first bet or I lose it. Suppose the first case. I win 8, but lose 9 on the second bet, for a net loss of 1. Suppose the second case, I lose 4, but win the second bet, which, however, only pays me 3. So again I suffer a net loss of 1.

SOCRATES: Congratulations, Lamprocles!

LAMPROCLES: Hey Dad, I can do it too!

THEAETETUS: The combination of bets never yields a win at any stakes, and when the stakes of the two bets are equal, the result's a book biased against the person making them. What is it about the two bets that makes them wrong? Do you see?

SOCRATES: Somehow they violate your axiom.

THEAETETUS: Yes, together with the PROBE principle. Note that a bet on A&B, together with a bet against B, has to guard against the fact that A&B implies B. Therefore, your degree of belief in A&B ought to be no greater than your degree of belief in B.[1] There's a corollary for bets: Since A&B implies B, if bets on A&B and against B aren't disadvantageous together, then PROBE(A&B)#PROBE(B). [READ "PROBE(x)" AS "THE PROBE OF A BET *ON* x."] Would you like me to prove this corollary? It's less than 20 steps.

SOCRATES: If my one bet on the premise and my other *against* the conclusion are not conjointly disadvantageous to me, then my PROBE on the first doesn't exceed the PROBE of my cobettor on the second. Sounds tricky, doesn't it, Son?

THEAETETUS: You'll understand if you see the proof. Let me show it to you.

SOCRATES: Not now. Put it in the notes for me to examine at my leisure.[2]

THEAETETUS: Will do. The bets in my example violate the corollary: The PROBE(A&B) was 1/3, but the PROBE(B) was 1/4.

LAMPROCLES: It should've been 1/3 or greater.

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[1]See Theorem V, Appendix to the first Dialogue.

[2]See Appendix, Theorem XVI.

THEAETETUS: Good. So the bets on A&B, but against B, were incoherent, and there are stakes guaranteeing the bettor a loss.

LAMPROCLES: I understand. I really do.

SOCRATES: In matters like these, the feeling of understanding is often illusory. How about a test? [BRINGS OUT THE THREE WALNUT SHELLS.] Remember these?

LAMPROCLES: Yes, and the 20 drachmas I lost to you.[3]

SOCRATES: You allowed odds of 1:4 that the nut was under the leftmost shell and 9:16 odds that it was under one of the outside shells.

LAMPROCLES: Dad, I do understand. I see now that if the nut's under the left shell, then it has to be under one of the outside shells. That's implied. It's under the left; therefore it's under the left or right. From premise to conclusion. You got me to bet in favor of the premise and against the conclusion. My PROBE in favor of the premise was 4/5; my—no, *your* PROBE in favor of the conclusion was 16/25. As you just said, if my bets in favor of the premise, but against the conclusion, aren't to be to my disadvantage, then my PROBE on the premise mustn't exceed yours on the conclusion. But 4/5 does exceed 16/25. So I lost my 20 drachmas. If we'd set your PROBE at 21/25 instead, I'd've made some money when the nut was under the middle shell.

SOCRATES: [WITH EMOTION, EMBRACES LAMPROCLES] This schnook of a son, this schlemiel! He'll get to keep some of his money after all!

LAMPROCLES: Aw c'mon, Dad. People are looking.

4. *The combination of bets must be coherent by fulfilling two conditions.*

THEAETETUS: I was saying that a bet on A, conditional on B, is equivalent to the combination of two unconditional bets, one on A&B, the other against B. To make the case for that claim, I add the proviso that the unconditional bets are coherent in the sense that:

I. BET(A&B) AND BET(NOT-B) ARE NOT CONJOINTLY DISADVANTAGEOUS. [READ "BET(x)" AS "A BET ON THE SIDE OF THE TRUTH OF x."] It follows that
PROBE(A&B)#PROBE(B).

A further proviso is that a win on the second bet exactly offset a loss on the first bet. In symbols:

II. PROBE(A&B)STAKES(A&B)=PROBE(B)STAKES(B)=PROBE(B)STAKES(NOT-B). [READ "STAKES(x)" AS "THE TOTAL STAKES IN THE BET ON x." STAKES(x) = STAKES(NOT-x), SINCE WE MEAN TOTAL STAKES.]

The left side of the equation represents what you might lose in your bet on A&B; the right's what you might win in your bet against B. You do see why we need this proviso, don't you?

SOCRATES: Tell us.

[3]See the beginning of the dialogue.

THEAETETUS: To make a situation parallel to each one that exists in a conditional bet. In a conditional bet there's a winning situation, a losing situation, and two situations in which the bet's off, neither winning nor losing. By setting a win in the bet against the conclusion, B, equal to a loss in the bet on the premise, A&B, we get the same four results in the combination of unconditional bets. Here, look at this table of payoffs—I introduce a vertical line to abbreviate "conditional on" in the behavioral contexts, PROBE, STAKES, AND BET:

a = POSSIBLE SITUATION *S*
b = PAYOFF IN *S* OF BET ON A, CONDITIONAL ON B
c = PAYOFF OF COMBINATION OF TWO BETS IN *S*, ON A&B, AGAINST B

1:	a	A&B is true
	b	[1–PROBE(A|B)]xSTAKES(A|B)
	c	{[1–PROBE(A&B)]xSTAKES(A&B)}–{PROBE(NOT-B)xSTAKES(NOT-B)}
2:	a	NOT-A, BUT B is true
	b	PROBE(A|B)xSTAKES(A|B)
	c	–PROBE(A&B)xSTAKES(A&B)–{PROBE(NOT-B)xSTAKES(NOT-B)}
3:	a	A, BUT NOT-B is true
	b	doesn't hold
	c	–PROBE(A&B)xSTAKES(A&B)+{[1–PROBE(NOT-B)]xSTAKES(NOT-B)}
4:	a	NEITHER A NOR B is true
	b	same as previous case
	c	same as previous case

Table 14: The four possible situations, with the payoffs of the conditional bet and of the combination of two unconditional bets in each of the situations. In 1b and 2b read (A|B) as "A, conditionally on B."

Note that in the last two cases the conditional bet doesn't hold, and the combination of bets nets 0. Netting zero and not holding at all are equivalent for assessing the worth of a bet in the situation, don't you agree?

SOCRATES: Yes. The loss of a bet on A&B equals a win of a bet against B. I see I must memorize your two provisos.

LAMPROCLES: I'll write them down.

5. *What is the PROBE of this coherent combination of bets?*

THEAETETUS: Our conditions on the combination bet ensure that the result of the subtraction in the payoff when A&B is true[4] is not a negative number. The expression simplifies to: STAKES(A&B)–STAKES(NOT-B) by proviso II.; by proviso I. the second term cannot exceed the first. We're assured that in a coherent combination of bets the win's not negative.

LAMPROCLES: [DOES CALCULATIONS CONFIRMING THEAETETUS.] Got it.

SOCRATES: Good. It's always desirable to have the quantity of one's winnings expressed in positive numbers.

THEAETETUS: Zero's not excluded, however.

SOCRATES: Oh.

THEAETETUS: The potential for loss is STAKES(NOT-B), if you use proviso II. to simplify table 14, case 2, line c. And the net total stakes for the combination of bets is just the stakes of the bet on A&B.

LAMPROCLES: How do you figure that?

THEAETETUS: Total stakes are just the sum of what you risk losing and what you stand to gain: STAKES(NOT-B) + {STAKES(A&B)–STAKES(NOT-B)} = STAKES(A&B).

LAMPROCLES: Got it.

THEAETETUS: Now I'll prove some facts about the combination of unconditional bets, making no reference to conditional bets. In particular, I've already shown that the combination bet has, in effect, a PROBE of its own. It is:

STAKES(B)/STAKES(A&B).

LAMPROCLES: How do you figure that?

THEAETETUS: Recall what a PROBE is: the portion risked of total stakes. The risk in the combination bet is the stakes of the bet against B; the total stakes of the combination bet are the stakes of the bet on A&B. The fraction's the portion.

LAMPROCLES: Got it.

THEAETETUS: Another way to express its PROBE is with this equation:

IF I. AND II. ARE TRUE OF BETS ON A&B, AGAINST B, THEN
PROBE(A&B, AGAINST B) = PROBE(A&B)/PROBE(B)[5]

I really like the proof of this. Let me show you . . .

SOCRATES: In the notes, so we can examine it at our leisure.

LAMPROCLES: [READS APPENDIX, PROOF OF THEOREM XVII, AS THEAETETUS WRITES IT.]

THEAETETUS: Another result connects the combination bet to degrees of belief:

IF I. AND II. ARE TRUE OF BETS ON A&B, AGAINST B, THEN
PROBE(A&B, AGAINST B) # prob(A&B)/prob(B)
IF PROBE(B) ... 0.

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[4]See table 14, case 1, line c.

[5]Read "(A&B, NOT-B)" as "the combination of the bet on A&B and the bet on NOT-B." Read "(A&B, AGAINST B)" the same way.

Since condition I. says PROBE(B) isn't disadvantageous, if it doesn't equal 0, neither can prob(B).

LAMPROCLES: [READS APPENDIX, PROOF OF THEOREM XVIII, AS THEAETETUS WRITES IT.]

SOCRATES: All very nice. But what do these results show about conditional bets?

6. *The principle of equivalence between a conditional bet and a combination bet.*

THEAETETUS: We need a principle to ensure that our combination bet's equivalent to our conditional bet. The PROBEs must be the same for the conditional bet and the combination of unconditional bets. So we stipulate:

NORMATIVE EQUIVALENCE OF CONDITIONAL BET WITH COMBINATION BET

IF BET(A&B) AND BET(NOT-B) FIT CONDITIONS I AND II, AND
IF A PERSON REGARDS THE COMBINATION AS FAIR AT PROBE(A&B, NOT-B),
HE MUST REGARD BET(A|B) AS FAIR AT PROBE(A|B)=PROBE(A&B, NOT-B).

We can ensure identical payoffs by specifying that the stakes of the conditional bet are the same as the stakes in the combination of bets.

SOCRATES: But *are* the PROBEs of the combination of two bets the same as the conditional bet's, really?

THEAETETUS: It's not a matter of discovery; the principle of normative equivalence, like I. and II., is a stipulation, Socrates.

SOCRATES: A question begging stipulation. Conditional bets and combinations of unconditional bets have different appeal. You've admitted as much. Suppose I think that makes a difference to the PROBE of a bet, so that a conditional bet and a pair of unconditional bets satisfying I. and II. on the propositions, A and B, and at the same stakes, may yet have different fair PROBEs for me.

THEAETETUS: I admit the conditional bet and this combination of two unconditional bets are different in that more money is tied up in securing the combination of bets than in the conditional bet, a possible motive for preferring a conditional bet over the combination of bets. And the fact that your degree of belief in B may not have a frequency as its basis may worry us about the combination bet, but makes no difference in the conditional bet. But I'm not concerned to show the two equivalent in every respect, only that conditions I. and II. are the sole determiners of the PROBEs. Are there other differences between the two methods of betting on A with B in the picture?

SOCRATES: I think so. Suppose B is some proposition like "there's been no cheating." The conditional bet makes sense. I might bet that the craps player will pass, assuming no cheating. But what combination bet is equivalent to this conditional bet? One bet's a bet on his passing AND there being no cheating. Even if I do come up with odds on that, how about the second bet, where I'm supposed to bet there is cheating?

THEAETETUS: A conditional bet may presume the condition satisfied unless there's evidence against it's being satisfied. In a combination bet, there's not the presumption that one has lost one's bet against the condition, unless you dig out the evidence to convince your co-bettor otherwise. Is that the difference you see?

SOCRATES: Precisely. In a conditional bet, the condition's holding can be an assumption, defeasible but not requiring proof. And even when the presumption isn't there, a conditional bet avoids the whole issue of the probability of the condition. That cannot be avoided in the combination bet, where odds must be set on the negative bet, the bet against the condition holding, or even in the positive bet which says it does hold.

THEAETETUS: I agree to this difference between the two bets also, but again insist it's irrelevant to comparing their PROBEs. Are there other differences?

SOCRATES: Aren't these enough?

7. *A biased book argument for the principle of equivalence between conditional bets and combination bets. The first lemma.*

THEAETETUS: Very well. I admit I cannot prove the principle of equivalence, but neither can I do without it. So the situation calls for a biased book argument.

LAMPROCLES: Oh boy.

SOCRATES: What do you mean? Does the principle have the status of an axiom?

THEAETETUS: Yes. It's a fundamental assumption, although not as fundamental as the PROBE principle, which made the biased book argument for our axiom possible. It also depends on the Principle of Indifference,[6] because the concept of a fair PROBE occurs in it. So it's entirely normative, and not a statement of our pre-normative psychological theory.

LAMPROCLES: Show me how to set up the bets, please.

THEAETETUS: Suppose there's a person willing to wager both a conditional bet and a combination of unconditional bets concerned with the propositions A and B. Suppose also that the unconditional bets fit conditions I and II.

SOCRATES: Already you've become unrealistic. Of the two alternative ways of betting on A with B in the picture, he may prefer one to the other.

THEAETETUS: So much the better, Socrates. And let him be foolish enough to let his preference dictate different PROBEs for the two. This person's fair PROBEs in the two situations differ from one another. We distinguish two cases:

FAIR PROBE(A|B) < FAIR PROBE(A&B, NOT-B)
FAIR PROBE(A|B) > FAIR PROBE(A&B, NOT-B)

Are we agreed that he violates the principle of the normative equivalence of the two kinds of bets?

LAMPROCLES: Yes.

THEAETETUS: Now let's punish him with a biased book. Consider the first case, and for the sake of definiteness, let's say his PROBE(A|B) is 1/4 and his PROBE(A&B, NOT-B) is 1/3. The PROBE of his cobettor in the conditional bet, betting against A conditionally on B, must be 3/4.

LAMPROCLES: Why's that?

THEAETETUS: It's obvious from the rules of betting that PROBE(A|B) + PROBE(NOT-A|B) = 1. Think of 1 as the whole stakes; then the proportions of the stake that each bettor contributes must sum to the whole stakes.

[6] See near the end of the first dialogue; it's there called the corollary norm for indifference.

LAMPROCLES: Got it.

THEAETETUS: If our principle of equivalence is correct, the sum of PROBE(NOT-A|B) and PROBE(A&B, NOT-B) should be 1. But they sum to 1/12 more than 1, for they are complementary. Does the situation sound familiar?

SOCRATES: Very.

LAMPROCLES: Not very to me.

THEAETETUS: When the PROBEs of bets on mutually exclusive and jointly exhaustive alternatives don't sum to 1, we can always set up a biased book. If they sum to more than 1, we take the "against" side; when they sum to less, we take the "on" side.

LAMPROCLES: Got it.

THEAETETUS: I bet as I did at the race track when I found this aberration. When I found PROBEs of mutually exclusive, jointly exhaustive propositions summing to more than 1, I took a bet against each horse at the stated PROBE for each.

SOCRATES: Yes, I remember.

THEAETETUS: The key is always to look for the bets on situations which are mutually exclusive and jointly exhaustive, and sum up the PROBEs. That's what I do here, except I have to be clear about the sides I take.

LAMPROCLES: The PROBEs sum to more than 1. So you bet against.

THEAETETUS: A conditional bet against NOT-A is a conditional bet on A. So I bet on A, conditionally on B, at the PROBE of 1/4. Let's have total stakes of 1200. So my risk is 300; I lose it if A's false. My gain if I win is 900. Is that clear so far?

LAMPROCLES: [CALCULATING FURIOUSLY WITH PENCIL AND PAPER] Yes.

THEAETETUS: I bet "against" in the combination bet by reversing the sides I take in the two bets. I bet against A&B and against NOT-B, that is, on B.

LAMPROCLES: How do you know the other guy will take the bets?

THEAETETUS: That's where the concept of fairness comes in. He ought to feel neutral about which side to take in a bet he thinks fair. And our norms permit him to take either side of a fair bet. Remember, by "fair" we mean that he thinks the bet fair to both sides.

LAMPROCLES: OK.

THEAETETUS: Let's have total stakes in the combination bet be 1200 also. That settles the stakes of the two bets in the combination, 1200 for the bet against A&B, and 400 for the bet on B. For,

STAKES(A&B, NOT-B)=STAKES(A&B)=1200,

and the

PROBE(A&B, NOT-B) = STAKES(NOT-B)/STAKES(A&B)=1/3.

Now we use the equation proved from conditions I. and II.:

PROBE(A&B, NOT-B)=PROBE(A&B)/PROBE(B).

Recall that we made no use of the principle of equivalence in its proof. So there's no begging of the question of its truth.

SOCRATES: It's enough for now to have your assurance that no question begging assumptions are made.

THEAETETUS: Only provisos I. and II., Socrates. Therefore,

PROBE(A&B) / PROBE(B) =1/3.

Since I shall be betting against A&B, my cobettor's PROBE in this bet will be 1/3 of my PROBE in my bet on B. But what's my cobettor's PROBE? What odds will he take for this bet?

LAMPROCLES: I don't know.

THEAETETUS: Let's ask him. What odds do you want in your bet on A&B, Socrates?

SOCRATES: I can choose anything?

THEAETETUS: Yes, although we reserve the right to refuse.

SOCRATES: OK; 5 to 1.

THEAETETUS: Fine. My PROBE in my bet against A&B is 5/6. So my risk is 1000; I lose it if A&B's true. My gain if I win is only 200. Now we fill out my bet on B. By hypothesis, Socrates has already committed himself to a PROBE of 1/3 on the combination of the two bets. It follows my PROBE(B) is 1/2. So my potential loss is 200, as is my potential gain. Summing up:

THREE BETS	TOTAL STAKES OF BET	PROBE OF BET
BET(NOT(A&B))	1200	5/6
BET(B)	400	1/2
BET(A|B)	1200	1/4

Table 15: Description of bets forming a major part of the biased book.

Now let's see what the results are for each of the four possible cases.

LAMPROCLES: Let me do it.[7] First, suppose A&B's true. I win my bet on B, gaining 200. I lose my bet against A&B, losing 1000. I win my bet on A, conditionally on B, winning 900. Net of the three bets is a gain of 100.

THEAETETUS: Nice.

LAMPROCLES: Second, suppose B, but not A, is true. I win my bet on B, gaining 200. I win my bet against A&B, winning another 200. I lose my bet on A, conditionally on B, losing 300. Net of the three bets is a gain of 100.

THEAETETUS: Nice.

LAMPROCLES: Third, suppose A, but not B, is true. I lose my bet on B, losing 200. I win my bet against A&B, winning my 200. My bet on A, conditionally on B, is called off. Net of the three bets is a 0.

THEAETETUS: That's correct. We'll come back to this 0 in a jiffy.

LAMPROCLES: Fourth, suppose neither A nor B is true. I lose my bet on B, losing 200. I win my bet against A&B, winning 200. My bet on A, conditionally on B, is called off. Net of the three bets is again 0.

CASE:	PAYOFFS OF BET(NOT(A&B)),	OF BET(B),	OF BET(A|B),		NET OF THREE
A&B:	–1000	+200	+900	=	+100
NOT-A,&B:	+200	+200	–300	=	+100
A&NOT-B:	+200	–200	off	=	0
NOT-A,&NOT-B:	+200	–200	off	=	0

Table 16: Payoffs of bets made when PROBE(NOT-A|B)+PROBE(A&B,NOT-B)>1.

THEAETETUS: Note that so far we've guaranteed that when B's true, as it is in the first two

[7] Summarized in table 16.

cases, we win 100. When B's false, we don't lose. That result's good as it is. But it isn't a biased book, a guaranteed gain in every case. One hedge bet against B at the stated odds of 1 to 1 with its total stakes not to exceed our net gain on the first three bets, however, does the trick.

LAMPROCLES: Let 100 be the total stakes for the fourth bet. If B's true, I lose 50. But I net 100 on my first three bets. So I still net 50. If B's false, I win 50. My other three bets net 0 in this case. So my net's 50 again. I win 50 no matter what happens. It's a sure thing!

8. *Second lemma of this biased book argument.*

THEAETETUS: Yes. We've only come half way, however. We've considered the person whose fair PROBE(A|B) < fair PROBE(A&B, NOT-B). Now we construct a biased book against the person whose fair PROBE(A|B) > fair PROBE(A&B, NOT-B). For the sake of definiteness, let's say PROBE(A|B)=1/2 and PROBE(A&B, NOT-B)=1/6. If the PROBEs ought to be equivalent, as we say in the principle of equivalence, then PROBE(NOT-A|B) and PROBE(A&B, NOT-B) should sum to 1. But they only sum to 2/3. A familiar situation?

SOCRATES: Yes.

LAMPROCLES: We bet "on" each at the stated PROBEs.

THEAETETUS: Correct, remembering that a bet on NOT-A, conditionally on B, is a bet against A, conditionally on B. Let the stakes again be 1200. So if A's false, I win 600, provided B's true; if A's true, I lose 600, provided B's true. If B's false, the conditional bet's called off. Lamprocles, do you want to construct the combination bets?

LAMPROCLES: OK. We set the same stakes for the combination bet as for the conditional bet. The stakes for the combination bet equal the stakes in the bet on A&B. So my stakes there are 1200 also, and so the stakes for the bet against B must be 200, by the rule

PROBE(A&B, NOT-B)=STAKES(B)/STAKES(A&B).

THEAETETUS: Good memory, Lamprocles.

LAMPROCLES: I'm wise where money matters, Theaetetus.

THEAETETUS: Bad memory, Lamprocles.

LAMPROCLES: OK, OK, so I lost a lot of money; so I realize I've got a lot to learn.

SOCRATES: The beginning of wisdom . . .

LAMPROCLES: My PROBEs for the bets in the combination, oh yeah . . . Here I have to query my victim.

SOCRATES: Happy to oblige. If I'm to bet against A&B, I shall require even odds.

LAMPROCLES: So again I win 600 if A&B is true, lose 600 if it's false. By hypothesis, Dad takes the PROBE(A&B, NOT-B) to be 1/6. So by Theaetetus's theorem, based on conditions I. and II., 1/6 PROBE(B)=1/2. So PROBE(B) is 3.

THEAETETUS: Which is impossible. Your father's trying to pull a fast one. After your victim gives you odds against A&B, you must test the bets for disadvantage according to condition I. The PROBE(A&B)'s never greater than PROBE(A&B, NOT-B), for then PROBE(B) would exceed 1, which is impossible. Such absurdities indicate your victim is subject to an even simpler biased book.

LAMPROCLES: Come clean, Dad, or I'll get you on another set of bets.

SOCRATES: OK; 5 to 1 against A&B.

LAMPROCLES: So I win 1000 if A&B is true, and lose 200 if it's false. PROBE(B) is 1.

Since I'm betting against B, I have no risk at all. Is that allowed?

THEAETETUS: It's not illegal, just foolish of your father.

LAMPROCLES: Thanks, Dad. The stakes in this bet are 200, which I win if B's false. If B's true I suffer no loss, thanks to Dad's poor choice of PROBEs.

SOCRATES: Lay off, Son.

THREE BETS	TOTAL STAKES OF BET	PROBE OF BET
BET(A&B)	1200	1/6
BET(NOT-B)	200	0
BET(A|B)	1200	1/2

CASE: PAYOFFS OF	BET(A&B),	OF BET(NOT-B),	OF BET(NOT-A |B),	NET OF THREE
A&B	+1000	0	-600	= +400
NOT-A,&B	–200	0	+600	= +400
A&NOT-B	–200	+200	off	= 0
NOT-A,&NOT-B	–200	+200	off	= 0

Table 17: Description of major bets in biased book when
PROBE(NOT-A|B)+PROBE(A&B, NOT-B) < 1.
Their payoffs are in the lower part of the table.

LAMPROCLES: Now to the four cases. First, suppose A&B's true. I lose my conditional bet, 600. I win my bet on A&B, 1000. I lose nothing in my bet against B. The net gain is 400. Second, suppose not A, but B's true. I win my conditional bet, gaining 600. I lose my bet on A&B, losing 200. I lose nothing in my bet against B. The net gain is again 400. Third, suppose A, but not B's true. My conditional bet's called off. I lose my bet on A&B, losing 200. But I gain 200 on my bet against B. So the net in this case is 0. Fourth, neither A nor B's true. The net's again 0. I win 400 if B is true, nothing if it's false.

THEAETETUS: Excellent.

LAMPROCLES: Now I construct the hedge bet against B with the total stakes not to exceed our net winnings. Dad, I want to bet against B with you, with 400 as our stakes. I know how sure you are of B's truth, and I'm prepared to give you concessionary odds you won't refuse. At even odds, for example, if B is true, I lose 200, but still keep 200 of my winnings on the first three bets. If B is false, I win 200, to compensate for the net of 0 on my first three bets.

THEAETETUS: Excellent, Lamprocles. We've shown that conformity to the principle of the equivalence of conditional and combination bets is indispensable to avoiding the squeeze play of a biased book. Now let's show that conformity suffices to prevent a squeeze play. When PROBEs of the combination bet and the conditional bet we think fair are the same, the sum of the PROBE of the first and the PROBE of the complement of the second must sum to 1 exactly. The person will therefore sense disadvantage in at least one of the PROBEs needed for a book biased against him.

LAMPROCLES: According to the PROBE principle?

THEAETETUS: You got it. Socrates, are you convinced that the principle of equivalence is true?

SOCRATES: Yes, indeed, Theaetetus. I concede your principle of equivalence.

9. *Bringing degrees of belief back into the picture.*

THEAETETUS: Good. But before I make use of it, recall our results connecting the combination bet's PROBE to PROBEs and probabilities:

IF BET(A&B) AND BET(NOT-B) FIT CONDITIONS I. AND II., THEN
PROBE(A&B, NOT-B) = PROBE(A&B)/PROBE(B)
PROBE(A&B, NOT-B) # prob(A&B)/prob(B)
if PROBE(B) … 0.

I suppose you still want its proof confined to the notes?

LAMPROCLES: [SHOWS SOCRATES APPENDIX, THEOREMS XVII AND XVIII.]

SOCRATES: Yes, please. All very nice. So what?

THEAETETUS: So it's true of the conditional bet also. That's what the principle of equivalence enables us to do, namely, extend these results to conditional bets. We've proven these:

THE MAIN RESULTS

IF BET(A&B) AND BET(NOT-B) FIT CONDITIONS I. AND II., AND
PROBE(A&B, NOT-B) IS DEEMED FAIR,

THEN PROBE(A|B) OUGHT TO BE DEEMED FAIR
IF AND ONLY IF:
PROBE(A|B)=PROBE(A&B)/PROBE(B)
PROBE(A|B)=prob(A&B)/prob(B)
IF PROBE(B) … 0.

Isn't that music?

SOCRATES: The result's very elegant.

10. *Recipe for constructing this sort of biased book.*

LAMPROCLES: Dad, I want to make sure I got this straight.

SOCRATES: Sure, Son.

LAMPROCLES: I mean, down cold, so I never forget it.

SOCRATES: Yes, of course.

LAMPROCLES: I'm going to make a chart, and you tell me if I do it right. OK?

SOCRATES: OK.

LAMPROCLES: First, there's something useless about the PROBE of a combination of bets. Nobody talks that way. I'd like to avoid it if I can.

THEAETETUS: Sure. Just talk about the three PROBEs, the bet on A, conditionally on B, the unconditional bet on both A and B, and the unconditional bet on B.

LAMPROCLES: Right. And if I get PROBEs on these three bets that people are willing to accept as fair, so that they'll take either side of the bet, the first thing I do is check if the PROBE of the bet on B is not less than the PROBE of the bet on both A and B. If it is, I have a biased book right there, without considering the conditional bet.

THEAETETUS: Right.

LAMPROCLES: If the PROBE of the bet on A and B together is not greater than the PROBE of the bet on B alone, then I divide the first PROBE by the second one. If the result of the division equals the PROBE of the conditional bet, I get no action.

THEAETETUS: Why? Are you only betting on sure things?

LAMPROCLES: Yeah. If the result of the division is greater or less than the PROBE of the conditional bet, then I set up the bets in one of these ways:

IF PROBE(A|B) < PROBE(A&B)/PROBE(B), THEN I THINK THAT THE SUM:
PROBE(NOT-A|B)+[PROBE(A&B)/PROBE(B)] > 1, AND SO I BET "AGAINST."

I MAKE THESE BETS:

BET ON A, CONDITIONALLY ON B, AT STAKES S
BET AGAINST A&B TOGETHER, AT STAKES S
BET ON B, AT STAKES: S[PROBE(A&B)/PROBE(B)]
BET AGAINST B, AT STAKES: AT MOST, NET WINNINGS OF FIRST THREE BETS.

IF PROBE(A|B) > PROBE(A&B)/PROBE(B), THEN I THINK THAT THE SUM:
PROBE(NOT-A|B)+[PROBE(A&B)/PROBE(B)] < 1, AND SO I BET "ON."

I MAKE THESE BETS:

BET AGAINST A, CONDITIONALLY ON B, AT STAKES S
BET ON A&B TOGETHER, AT STAKES S
BET AGAINST B, AT STAKES: S[PROBE(A&B)/PROBE(B)]
BET AGAINST B, AT STAKES: AT MOST, NET WINNINGS OF FIRST THREE BETS.

Table 18: The two alternative biased books.

How's that look, Dad?

SOCRATES: Good. Let's run a bunch of numbers through. The PROBEs are 2/5 on A, conditionally on B, 3/5 on A&B, and 4/5 on B. What do you do?

LAMPROCLES: OK. That's the first case, because PROBE(NOT-A|B) is 3/5, and PROBE(A&B, NOT-B) is 3/4, summing to 7/20 over 1. I bet the "against" side in each, which is: on A, conditionally on B, on NOT(A&B), and on B. Set stakes at 100 in the first two bets, forcing 75 in the third. If A&B, I win the first, collecting 60; lose the second, giving up 60; win the third, collecting 15. Net: 15. If not-A but B, I lose the first, giving up 40; win the second, collecting 40; win the third, collecting 15. Net: 15. If not-B, the first bet is called off; lose the second, giving up 60; win the third, collecting 60. Net: 0. Having worked that out in advance I make a supplementary bet against B at stakes of, oh say, 7 1/2. If B, I give up 1 1/2 of the 15 I netted on the first three bets, now 13 1/2. If not-B, I win 6. I always win.

SOCRATES: Lamprocles!

LAMPROCLES: Yeah, Dad. I can do it. And, Dad, I'm gonna make you real proud of me.

SOCRATES: How, Son?

LAMPROCLES: I'm gonna become the best goddam hustler in Greece.

SOCRATES: It's times like these that bring tears to a man's eyes.

LAMPROCLES: I'm a man of high ambition, Dad. Now for some action. [RISES TO LEAVE.]

THEAETETUS: Here, you're forgetting your pad and pencil. Who ever heard of a bookie without pad and pencil.

LAMPROCLES: My theorems! Thanks. [TUCKS PENCIL BEHIND HIS EAR; LEAVES.]

[A Person's Fair Conditional PROBE Equals his Conditional Probability]
[In Three Parts]

1. *Does a person find a conditional bet advantageous or not by comparing its PROBE with the degree of something in his mind?*

THEAETETUS: Good luck, Lamprocles. I hope you don't need it.

SOCRATES: Lamprocles?

THEAETETUS: Oh! The moon's just appeared in the window; how bright it is! But I prefer torchlight for looking at you, Socrates; reflected moonlight on your face is, well, ghastly. How late's it gotten? . . . The crickets have stopped singing, Socrates, and so have the tree leaves. Our company's reduced to moonlight and silence. . . . Socrates? [AN OWL HOOTS OUTSIDE.]

THEAETETUS: Are you shivering, Socrates?

SOCRATES: It's nothing. I was just thinking.

THEAETETUS: Think about this. A conditional bet that's not disadvantageous has a PROBE(A|B) # prob(A&B)/prob(B), if prob(B)>0, and we know, of any conditional bet that's fair for a person, that:[8]

FAIR PROBE(A|B) = prob(A&B)/prob(B), if prob(B)>0.

The observable is on the left side of the equality, the mental on the right.

SOCRATES: That's curious. We've defined a person's sense of fairness as the outcome of a comparison between a PROBE of a bet on a proposition and a degree of belief in that proposition.[9] Is that the case also with a person's sense of the fairness of a conditional bet? Is its PROBE compared to a degree of belief in a proposition?

THEAETETUS: Are you asking whether a person's sense of the fairness of a conditional bet is the outcome of his mental comparison of the bet's PROBE to a degree of *one belief*?

SOCRATES: Yes. If we generalize our way of thinking of fairness to the fairness of a conditional bet, it'd be a comparison of one PROBE to one mental thing, so that when the PROBE was judged fair, it'd be a shadow cast by that one thing equal to it in length, I mean a manifestation of it in behavior. What's that one thing?

THEAETETUS: I see what you're driving at. Underlying our notion of the way people sense

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[8]See Appendix, theorem XIX

[9]See the first dialogue.

disadvantage and fairness of PROBEs in ordinary bets is the assumption that they compare them to their degrees of belief. To make a long story short, I don't think that's the way it works for conditional bets. Oh, yes, there is a degree of something mental, but it's not a belief of one proposition. Whatever it is, let's symbolize the degree of it thus: prob(A|B). This is our first use of "|" in a nonbehavioral context.

2. *Introducing the subjectivist concept of conditional probability.*

SOCRATES: I recall. Now we have a symbol for an entity, or pair of entities, that may or may not exist, mental items we hypothesize underlie people's sense of fair PROBEs of conditional bets. Were I the namer, I'd choose a big question mark, but I accept the symbol we've been using with PROBE, STAKES, and BET, the behavioral stuff. I don't know how to pronounce it, though.

THEAETETUS: I pronounce "prob(A|B)" as "my degree of whachumucallit that A conditionally on B" or just "my probability of A conditionally on B," supposing prob(*) is my mind now. Maybe also "the degree to which B, if true, would probabilify A for me."

SOCRATES: Can I also pronounce the vertical line as "if"?

THEAETETUS: Sure. Of course *then* you're going to read it as a degree of belief in the proposition "A if B" and give me an argument over my demurral about its object being a single proposition. But let's postpone that debate until we have the mathematical properties of this concept straight. The whole symbol stands for a "conditional probability," let's say, however we analyze what's inside the parentheses. I propose this definition of it:

if prob ... 0, prob(A|B)=prob(A&B)/prob(B).
Prob(A|B) is undefined when prob(B)=0.

SOCRATES: I prefer that we not presuppose its existence. By defining it as a ratio which exists when its denominator does, you define this entity in a way which begs the question of its existence or else evades the question of its representing a mental state.

THEAETETUS: How so?

SOCRATES: The definition is reductive. We concede that the defining terms are names of the degrees of two beliefs. If the ratio of those two degrees is the whole story about prob(A|B), well we can forget about new discoveries of what's in the mind; it's made of the same old stuff. Can you make this entity more intriguing?

THEAETETUS: No problem. I suggest we only use a normative principle in our reasoning:

NORM FOR CONDITIONAL PROBABILITY—prob(A|B)

if prob(A|B) exists, then it ought to be that
prob(A|B) = prob(A&B)/prob(B), and
prob(A&B) # prob(B).

SOCRATES: OK. If prob(A|B) doesn't exist if prob(B)=0, then your equation can't be wrong if your norm isn't.

THEAETETUS: Now there's a mouthful of "ifs."

SOCRATES: At any rate, the norm doesn't preclude the possibility that the equality sign only relates three distinct mental entities, prob(A&B), prob(B), and a third thing, prob(A|B). It doesn't *identify* the third thing with the ratio of the first two, as being nothing but that ratio, but says only that it ought to be the same *quantity* as the ratio. As an analogy to what I mean, consider the fraction, 1/10 of a drachma. A quantity of value is equal to that fraction, but the fraction does not define into existence a coin of that value. Do you agree?

THEAETETUS: I do, Socrates. Just as there's no coin equal to the fraction, so there may not be any one mental thing equal to the ratio of two mental things, prob(A&B)/prob(B). Our indifference principle of fair PROBEs makes the same distinction, for a fair PROBE equals a degree of belief without itself being a degree of belief. It's just the portion of the stakes which the bettor risks losing. It's not mental at all. Now here's that sort of equation again:

FAIR PROBE(A|B) = prob(A|B)

SOCRATES: It seems that prob(*) and prob(*|*) may be very different things.

THEAETETUS: In their interpretations, yes. Formally, however, they must have the same properties.

SOCRATES: What are they?

THEAETETUS: Just as we proved any rational prob(*) obeyed the axiom of probability, we must prove that any rational prob(*|*) also obeys it. In other words, consider a set of mutually exclusive and jointly exhaustive propositions, together with some one condition. The probabilities of each proposition, conditional on that one condition, must sum to the maximum. It's not difficult to show that conditional probabilities do conform to the axiom of probability. I've written the proof down.[10]

3. *We calculate the probability of conjunctions of propositions by using conditional probability.*

SOCRATES: Then, my fellow night-owl, you must tell me more about this intriguing concept before we decide whether it's a concept of a degree of belief in some unitary thing, or perhaps it's just a useful fiction. What intrigues you?

THEAETETUS: It's got to connect to a real mental thing, and I'll argue for that. First of all, let's assume that if prob(A|B) exists, then prob(B)>0. That way there's a connection between the norm and the definition I proposed. Then I derive from my definition, or the norm, a law:

THE NORM FOR THE PROBABILITY OF THE CONJUNCTION OF TWO PROPOSITIONS

IF prob(A|B) AND prob(B|A) EXIST,
prob(A&B) = prob(A)prob(B|A) = prob(B)prob(A|B)

SOCRATES: I see that's an immediate consequence of the norm for conditional probability, together with the equivalence of (A&B) and (B&A). In fact, the two norms are derivable from each

[10]See Appendix, Theorem XX.

other. They express the same proposition, the same norm.

THEAETETUS: That's right. Can you see how to generalize this norm to conjunctions of any number of propositions? Here it is:

THE GENERAL PRODUCT LAW

IF THE CONDITIONAL PROBABILITIES EXIST, THEN
THE PROBABILITY OF A CONJUNCTION OF PROPOSITIONS IS THE PRODUCT OF
THE PROBABILITY OF THE FIRST,
THE PROBABILITY OF THE SECOND CONDITIONALLY ON THE FIRST,
THE PROBABILITY OF THE THIRD CONDITIONALLY ON THE CONJUNCTION OF THE FIRST TWO,
THE PROBABILITY OF THE FOURTH CONDITIONALLY ON THE CONJUNCTION OF THE FIRST THREE,
AND SO ON.

SOCRATES: I don't see the proof of the generalization, but don't give it now.

THEAETETUS: Oh, it's too simple not to give it. We use the rule of replacement of proposition letters, since the law holds true for all propositions and not just the ones we use in our illustration of the law. We use the rule recursively, that is, over and over. For example, we replace "A" with "A&B," "B" with "C" in the norm for two propositions thus. The first line is the original one, the second shows the replacements:

if prob(A)>0, prob(A &B) = prob(A)prob(B | A)
if prob(A&B)>0, prob(A&B&C) = prob(A&B)prob(C|A&B)

We note that if prob(A&B)>0, then prob(A)>0. So we may use the first equation to make a substitution of equals for equals in the second equation thus:

if prob(A&B)>0, prob(A&B&C)=prob(A)prob(B|A)prob(C|A&B).

If you want the case of four propositions, replace letter "A" with "A&B" again, "B" with "C," and "C" with "D" in this equation thus:

if prob(A&B&C)>0, prob(A&B&C&D)=prob(A&B)prob(C|A&B)prob(D|A&B&C).

Since prob(A)>0 if prob(A&B&C)>0, we may again make that substitution of equals for equals in this equation and get:

if prob(A&B&C)>0, prob(A&B&C&D)=prob(A)prob(B|A)prob(C|A&B)prob(D|A&B&C) . . .

SOCRATES: Theaetetus, I see we can repeat this procedure endlessly, for conjunctions of any number of propositions.[11] But, tell me, what do you find intriguing?

[The Mind Detects Degrees of Synchronic and Diachronic Relevance Between Propositions]
[In Fifteen Parts]

1. *When two conjuncts are mutually irrelevant, the product law has a simpler form.*

THEAETETUS: The intriguing thing is that this product law can be simplified in some cases, but not in others. Sometimes the probability of a conjunction is just the product of the probabilities of its conjuncts, without conditional probabilities ever entering the calculation.

SOCRATES: How can that be? Are there exceptions to the product law?

THEAETETUS: You know there can't be exceptions to any theorems, Socrates! But sometimes the probability of one conjunct conditionally on another is no different from its unconditional probability. In that case it makes no difference which you use to compute the probability of the conjunction.

SOCRATES: Give me an example.

THEAETETUS: OK. I'll throw these two dice under cover so that you can't see the numbers. There. [DICE HIDDEN FROM SOCRATES'S VIEW] Tell me the probability of the conjunction: 6 on the pink die, and 4 on the yellow die.

SOCRATES: The product law tells me the probability of the conjunction of two propositions is the product of the probability of one conjunct and the probability of the other conjunct conditionally on the first.

THEAETETUS: Correct.

SOCRATES: But isn't the probability 1 or 0, since you've already rolled? Let's peak.

THEAETETUS: No, no, no. A probability is someone's degree of belief at a time, given what he knows and fails to know just then. So you tell me, knowing only what you know now, what your degree of belief is that there's a 6 on the pink and a 4 on the yellow.

SOCRATES: My probability of a 6 on the pink die is 1/6. So is my probability of a 4 on the yellow die. But the product law doesn't say I should multiply the two together.

THEAETETUS: No, it doesn't.

SOCRATES: If I start my calculation with the probability of 6 on the pink die, to complete the multiplication I need the probability of 4 on the yellow die, conditionally on the 6 appearing on the pink die. Then I multiply the one's unconditional probability by the other's conditional probability.

THEAETETUS: Correct. Now what's your probability of a 4 on the yellow die, conditionally on the 6 appearing on the pink die?

SOCRATES: Are you asking me to calculate it?

THEAETETUS: No. I'm asking you to give it, so that you can calculate the probability of a conjunction.

SOCRATES: I see. It's a question of which comes first in a process of calculation. If I calculate the probability of a conjunction, then I already have the conditional probability.

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[11]For another proof by mathematical induction, see Appendix to Dialogue I, Theorem XI.

THEAETETUS: Right.

SOCRATES: I see a reason why prob(A|B) should be interpretable independently of its equaling the fraction prob(A&B)/prob(B) when prob(B)>0. We use it to compute prob(A&B). So I must have it to begin with.

THEAETETUS: Yes, Socrates. That's one part of the idea. Prob(A|B) must be some mental unit, if it's ever to be at the origin of a calculation of a probability of a conjunction.

2. *How degrees of belief in propositions affect their degree of relevance to each other.*

SOCRATES: What then is my probability of a 4 on the yellow die conditionally on a 6 appearing on the pink one? Theaetetus, try as I may, I can't see any *relevance* of what appears on the one die to what appears on the other.

THEAETETUS: That's exactly right, Socrates. What you believe about the pink die's number doesn't make a bit of difference to what you believe about the yellow die's. The probability of a 4 on the yellow die is 1/6, if a 6 appears on the pink one and also if a 6 does not appear on the pink one. It's just 1/6. So here's a case where the probability of the conjunction is just the product of the probabilities of its conjuncts.

SOCRATES: The probability of a 4 on the yellow and a 6 on the pink is 1/6 x 1/6, or 1/36. I think that was obvious all along. I only made a pretence of calculating it.

THEAETETUS: Yes, Socrates. But don't you see what the product law told you about that degree of belief in a conjunction? It presupposes a judgment of irrelevance in this particular case. You couldn't see that 1/36 was the conjunction's probability unless you tacitly assumed the irrelevance of one die's number to the other die's. To reinforce my point, consider an example where you make an assumption of relevance, not irrelevance. Earlier you wanted to bet that Lamprocles would make his point the hard way, conditionally on his establishing a point. Let's consider the two propositions: He establishes his point, and he makes his point the hard way. Do you feel that establishing a point and making it the hard way have no relevance to one another?

SOCRATES: No. I admit to relevance in this case.

THEAETETUS: Good. To positive or negative relevance?

SOCRATES: What do you mean by that?

THEAETETUS: Would it come to seem *more* likely he'd make the point the hard way, if you were to come to believe he'd established a point, than if you had not come to believe he had, but you'd just persisted in your current opinions? Or would it come to seem *less* likely? Surely the situation's unlike the case of a single throw of a pair of dice. If in craps you were to come to believe Lamprocles won or lost on his first throw, your degree of belief in his making any point the hard way must drop from its previous level. That's negative relevance. If you were to come to believe he established 5 or 9 as his point, that too has a negative relevance, for they cannot be made the hard way. But if you thought he established 4, 6, 8, or 10, that's better than the other alternatives. And if you thought he established a 6 or 8, that's less positively relevant than if he'd established a 4 or 10, because there are more of the other ways of making a 6 or 8 than there are of making the 4 or 10.

SOCRATES: Maybe there's a simpler illustration of positive and negative relevance?

THEAETETUS: OK. [PRODUCES A DECK OF FIFTY TWO PLAYING CARDS.] Half these cards are red; the other half black. I deal two face down. What's the probability of this one being black? [POINTS TO ONE OF THE DEALT CARDS, STILL FACE DOWN.]

SOCRATES: My degree of belief, while it's face down: 1/2.

THEAETETUS: Would it make any difference to your degree of belief if I turned the *other* card face up?

SOCRATES: I think so. I'd come to know which color it was, and that would affect my degree of belief that the remaining card was black. For example, if the card you turned up were black, then the probability of the remaining card being black would be the proportion of black cards among the fifty one remaining cards whose color I don't know. So the probability of the card being black, conditionally on the other card being black is 25/51. That's less than 1/2.

THEAETETUS: So one card's being black is negatively relevant to the remaining one's being black.

SOCRATES: Ah, so!

THEAETETUS: Just so.

SOCRATES: On the other hand, it might turn out to be red. In that case the remaining card has a 26/51 chance of being black, for that'd be the proportion of black cards among the remaining 51 cards. The fraction 26/51 is greater than 1/2. So the card's being discovered to be red is *positively* relevant to the remaining card's being black. Yes?

THEAETETUS: Yes. With all this talk about what's yet to happen, we may be obscuring a feature of the relevance I'm talking about. It's *synchronic* relevance. I mean, it's the relevance relationships between propositions within the mind of a person at one moment, within the function prob(*). In our earlier discussions, we used prob(*) rather loosely; let's start using it only for those people whose degrees of belief are coherent, in that they obey the axiom of probability. If they do, prob(*) will be a *function* in the strict sense that for any proposition it dictates exactly one degree of belief. But that function is just one of an infinity of such prob(*)-functions, descriptive of other minds or of yours at other moments.

SOCRATES: Ah, but which is the right one?

THEAETETUS: They're all legitimate, because each one conforms to the axiom of probability. They differ in how well informed they are about objective chance, but they're all equally rational. We could introduce some of them and distinguish them from the prob(*) that describes the current mind of the person we're discussing, by using prime marks: prob'(*) and prob''(*). Am I clear?

SOCRATES: Clear enough, but there is just one that is right. Just one prob(*)-function depicts the maximally knowledgeable person, the God, at all times. As for ignorant me, there's prob(*), which is me now, provided I'm currently rational, and prob'(*), which is me at another time, provided I'm rational then. I shouldn't conflate what I think about a proposition now with what I thought of it before or will think of it later. The synchronicity you stressed requires our discussion staying inside prob(*); don't mix it with prob'(*).

THEAETETUS: That's it. Let that suffice for now. Returning to the cards, even if these propositions say what will happen or what has already happened, we're talking about the relationship these propositions have to other propositions now, given just what the person believes now.

SOCRATES: But, if my reasoning about the colors of the two cards was correct, it's OK to think of what I *now* believe would be the effects of my next step in learning, if there is to be one, and that gives me my current synchronic relevance relations. Is that right?

THEAETETUS: Wait; you rush ahead. Forget about the cards. I can define the kinds of relevance in terms of probability thus:

DEFINITIONS OF SYNCHRONIC RELEVANCE, APPLICABLE TO TWO PROPOSITIONS

A IS POSITIVELY S-RELEVANT TO B,	IF AND ONLY IF prob(B|A) > prob(B)
A IS NEGATIVELY S-RELEVANT TO B,	IF AND ONLY IF prob(B|A) < prob(B)
A IS S-RELEVANT TO B,	IF AND ONLY IF A IS POSITIVELY OR NEGATIVELY S-RELEVANT TO B
A IS S-IRRELEVANT TO B,	IF AND ONLY IF EITHER prob(B|A)=prob(B), OR prob(A)=0

The product law can be simplified only when you make the judgment that the conjuncts are s-irrelevant to one another. We can leave the "s-" out, as long as we're clear that we're discussing synchronic relations, internal to one probability function.

3. *Irrelevance is reciprocal, so that the relation of synchronic irrelevance is symmetrical.*

SOCRATES: Now there's real definition, Theaetetus. You introduce concepts: positive and negative relevance. But one slight clarification, please. Is the relevance of one proposition to another mutual? I mean, is it reciprocated, or do we have unrequited relevance?

THEAETETUS: Requited, but first consider irrelevance. I'll be saying "irrelevant to one another" because, if one's irrelevant to the other, then the other's irrelevant to it. Irrelevance of one proposition to a second is always symmetrical, that is, mutual, as this proof shows:

Suppose A is irrelevant to B.
Either prob(B) = prob(B|A), or prob(A) = 0.

Suppose the first alternative.
prob(B) = prob(B&A)/prob(A)
prob(B)prob(A) = prob(B&A)
prob(B&A) = prob(A&B)
prob(B)prob(A) = prob(A&B)
Either prob(B) > 0, or prob(B) = 0
if the former, prob(A) = prob(A&B)/prob(B), which says
prob(A) = prob(A|B), i.e., B is irrelevant to A.
if the latter, then still B is irrelevant to A.

Now suppose the second alternative, prob(A) = 0.
prob(A&B) = 0
either prob(B) > 0, or prob(B) = 0.
if the former, prob(A&B)/prob(B) = 0.
prob(A) = prob(A|B), i.e., B is irrelevant to A.
if the latter, then still B is irrelevant to A.

Therefore on any alternative, B is irrelevant to A.

So, if A is irrelevant to B, B is irrelevant to A. QED

4. *The simpler form of the product law for two mutually irrelevant conjuncts, and corollaries.*

SOCRATES: I see how the product law for conjunctions of two propositions is simplified when the conjuncts are irrelevant to each other.

PRODUCT LAW FOR TWO CONJUNCTS IRRELEVANT TO ONE ANOTHER

If prob(B|A) = prob(B), then
prob(A&B) = prob(A)prob(B|A) = prob(A)prob(B).

THEAETETUS: Not only prob(A&B) = prob(A)prob(B) when A and B are irrelevant; we also have these three corollaries:

if prob(B|A) = prob(B), then
(i) prob(NOT-A,&B) = prob(NOT-A)prob(B)
(ii) prob(A&NOT-B) = prob(A)prob(NOT-B)
(iii) prob(NOT-A,&NOT-B) = prob(NOT-A)prob(NOT-B)

These derive from another property of irrelevance, namely, if A's irrelevant to B, then so's NOT-A. I prove this result thus:

Suppose A is irrelevant to B.
Either prob(B) = prob(B|A), or prob(A) = 0.

Suppose the first alternative.
prob(B) = prob(B&A)/prob(A)
prob(B) = prob(B&A) + prob(B&NOT-A)
prob(A) = 1 – prob(NOT-A)
[prob(B&A)+prob(B&NOT-A)][1–prob(NOT-A)]=prob(B&A)
. . . algebra[12] . . .
[prob(B&A)+prob(B&NOT-A)]prob(NOT-A)=prob(B&NOT-A)
prob(B) = prob(B|NOT-A), i.e., NOT-A is irrelevant to B.

Suppose the second alternative.
prob(NOT-A) = 1
prob(B&NOT-A) = prob(B)
prob(B) = prob(B|NOT-A), i.e., NOT-A is irrelevant to B.

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[12]Compare steps 5-7 of Theorem XXVIII in the Appendix.

Thus, on any alternative, NOT-A is irrelevant to B.
So if A is irrelevant to B, NOT-A is also. QED

SOCRATES: That proof leaves some of the algebra to me, but I understand it. You've proved two theorems about irrelevance: If A's irrelevant to B, then B's irrelevant to A, and if A's irrelevant to B, then NOT-A is too.

THEAETETUS: With those two I derived my three corollaries.

SOCRATES: I can see how you did.

5. *Do the product laws clue us in about what mental thing prob(A|B) represents?*

THEAETETUS: Now let's consider another reason why the conditional probability ought to be interpretable as signifying some mental unit.

SOCRATES: Another reason, besides the fact that you may need it to compute the probability of a conjunction?

THEAETETUS: Yes, Socrates. Even if you do know the probability of a conjunction of two propositions *without* computing it, nevertheless in knowing it, when you also know the probabilities of the conjuncts, you've made a judgment of either irrelevance or relevance of the conjuncts to each other. That judgment, you may now think, must be a comparison of a conditional probability to an unconditional one. So you cannot escape having a judgment in which a conditional probability figures.

SOCRATES: I like your argument, Theaetetus. To discriminate when the probability of a conjunction of two propositions is the product of the probabilities of the conjuncts, it's both sufficient and indispensable to judge whether the conjuncts are irrelevant to one another.

THEAETETUS: Yes, that's it. Suppose the two conjuncts are irrelevant to each other. Then either their probabilities are both positive, and their product's the probability of their conjunction, or they're not both positive, in which case their product's 0, which is also the probability of their conjunction. Now reverse direction. Suppose their product equals the probability of their conjunction. Then either both's probabilities are positive, in which case each's probability equals its probability conditionally on the other, or one's probability is 0, and it's irrelevant to the other. Either way each is irrelevant to the other.

SOCRATES: And conditional probabilities are indispensable in calculating relevance?

THEAETETUS: Did I say that? No; that's the illusion created by my definitions of relevance! My argument shows that we can define the relevances without new-fangled entities called conditional probabilities. Here's how:

ALTERNATE DEFINITIONS OF SYNCHRONIC RELEVANCE,
APPLICABLE TO TWO PROPOSITIONS

A IS POSITIVELY S-RELEVANT TO B, IF AND ONLY IF prob(B&A) > prob(B)prob(A)
A IS NEGATIVELY S-RELEVANT TO B, IF AND ONLY IF prob(B&A) < prob(B)prob(A)
A IS S-RELEVANT TO B, IF AND ONLY IF A IS POSITIVELY OR NEGATIVELY S-RELEVANT TO B
A IS S-IRRELEVANT TO B, IF AND ONLY IF prob(B&A) = prob(B)prob(A)

We could even define a measure of the degree of synchronic relevance this way:

$$\text{s-rel}(A|B) = \text{prob}(A\&B) - \text{prob}(A)\text{prob}(B).$$

So the clue from relevance to something new in the mind is this: The synchronic relevance between two propositions seems to be a matter of degrees of their conjunctivity. The purest conjunctivity is irrelevance.

6. *The definition of irrelevance can be generalized to cover any number of propositions; interchangeability.*

SOCRATES: You make it sound as if the only utility of the concept of conditional probability is to find irrelevancies in order to simplify the calculation of the probabilities of conjunctions.

THEAETETUS: Perhaps I have the biases of a calculator, Socrates. But then I use irrelevance for many other calculations, provided I generalize the idea.

SOCRATES: Speaking of generalizing, what of conjunctions of more than two propositions?

THEAETETUS: In cases of three propositions conjoined, the argument I just gave about having made a judgment of relevance doesn't work. A person might know the probabilities of each of the three, and also of their conjunction. But from that information about him we cannot deduce that he's judged them mutually irrelevant or relevant.

SOCRATES: That's odd.

THEAETETUS: Knowing that each proposition is irrelevant to each of the others is no longer sufficient. Look at this three circle diagram, Socrates. You can verify that each is irrelevant to each other. But the probability of the conjunction of the three is not the product of the probabilities of each:

Diagram 11: Three propositions, all pairs irrelevant.

SOCRATES: Each proposition's probability is .5; and so are their probabilities conditionally on each of the other two separately. So they're each irrelevant to each of the others. Yet the product of their probabilities is half the probability of their conjunction.

THEAETETUS: The general product law forewarned us of this eventuality, though. According to the law, we need an irrelevance of the conjunction of two to the third to calculate the probability of the conjunction directly from the probabilities of the three propositions, but each of them is relevant to the conjunction of the other two.

SOCRATES: Let's see; prob(A|B&C) is prob(A&B&C) divided by prob(B&C): 1/4 divided by 1/4, in other words, 1. The probability of A "rises"; positive relevance.

THEAETETUS: Yes. There are many irrelevancies that might hold between three propositions, but certain pairs are sufficient without the others for using their probabilities to calculate their conjunction's probability. For example, you need to judge the first two irrelevant to each other, and the conjunction of the first two irrelevant to the third, so that prob(B|A) = prob(B), and prob(C|A&B) = prob(C). Then prob(A&B&C) = prob(A)prob(B)prob(C). It's just straightforward substitution of equals for equals, with the judgments of irrelevance supplying the equalities.

SOCRATES: That's six irrelevancies among three propositions:

between A and B,
between A and C,
between B and C,
between A and B&C,
between B and A&C, and
between C and A&B.

To simplify the calculation of their conjunction's probability, you said you needed only the first and last irrelevance. But parallelism suggests that other combinations of two of them might also be sufficient, such as the second and fifth, or the third and fourth.

THEAETETUS: That's correct. But irrelevance is not indispensable in simplifying the calculation of conjunctions of three propositions, unlike conjunctions of two propositions. Here's another diagram that shows no irrelevancies at all. Yet the probability of the conjunction of the three propositions does equal the product of their probabilities.

Diagram 12: Irrelevancies are *not* indispensable for the probability of a conjunction equaling the product of probabilities of conjuncts.

Although prob(A&B&C) = prob(A)prob(B)prob(C), I can't replace the two occurrences of A in that equation with NOT-A's and still have a truth, nor replace the two B's with NOT-B's, nor the two C's with NOT-C's. I want that property desperately. At least I want to know the conditions under which I do have that property of interchangeability.

SOCRATES: [WHILE VERIFYING THEAETETUS'S CLAIMS ABOUT THE DIAGRAM USING THE CHECKLIST OF IRRELEVANCIES ABOVE] Desperation is the mother of invention. So go to it.

THEAETETUS: I invent a generalization of our definitions of irrelevance for any number of propositions.

DEFINITION OF COMPLETE SYNCHRONIC IRRELEVANCE, APPLICABLE TO N PROPOSITIONS

N PROPOSITIONS ARE COMPLETELY S-IRRELEVANT IF AND ONLY IF

EVERY ONE IS S-IRRELEVANT TO EVERY ONE OF THE OTHERS INDIVIDUALLY,
AND
EVERY CONJUNCTION OF TWO IS S-IRRELEVANT TO EVERY OTHER INDIVIDUALLY,
AND
EVERY CONJUNCTION OF THREE IS S-IRRELEVANT TO EVERY OTHER INDIVIDUALLY, AS WELL AS TO EVERY CONJUNCTION OF TWO OF THE OTHERS,
AND SO ON, UP TO
CONJUNCTIONS OF N–1 OF THEM, EACH OF WHICH IS S-IRRELEVANT TO THE REMAINING ONE.

With this definition I really fly, Socrates. Not only are the probabilities of conjunctions in sets of completely irrelevant propositions equal to the products of the probabilities of their conjuncts, but the equations hold true when the conjuncts are replaced by their negations. That property makes for some very satisfying theorems, Socrates. You better believe it!

SOCRATES: So the concept of irrelevance is emerging as the really fundamental idea, in the sense that the mathematical elaboration of the theory depends on it. Am I right?

THEAETETUS: I think so.

7. *Relevance is also mutual, so that the relations of positive and negative relevance are symmetrical.*

SOCRATES: I wonder whether you might not be missing philosophy by building mathematical castles on complete irrelevance, ignoring positive and negative relevance. Remember, we want to know if prob(A|B) can be interpreted as meaning some mental thing. I propose we concentrate on relevance and its interplay with irrelevance. For, before we conclude anything about the nature of prob(A|B) from the phenomenon of relevance, we had better be sure we understand that phenomenon correctly. I'm not yet satisfied we do.

THEAETETUS: The theorems based on relevance aren't nearly as interesting as those based on irrelevance, Socrates.

SOCRATES: Indulge an old man and poor student, Theaetetus. What's the hurry? May we not take some rest and survey the new sights your conditional probability opens to our view?

THEAETETUS: Alright; I'd love to, Socrates. I do have some theorems about relevance. The first's that both kinds of relevance, like irrelevance, are symmetrical. By symmetry I mean mutuality. If A's positively relevant to B, then B's positively relevant to A, if the conditional probabilities exist. The proof's quick and easy. Suppose A's positively relevant to B. Then:

prob(B|A) > prob(B),	by definition of positive relevance. So,
prob(B&A)/prob(A) > prob(B),	by the norm for prob(A|B). So,
prob(B&A) > prob(A)prob(B),	by algebra. But,
prob(A&B) = prob(B&A),	by logic. So we may substitute thus:
prob(A&B) > prob(A)prob(B).	Dividing this by prob(B),
if prob(B)>0, then prob(A&B)/prob(B) > prob(A).	But,
if prob(A|B) exists, then prob(A|B) = prob(A&B)/prob(B),	by norm. So,
prob(A|B) > prob(A),	by substitution of equals.

The last line says, if prob(A|B) exists, B is positively relevant to A. That follows from the premise that A is positively relevant to B. I can prove symmetry of negative relevance also.

8. *Just as probability is subjective, so also the relevance relations are subjective.*

SOCRATES: I want to see what this symmetry of relevance amounts to in the concrete. Deal me a card. [THEAETETUS PICKS UP THE TWO CARDS ON THE TABLE, SHUFFLES THE FULL DECK, AND DEALS SOCRATES ONE CARD *FACE UP*.] Thank you. What I see here is relevant to what I should expect if you deal me another card. This card's red. That's positively relevant to the next card's being black.

THEAETETUS: What?

SOCRATES: Sure. For the probability of a second card's being black, conditionally on the first's being red is 26/51, whereas the probability of the second card's being black, without regard for the color of the first, is 26/52.

THEAETETUS: Ah, well, no. How can you say the probability of the first card's being red is positively relevant to the second card's being black? In fact, it's irrelevant.

SOCRATES: How can that be? Surely withdrawing a red card from the deck increases the chance of drawing a black one next.

THEAETETUS: There's no doubt about that. But the question is what's synchronically relevant to what, according to our definition. Let me ask you, what's the probability of my having dealt you a red card just a minute ago?

SOCRATES: Why 1/2, of course.

THEAETETUS: Can you honestly tell me, as you look at that card in front of you, that very red card, that your degree of belief that it's red is only 1/2?

SOCRATES: Oh, by heaven, you've caught me. No. I'm certain you dealt me a red card. That means its probability is 1. Before you dealt me the card, before I saw it, my degree of belief that it would be a red one was 1/2. But not any longer.

THEAETETUS: With our conception of probability as a person's degree of belief at a particular time, we can't just mindlessly say things like the probability of having been dealt a red card is 1/2. It depends on the person, not on the constitution of the deck of cards.

SOCRATES: I can illustrate that point for you. Where are those dice you hid from me earlier? I gave them a probability of 1/36 of 6 on the pink and 4 on the yellow.

[THEAETETUS REVEALS THE PINK DIE. IT IS 6.]

SOCRATES: Whoops. My degree of belief in a 6 on the pink *and* a 4 on the yellow just jumped from 1/36 to 1/6.

[THEAETETUS REVEALS THE YELLOW DIE. IT IS 2.]

SOCRATES: Whoops. My degree of belief in the conjunction just took a dive to 0.

THEAETETUS: That's the idea. In the function notation I introduced earlier, first prob(*) described your degrees of belief, then prob'(*), and now prob''(*) does.

SOCRATES: Earlier I suggested something like this, but you stopped me and accused me of jumping ahead.

THEAETETUS: And rightly so. I mean no disrespect Sir, but you were ahead of yourself, as your first impressions of probabilities in this sequence just now were not even in accord with your suggestion. Here's another fact to take into account: Do you think any of your conditional probabilities change?

SOCRATES: Are my first impressions going to be wrong again?

THEAETETUS: If they say no, then they're wrong again. Let's go abstract and consider the first of our diagrams [11] where prob(B|C)=.5. If prob'(*) results from prob(*) by learning just A, and with certitude, then prob'(B|C)=1. So some conditional probabilities do change. In the second diagram [12], prob(B|C)=.06, but prob'(B|C)=.24.

9. *Our certitudes are synchronically irrelevant to all propositions, by virtue of their certitude.*

SOCRATES: Back to that red card. If prob'(*) symbolizes my degrees of belief before I saw the card, and prob"(*) symbolizes them after I saw it, then prob'(you deal me a red card) = 1/2, but prob"(you deal me a red card) = 1.

THEAETETUS: That's it exactly.

SOCRATES: But even so, why is my having been dealt a red card irrelevant to being dealt a black one next? This is strange: If you deal it down, it's relevant; if you deal it up, it's irrelevant!

THEAETETUS: What's the probability of drawing a black one next, I mean as a second card? Remember that I want your *current* degree of belief that you will draw a black one next.

SOCRATES: My current degree of belief that I'll draw a black one next is 26/51. I can't just mindlessly say the probability of a black card on the second draw is 1/2, can I?

THEAETETUS: No. Now what's your current degree of belief that you'll be dealt a black card next, conditionally on your having been dealt a red one first?

SOCRATES: My current degree of belief that I was dealt a red card first is going to be my denominator. That's 1. My numerator's my current degree of belief that I'm dealt a red card on the first AND a black card on the second. Do we have a theorem that if prob"(A)=1, then prob"(A&B)=prob"(B)?

THEAETETUS: Yes, we do.[13]

SOCRATES: Then my current degree of belief that I'm dealt a red card first AND a black card second is equal to my current degree of belief that I'll be dealt a black card next, which is 26/51. Dividing that by 1 changes nothing. Therefore, using my current degrees of belief as probabilities, the probability of a black card next is equal to the probability of a black card next conditionally on having been dealt a red card first.

THEAETETUS: Precisely. So your having been dealt a red card first is not now positively relevant to your drawing a black card next. It's irrelevant.

10. *Our uncertainty of a proposition is indispensable for its synchronic relevance, for us, to any proposition.*

SOCRATES: What? Doesn't that strike you as a counterintuitive result?

THEAETETUS: It's not counterintuitive. Take the symmetrical case. Your being dealt a

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[13]See the first Appendix to Dialogue I, Theorem XIV.

black card next is irrelevant to your having been dealt a red card *first*, isn't it?

SOCRATES: Sure. It can't make any difference to my current degree of belief that I've *already* been dealt a red card first, whatever I may be dealt next.

THEAETETUS: OK then. That's not counterintuitive, and the symmetry of irrelevance is not counterintuitive. So if the symmetrical case is contrary to your intuitions, so much the worse for your intuitions. Just to sharpen the point more, let's do things differently. [RETURNS SOCRATES'S RED CARD TO THE DECK AND SHUFFLES IT.] Do you think being dealt a red card first is positively relevant to being dealt a black one second?

SOCRATES: Things have changed, haven't they?

THEAETETUS: What's your current degree of belief that you're dealt a red card first? Is it still certitude?

SOCRATES: No. You haven't yet dealt a card for me to see. So now my degree of belief that it'll be red is . . . well I saw you shuffle the deck and I know half the cards are red, so 1/2. We're back to prob'(*). Once burnt, twice shy: I won't use my intuitions to decide whether getting a red first is now positively relevant to getting a black second; I'll calculate it. The probability of getting a black second conditionally on getting a red first, using my current degrees of belief, is 26/51. The probability of getting black second, using my current degrees of belief is . . . is what? Without knowing what I'll be getting first, I find it hard to say what my degree of belief in black second will be.

THEAETETUS: Here's how to think of your current belief. There are only two mutually exclusive routes to black second. One is red first AND black second; the other is black first AND black second. Your degree of belief in black second should be just the sum of your degrees of belief in each route to it. Using R_1 for red first, B_1 for black first, and B_2 for black second, the calculation is:

B_2 if and only if $((R_1\&B_2) \text{ OR } (B_1\&B_2))$, by logic. So,
$prob'(B_2) = prob'((R_1\&B_2) \text{ OR } (B_1\&B_2))$. by theorem VI. But,
$(R_1\&B_2)$ and $(B_1\&B_2)$ are mutually exclusive, so by theorem IV,

$$prob'(B_2) = prob'(R_1\&B_2) + prob'(B_1\&B_2)$$
$$prob'(R_1\&B_2) = prob'(R_1)prob'(B_2|R_1)$$
$$prob'(B_1\&B_2) = prob'(B_1)prob'(B_2|B_1)$$

Therefore,

$$prob'(B_2) = prob'(R_1)prob'(B_2|R_1) + prob'(B_1)prob'(B_2|B_1)$$

Use this last formula, since I'm sure your intuitions supply you all the numbers on the right side of the equality sign.

SOCRATES: $prob'(R_1)= 1/2$; $prob'(B_2|R_1)= 26/51$; $prob'(B_1)= 1/2$; $prob'(B_2|B_1)= 25/51$. Thus: $(1/2)(26/51) + (1/2)(25/51) = 51/102 = 1/2$. My current degree of belief, before I know what I'll get first, that I'll get a black card second is 1/2.

THEAETETUS: Precisely.

SOCRATES: But my current degree of belief that I'll get black second conditionally on getting red first is greater than that, 26/51. So getting red first is positively relevant to getting black second, but only before I get either. It's remarkable, Theaetetus, what certitudes do to relevance! All certitudes are irrelevancies, even if they were relevant before they became certain. You see?

11. *Synchronic relevancies are evanescent as we move from prob'(*) to prob''(*).*

THEAETETUS: Yes. If now I were to deal you a red card face up, then with your new degrees of belief, prob"(*), getting red first would be irrelevant to getting black second.

SOCRATES: It's a flimsy relevance that says my first card's color is or isn't relevant to my second card's, depending on whether it's dealt face down or face up, and it's an evanescent relevance that disappears as soon as I turn a face-down card face up. Honestly, you've got to feel suspicious.

THEAETETUS: Do you think that getting a black card second is positively relevant to getting a red one first?

SOCRATES: Do I believe that now, with my current degrees of belief? Uh, oh. I was about to say no, because how can the card you get second give you information about your first card, information that you don't already possess once the first's been dealt? But it hasn't been dealt!

THEAETETUS: Right.

SOCRATES: But it isn't normal to learn the color of your second card before you learn the color of your first card. My intuitions resist admitting the positive relevance of black on the second to red on the first. It's not a possible situation!

THEAETETUS: [HE OPENS THE DECK OF CARDS SO THAT SOCRATES SEES THE COLOR OF THE SECOND CARD DOWN FROM THE TOP OF THE DECK. IT IS BLACK. THEN HE DEALS SOCRATES HIS FIRST CARD FACE DOWN.] Should I shuffle the deck now before dealing you the second card? I'm inclined not to. That's a promise in fact. [HE DEALS SOCRATES HIS SECOND CARD FACE UP, THE ONE HE SAW.]

SOCRATES: You've caught me again. It's a possible situation to have information about my second card's color before I have it about my first card. But, admit it, it's quite unusual. And the information becomes irrelevant when I turn over this first card. [TURNS FIRST CARD FACE UP. IT IS RED.] I want to protest that if it's irrelevant now, it should've been deemed irrelevant even before I turned over the first card. Let me summarize what you've demonstrated to me, and my reactions:

BEFORE I SEE MY FIRST CARD IS RED; prob'(*):

- $prob'(R_1) = 1/2$
- $prob'(B_2) = 1/2$
- $prob'(B_2|R_1) = 26/51$
- $prob'(B_2|R_1) > prob'(B_2)$
- R_1 is positively relevant to B_2.
- B_2 is positively relevant to R_1 —counterintuitive?

AFTER I SEE MY FIRST CARD IS RED; prob"(*):

- $prob''(R_1) = 1$
- $prob''(B_2) = 26/51$
- $prob''(B_2|R_1) = 26/51$ — well, the degree of something stayed the same!
- $prob''(B_2|R_1) = prob''(B_2)$ i.e. what had been relevance is now irrelevance
- R_1 is irrelevant to B_2 —counterintuitive?
- B_2 is irrelevant to R_1.

Table 19: Do these results fit your intuitions?

If this is your position, you'll have to defend it more thoroughly. For I see now why I find the symmetries of relevance and irrelevance counterintuitive. I don't believe relevance and irrelevance are ephemeral relations between propositions. But your definitions make them ephemeral. Seeing my first card to be red sufficed to change the relation between two propositions from one of positive relevance to one of irrelevance. The ephemerality is itself counterintuitive. I think we can have a more stable relation of relevance if we don't define it in a way that implies its symmetry. So, I want, yes, a diachronic relevance relation! What have you to say to that, my sleepless wunderkind?

12. *Even within prob(*), taking a third proposition into account can put two propositions, by themselves irrelevant to one another, into a relation of relevance.*

THEAETETUS: I must admit the evanescence of synchronic relevance and irrelevance, Socrates. I have two models that show how knowledge of a third proposition can "change," speaking loosely, the synchronic relevance of two propositions to irrelevance, or vice versa, even within one prob(*):

Diagram 13: Two models; in the left synchronic relevance between A and B "disappears" if C is taken into account; in the right the synchronic irrelevance between A and B "disappears" if C is taken into account. (Of course, nothing is changing really.)

In the left model A and B are positively relevant to each other because the probability of their conjunction is .23, although each has a probability of .4, so the product of their probabilities is only .16.

SOCRATES: I see, if I ignore C. The probability of either "rises" to .575, given the other.

THEAETETUS: Yes. But if C is taken into account and no more, the probabilities of A and B are each seen to be .6, conditionally on C, but the probability of their conjunction, conditionally on C, is .36, and is equal to the product of their conditional probabilities, which means they're not conditionally relevant to each other.

SOCRATES: How do you know what the probabilities of A and B "become" relatively to each other when C is given?

THEAETETUS: Sorry. I mean $prob(A|C) = prob(B|C) = .6$, and $prob(A\&B|C) = .36 = prob(A|C)prob(B|C)$. I should define synchronic relevance and irrelevance of two propositions conditionally on a third. Given the definitions, this model shows that A and B, while positively relevant to each other absolutely, are irrelevant to each other conditionally on C:

DEFINITIONS OF CONDITIONAL SYNCHRONIC RELEVANCE

A AND B ARE

POSITIVELY S-RELEVANT TO EACH OTHER, CONDITIONALLY ON C,

IF AND ONLY IF $prob(A|B\&C) > prob(A|C)$;

NEGATIVELY S-RELEVANT TO EACH OTHER, CONDITIONALLY ON C,
IF AND ONLY IF prob(A|B&C) < prob(A|C);
S-RELEVANT TO EACH OTHER, CONDITIONALLY ON C,
IF AND ONLY IF EITHER POSITIVELY OR NEGATIVELY S-RELEVANT;
AND S-IRRELEVANT TO EACH OTHER CONDITIONALLY ON C,
IF AND ONLY IF EITHER prob(A|C) = 0 OR prob(A|B&C) = prob(A|C)

SOCRATES: Doubts about your definitions of synchronic relevance still bother me.

THEAETETUS: In the diagram's second model, the situation's reversed. Disregarding C, A and B are irrelevant to each other. Their probabilities are .3; the probability of their conjunction is .09, equal to the product of their probabilities. But taking C into account, they're seen to be conditionally negatively relevant to each other. I mean prob(A&B|C) < prob(A|C)prob(B|C). And, meaning no disrespect, Sir, but you must not use the language of change, like "rise" and "become." I gave you licence to do that, with my own loose talk, but, Sir, you want to put change where I don't, and you want to keep it out where I want to put it in. Our intuitions are clashing on two fronts.

SOCRATES: I'm afraid so, but surely we agree that the two models can be used to model learning over time, do we not? I might learn just C and learn it with certitude. Then my prob(*) is replaced by my prob'(*), where all prob'(*) = prob(*|C), and all prob'(*|*) = prob(*|*&C). This is what suggests itself when I reflect on my learning from the card deals, and in this situation one class of conditional probabilities do remain the same through the transition.

THEAETETUS: I'd say you just stated principles that are definitive of rational revision of belief. The equations don't make the conditional notation definitive of the primed notation, as if one notation were merely an abbreviation for the other. That would destroy diachronicity.

13. *The evanescence of synchronic relevance is illustrated as we progress through a game of chance.*

TAVERNER: Last call for drinks.

THEAETETUS: Don't you buy back?

TAVERNER: Why not, if we make it a wager?

THEAETETUS: You toss a die. If we two can each better your score, you buy our last drinks, OK?

TAVERNER: OK.

SOCRATES: What's the probability that we'll both better his score?

THEAETETUS: We'll make a table, and compute it.

ROW A = POSSIBLE TAVERNER'S SCORES
ROW B =THEIR PROBABILITIES
ROW C = PROBABILITY OF ONE OF THEM BEATING TAVERNER'S SCORE, IF SCORE IS AS IN ROW A.
ROW D = PROBABILITY OF BOTH BEATING THE TAVERNER'S SCORE, IF SCORE IS AS IN ROW A.
THE LAST COLUMN LISTS THE PROBABILITIES OF BEATING THE TAVERNER, WHEN THE TAVERNER'S SCORE IS UNKNOWN.

							weight-ed ave. of probs
A	1	2	3	4	5	6	
B	1/6	1/6	1/6	1/6	1/6	1/6	
C	5/6	4/6	3/6	2/6	1/6	0	15/36
D	25/36	16/36	9/36	4/36	1/36	0	55/216

Table 20: A game in which either's beating the taverner's score is positively relevant to the other's beating it, until they know what the taverner's score is, in which case either's win is irrelevant to the other's winning. Compare table 11.

SOCRATES: Explain.

THEAETETUS: Each of the taverner's ways of scoring is as likely as any other, namely 1/6. So without knowing what his score is yet, your probability of beating him is 15/36, and so's mine.

SOCRATES: That's the "weighted average" stuff on the table. My probability of bettering his score is the weighted average of the probabilities of my different ways of outscoring him on row C, the weight being the chance of my beating him that way on row B. Dear me.

THEAETETUS: Only the name "weighted average" is new. The technique's the same as the one you used to compute the chance of a black card being drawn second from a deck, from which not even the first card has been drawn yet. Don't you remember?

SOCRATES: Oh. It's seeing when two cases are the same that's hard. I outscore him if and only if (he rolls 1 & I roll higher than 1 OR he rolls 2 & I roll higher than 2 OR . . . and so on up to . . . OR he rolls 6 & I roll higher than 6). By the same reasoning as before[14] I reach the calculation:

$$\text{prob(I outscore him)} = 1/6(5/6)+1/6(4/6)+1/6(3/6)+1/6(2/6)+1/6(1/6)+1/6(0) = 15/36.$$

THEAETETUS: With cards there's a judgment of relevance between what's drawn first and what's drawn second. With dice, there's no relevance between his rolling a 1, say, and your rolling something higher. But, if the proposition is that you outscore him, then there is a relevance between what he scores and that.

SOCRATES: I see.

THEAETETUS: But look; the probability you assign to our both scoring higher than the taverner, before we know his score, is 110/432, which is greater than the product of our individual chances of scoring higher, 75/432.

SOCRATES: Now how did you . . . Oh. It's that the 55/216 in row D restated for comparison. So you got that by the same procedure you got our individual probabilities. We win if and only if (he rolls 1 & we both roll higher than 1 OR . . . and so on up to . . . OR he rolls 6 & and we both roll higher than 6). So,

[14]See page 105.

prob(we win) = 1/6(25/36)+1/6(16/36)+1/6(9/36)+1/6(4/36)+1/6(1/36)+1/6(0) = 55/216
= 110/432.

THEAETETUS: So there's positive relevance between one of us outscoring him and the other outscoring him too, before either of us rolls.

SOCRATES: Positive, you say?

THEAETETUS: Yes, the probability of the conjunction (You outscore him & I outscore him) is greater than the product of the probabilities of each of the conjuncts.

SOCRATES: Let me figure it with conditional probabilities. The probability I beat his score, given you beat his score, is the probability we both outscore him, 55/216, divided by the probability you outscore him, 15/36. That's . . . 22/36, and higher than my 15/36 probability that I'll outscore him without my having the information that you outscored him. Positive relevance.

THEAETETUS: That's right.

TAVERNER: Gentlemen! Ready or not, I'm about to roll the dice.

SOCRATES: [MUTTERING] If I learn Theaetetus outscored him, I'd know that he didn't throw a 6. Maybe, of the remaining 5 possibilities, I even learn that there's a greater than 1/5 chance he threw a 1, less than a 1/5 chance he threw a 5. That would affect my degree of belief that I too could outscore him.

THEAETETUS: Close your eyes, Socrates.

SOCRATES: I won't peak.

[TAVERNER THROWS A 4. THEAETETUS THROWS A 5. SOCRATES DOESN'T SEE THE RESULTS.]

THEAETETUS: I won my toss, Socrates.

SOCRATES: [EYES STILL CLOSED] Thanks for the information.

THEAETETUS: You previously calculated your chance of bettering his score as 15/36. Without opening your eyes, what do you think it is now?

SOCRATES: It's up considerably; 22/36.

THEAETETUS: As long as you don't know what the taverner's score is, information about my beating it is relevant to your estimates of your own chance of beating it. But there's even better information to be had, which will render the information about my outscoring him obsolete, indeed irrelevant. Open your eyes.

SOCRATES: I see he threw a 4; so I know my chances of bettering his score are down to 12/36. They're 1/3, regardless of the fact that you bettered it on your toss by throwing the 5.

TAVERNER: So toss already. [SOCRATES TOSSES, SCORES A 6. TAVERNER PICKS UP THE DIE, EXAMINES ALL SIDES, RUBS IT AGAINST THE OTHER DIE, BALANCES IT BETWEEN HIS FINGERS, LOOKS SOCRATES IN THE EYE FOR FIVE SECONDS.] Ah, I'll get your drinks; they're on the house.

14. *There is a stable relation of diachronic relevance.*

THEAETETUS: You don't like evanescence. I'll give you some stability. The degrees of belief you had before I told you that I outscored him let's call your prob(*); after I told you, your degrees of belief are your prob'(*), and after you learned the Taverner's score your degrees of belief are your prob''(*).

SOCRATES: Which are all different from my current state of prob'''(*), since I know I outscored him.

THEAETETUS: Notice that one quantity stayed constant from prob(*) to prob'(*):

prob(Socrates will outscore him|Theaetetus outscored him) = 11/18
prob'(Socrates will outscore him|Theaetetus outscored him) = 11/18

The condition in these conditional probabilities was the very fact you learned which made you go through the transition from prob(*) to prob'(*). We're comparing before and after, and there's no difference.

SOCRATES: I see. In the second transition, if my condition is that the taverner scored a 4, which is what I learned, then the "before and after" conditional probabilities were:

prob'(Socrates will outscore him|the taverner scored a 4) = 1/3
prob''(Socrates will outscore him|the taverner scored a 4) = 1/3.

They too are the same. And, I suppose, I can even put down the trivial constancy through my latest transition:

prob''(Socrates outscores him|Socrates outscores him) = 1
prob'''(Socrates outscores him|Socrates outscores him) = 1.

Stable. So what?

THEAETETUS: I can define a stable diachronic relevancy; that's what you wanted. First, though, let's assume that, when you learn any of these conditions, there's always room for at least a smidgeon of doubt. When you learn anything, you do not become absolutely certain of its truth. That's so that the next bunch of conditional probabilities I'll state don't have conditions with probabilities of zero, which would render them undefined, according to my definition of conditional probability. They're your conditional probabilities *after* you learn the condition they deny:

prob'(*|Theaetetus did *not* outscore him)
prob''(*|the taverner did *not* score a 4)
prob'''(*|Socrates did *not* outscore him)

SOCRATES: I think I can manage not to be certain of the falsity of these conditions. My eyes do sometimes play tricks on me.

THEAETETUS: Then notice that these conditional probabilities too will each form a pair with their before-counterparts to form "before and after" exact equalities:

prob(*|Theaetetus does not outscore him)
prob'(*|the taverner does not score a 4)
prob''(*|Socrates does not outscore him)

With that observation granted, here's my definition of diachronic relevance:

DEFINITION OF B'S DEGREE OF DIACHRONIC RELEVANCE TO A

B IS D-RELEVANT TO A TO THE DEGREE d-rel(A|B), WHERE
d-rel(A|B) = prob(A|B) – prob(A|NOT-B), IF 0 … prob(B) … 1.
OR, EQUIVALENTLY,[15] WHERE
d-rel(A|B) = s-rel(A|B) / prob(B)prob(NOT-B), IF 0 < prob(B) < 1.

[15]The equivalence is proved in the Appendix.

This relation would not persist through a change in probability function, if we were to learn B with certitude; one of the "after" conditional probabilities would be undefined, since we would have learned with certitude that its condition is false. But if we learn B with less than certitude, then this relation does persist. You could put prime marks after each "rel" and "prob," and, if the change from prob(*) to prob'(*) was due wholly to the learning of B with less than certitude, then the equations would remain true. So I think it right to call it diachronic relevancy.

SOCRATES: But if I learn some third proposition instead of B then the diachronic relevance between A and B may not persist; is that right?

THEAETETUS: Right. It's guaranteed to persist unchanged only when B is the proposition learned and the only proposition learned in the transition. So the transition consists of one step.

SOCRATES: You give me an idea; perhaps you were just going to state it. What sort of thing stays the same through a change? Not the starting state; not the ending state, but rather the rule of the change.

THEAETETUS: You're referring to learning when you refer to change, namely, changes in degrees of belief caused by perception and the like. Surely that kind of learning is not governed by such rules as these. You look; you learn. Which way you turn your eyes is not governed by rules.

SOCRATES: But just as surely, when I learn something by seeing something, what I learn is relevant to many other things I believe, and so affects their probabilities.

THEAETETUS: Ah! The network of relevancies . . .

SOCRATES: . . . makes learning holistic: I see one thing, and a gazillion of my beliefs change.

THEAETETUS: Still, we must distinguish between two types of learning. One is typified by perception. Your new degrees of belief originate from outside your probability function, and renders it obsolete; another function describes them. We can call these changes *exogenous*, because they arise independently of your prob(*), that is, the prob(*) of the one about to learn something. But prob(*) contains relevancy relations between what you're about to learn exogenously and every other proposition. Some relations are relations of irrelevance; your new prob'(*) will leave those propositions at the probability they had in prob(*). Others are relations of positive relevance; your new prob'(*) will raise the probabilities of those propositions. Still others are relations of negative relevance; your new prob'(*) will lower the probabilities of those propositions. Still other changes will affect the very relevancies among these other propositions. I showed that earlier.

SOCRATES: I recall your abstract demonstration, using the earlier diagrams.

THEAETETUS: All these changes are *endogenous*, because the relevancy relations *in* prob(*) dictate them; they arise from within prob(*) although they're triggered by the exogenous learning. So you're right, Socrates, about rules, but only for the derivative and endogenous learning, not for the function-originative exogenous type. Now what were you about to say?

SOCRATES: A rule of endogenous changes in degrees of belief can't itself change as long as it's governing the changes. So we may have found an interpretation for conditional probabilities: They are components in the rules of endogenous change. When change is occurring according to a rule, that rule is not then subject to change, not even in these of its components, although the other rules, not then in charge, may change.

THEATETUS: Yes, yes, but I think you may have only found the superficial manifestation of stability. What keeps the conditional probabilities in the rules from changing? The rule of one step of endogenous change is, for changes in any arbitrary A, due to an exogenous change in belief of B, short of becoming certain of its truth or falsehood,

$$\text{prob}'(A) - \text{prob}(A) = \text{d-rel}(A|B)[\text{prob}'(B) - \text{prob}(B)].$$

Given our "before" degree of belief in any A, we can compute our "after" degree, if we have our "before and after" degrees for B, which changed in a function-originative or exogenous way, and if we have its relevance to A. Either synchronic or diachronic relevance will do, since they're interdefinable. To prove all this would involve a lot of new concepts . . .

TAVERNER: Not now, fellow.[16] Here's your drinks. Down the hatch, gentlemen! It's closing time.

THEAETETUS: Thanks. Socrates, my concession to you that we can use conditional probabilities, which I insist are synchronic, to understand change I hope will prompt you to concede that my d-rel meets your intuition's demands that some aspect of relevance remain the same.

15. *Five arguments that prob(A|B) stands for the mind's sense of the degree of relevance between A and B.*

SOCRATES: No, I do not trade one truth for another; I follow arguments. [THEAETETUS REDDENS AND STAMMERS IN A MINOR KEY.] I do think your arguments suffice to win the agreements you want, however. [THEAETETUS SMILES AND STAMMERS IN A MAJOR KEY.] But your d-rel is constant only for one step of learning, and . . . cowabonga!

THEAETETUS: "Cow. . ."! [STUNNED INTO OPEN-MOUTHED SILENCE]

SOCRATES: You've been leading me along from one topic to the next, and I only just remembered what it was we were examining in the first place. Theaetetus, where are we in your argument that prob(A|B) stands for something in the mind? [i] The argument I just suggested about rules of change not participating in the change they rule is a strong pointer to the conclusion that prob(A|B) names the degree of something real.

THEAETETUS: It does, and also because [ii] we make judgments of synchronic relevance and irrelevance all the time. And although there's also the diachronic kind of relevance [which is all I need concede to your carping about evanescence), we do judge the synchronic kind that is a degree of conjunctivity. Often that judgment depends on our having in mind this prob(A|B), whatever it may be. [iii] We also make judgments of fairness of conditional bets in terms of it. How, unless this term in the comparisons is present in our minds?

SOCRATES: Your argument is not without weight; you've even got me used to judging synchronic relevance your way, despite my qualms. But your argument's far from conclusive. We may base our judgments of A's relevance to B only on comparisons of risk between a bet on A and a combination of bets, one on A&B, the other against B.

THEAETETUS: But [iv] how are the fair odds on a combination of bets determined? And what's more, [v] an idea we have of the maturity of a belief is captured in my notion of relevance. You see that, don't you? As a belief matures, as we acquire wisdom, the degree of our belief in it becomes less and less susceptible to significant change as a result of further experience. What's that, I ask you, if not decreasing relevance of further experience?

SOCRATES: It's amazing how pull new ideas as if out of thin air. Maturity? My beliefs in

[16] This subject is continued in another dialogue, not included.

logical truths are mature in your sense, but hardly more, perhaps that I exist now.

THEAETETUS: Yes, and we sense our growth in wisdom by our being less and less surprised by the world, despite our keeping alive our susceptibility to surprise. That stability of contingent belief is none other than our possibilities of experience approaching irrelevance to what we already believe, not by our becoming inattentive, but rather by our having anticipated the world.

SOCRATES: Oh, I'd have to examine that further, but you've got five reasons to treat prob(A|B) as naming a mental base of comparisons. Not bad.

[Conditional Probabilities Cannot Be Degrees of Belief in One Proposition]
[In Eleven Parts]

1. *Could prob(A|B) be identical to prob(A or not B), which is identical to prob(A if B)?*

THEAETETUS: Score! Back to craps for one last round. Care to make a conditional bet?

SOCRATES: Concerning the intuitive meaning of prob(A|B) there's difficulty still. Mental states are directed to objects; what objects is this state directed toward? It's not a degree of belief that A, that I'll win my bet, period. But it might be the degree of belief that I'll not lose my bet, that either I'll win it, A, or get out of it, NOT-B. For example, suppose I bet that if you establish your point as 4, you'll make it the hard way, that is, make it by rolling <2,2> before a 7. Then, if a PROBE in this bet seems fair to me, perhaps it represents my degree of belief that I'll not lose. Either you do make 4 the hard way, or you don't establish 4 as your point in the first place.

THEAETETUS: Is that it? Is it the probability of (A OR NOT-B)?

SOCRATES: Let B be "You establish your point as 4"; let A be "You make it the hard way," that is, you roll a <2,2> before you roll a 7. Then (A OR NOT-B) means, "Either you make your 4 the hard way, or you don't establish a 4 at all." I conjecture that

prob(A|B) = prob(A OR NOT-B).

THEAETETUS: I can prove your conjecture wrong. Let's begin with another theorem I'll prove in the notes. It says the probability of a proposition expressed by (A OR NOT-B) equals the sum of the probabilities of its two disjuncts, minus the probability that the two disjuncts are true together.[17]

SOCRATES: You didn't say whether these disjuncts are mutually exclusive. For mutually exclusive disjuncts, we just added up the probabilities of the parts of an "or" sentence to find the probability of the whole sentence, I mean, of the proposition the sentence expresses.

THEAETETUS: Right, that's *only* when the disjuncts are mutually exclusive. Here I'm giving you a rule that applies to all propositions expressed with "or" and just two disjuncts, whether or not they're incompatible with each other. Since we sum the two probabilities, remember that "within" the probability of each disjunct is the probability of its being true *with* the other disjunct. That probability gets counted twice if we just add the probabilities of the two disjuncts together. So we subtract it once, to avoid counting it twice.

SOCRATES: Got it.

THEAETETUS: So, as our first step we put this theorem:

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[17]See the first Appendix of Dialogue I, Theorem IX.

prob(A OR NOT-B) = prob(A)+prob(NOT-B) – prob(A&NOT-B).

Here's another equivalence for you to check: A&B is equivalent to NOT(EITHER NOT-A, OR (BOTH A & NOT-B)).

SOCRATES: I'll calculate it from a diagram.

Diagram 14: Two-circle diagram to check equivalence of A&B to NOT(EITHER NOT-A, OR (BOTH A & NOT-B)). The rules are in the first Dialogue's second Appendix, section 2.

A&B	NOT(EITHER NOT-A, OR (BOTH A & NOT-B))
3-4&2-4	NOT(EITHER NOT 3-4, OR (BOTH 3-4 & NOT 2-4))
4	NOT(EITHER NOT 3-4, OR (BOTH 3-4 & 1-3))
	NOT(EITHER NOT 3-4, OR (3))
	NOT(EITHER 1-2, OR 3)
	NOT(1-2-3)
	4

They're equivalent. So they're equal in probability.

THEAETETUS: Excellent, and you'll follow these steps if you recall my theorems:

prob(A&B) = prob(NOT(EITHER NOT-A, OR (BOTH A & NOT-B)))
prob(A&B) = 1 – prob(EITHER NOT-A, OR (BOTH A & NOT-B))
prob(A&B) = 1 – [prob(NOT-A) + prob(BOTH A & NOT-B)]
prob(A&B) = 1 – prob(NOT-A) – prob(BOTH A & NOT-B)
prob(A&B) = prob(A) – prob(BOTH A & NOT-B)

So prob(A&B) is the same as prob(A) – prob(A&NOT-B). Make that substitution, equals for equals, of prob(A&B) for the prob(A) – prob(A&NOT-B) in the theorem I gave you before, and we get:

prob(A OR NOT-B)=prob(A&B)+prob(NOT-B)

But prob(A&B) is the same as prob(A|B)prob(B), provided prob(A|B) is defined, that is, provided prob(B)>0, as you can see from the definition of prob(A|B) I proposed earlier. So we make a substitution for prob(A&B) and get:

prob(A OR NOT-B)=prob(B)prob(A|B)+prob(NOT-B), if prob(B)>0.

That shows the probability represented by a fair conditional PROBE is not the same as the probability that you won't lose the conditional bet, except in one case, the case where prob(B)=1.

SOCRATES: I see. If prob(B)=1, then prob(A OR NOT-B) = 1prob(A|B)+0. But in any other case, there'll be an inequality. To reject the equation I proposed by the method Shark taught me, I assign probabilities to regions of a diagram:

Diagram 15: illustrating the inequality of prob(A OR NOT-B) and prob(A|B).

In this case, prob(A OR NOT-B)=.8, but prob(A|B)=.4/.6, that is, 2/3.

THEAETETUS: In your example, prob(B)=.6. The closer you make it to 1, the closer the truth gets to your conjecture.

2. *What is a propositional operator?*

SOCRATES: Not so fast.

THEAETETUS: What's the matter?

SOCRATES: We've only considered one interpretation of a conditional. Since "A if B," under many interpretations of "if," only implies "A OR NOT-B" and is not implied by it, prob(A if B) cannot exceed prob(A OR NOT-B), but does not have to equal it. You've not refuted that a conditional probability is the probability of some conditional but only of one of them. What *is* a conditional probability the probability of?

THEAETETUS: Wait.

SOCRATES: I'm waiting.

THEAETETUS: It can't be identified with a degree of belief in any conditional proposition either, if indeed conditionals are propositions.

SOCRATES: You mean, the fair conditional PROBE does not represent my degree of belief in the proposition, "if you establish your point as 4, you'll make it the hard way"?

THEAETETUS: If indeed that sentence expresses a proposition, then, no, the fair PROBE does not represent its probability.

SOCRATES: First explain your doubts about the sentence's expressing a proposition.

THEAETETUS: The doubt I have is whether we should treat the word, "if," as though it were like the words, "and," "or," and "not." These words are operators, but is "if"?

SOCRATES: Before I can answer, you'll have to explain to me what an operator is.

THEAETETUS: An "operator" is a term we attach to other words to make bigger words or phrases, a sort of glue, a linguistic glue. Words and phrases belong to various *categories*, like "noun," "verb," and "sentence." Operators are the sort of glue-word that creates a phrase of the *same* category as the words it's attached to.

SOCRATES: I need some examples. Is this the idea, that if I put the word "not" in the sentence, "I'm seated," I create something of the same category as the original, namely, another sentence, "I'm not seated"?

THEAETETUS: That's correct. If I put the word, "or," between two sentences, I create, not a noun or a verb, but another sentence. If I put it between two nouns, however, I create a noun phrase, and not a sentence or a verb phrase.

SOCRATES: Let's see. Here's one sentence: "Shark's awake." Here's another: "Shark snorts and whistles." And here's a third, created from the first two, by using an operator: "Shark's awake, or Shark snorts and whistles." I understand. Here's another, created from the first one: "Shark's not awake."

THEAETETUS: A quick study. An example of a glue-word that's not an operator is the word, "that." If you attach the word, "that," to the beginning of a sentence, you create a noun phrase, and not another sentence.

SOCRATES: Here's a sentence, my friend: "I appreciate your generosity." And here's the noun phrase: "that I appreciate your generosity." That I appreciate your generosity is obvious. [HE DRINKS.] The subject of my last sentence was the noun phrase, "that I appreciate your generosity."

TAVERNER: Drink up, folks. You may want to avoid the crowd going home; they're trouble, I'll tell you.

3. *What is the recursive application of a propositional operator?*

SOCRATES: But what I don't appreciate is your opacity, Theaetetus. What's it all got to do with interpreting a fair conditional PROBE?

THEAETETUS: I'm getting there. Notice that operators are recursively applicable, whereas other glue-words, like the word, "that," are not.

SOCRATES: Run that by me again, will you?

THEAETETUS: Recursion's a simple idea. Think of the operator as re-usable. We never run out of "or's" and "not's." The output of one use of the operator can be input for another use of it, whose output can be input for still another use of the operator, and so on, without end. That's recursion.

SOCRATES: I see. Since the result of putting a "not" in a sentence is another sentence, I can put a second "not" in it to make a third sentence. For example, "I'm seated," "I'm not seated," "I'm not not seated," "I'm not not not seated," "I'm not not not not seated." This could go on forever.

THEAETETUS: Yes. But each sentence makes sense, doesn't it? Since you are seated, every sentence with an even number of "not's" in it expresses the same true proposition, and every sentence with an odd number of "not's" expresses the same false one.

SOCRATES: We've started distinguishing between sentence and proposition, haven't we? It's true that I'm not not seated. It's even true that I'm not not not not seated. But there's only one fact of the matter: I am seated. There may be any number of distinct sentences generated by this recursive process, but only two propositions are ever expressed, the true one and the false one.

THEAETETUS: That's correct for your example.

SOCRATES: How does recursion work with "or"?

THEAETETUS: "Or" operates on two or more terms of the same category at once, joining them into a single term of the same category as the originals, whereas "not" operates on one term at a time. We call "not" a monadic operator, and "or" a polyadic operator, to note this difference. But it makes no difference to the recursiveness of an operator, whether it's monadic or polyadic. The output of one use of a polyadic operator can always be one of the many inputs in the next use of the operator.

SOCRATES: For example, . . .

THEAETETUS: It'll help if you use the word, "either" as a first correlative, when you use "or" to connect just two terms. With the word "either" we restrict "or"'s polyadicity to the dyadic case. But we're recompensed by having a term to mark the order in which the different "or's" have been used to create compound sentences.

SOCRATES: I'll illustrate what I think you mean. I'll just use schematic letters in place of sentences this time. Let my original sentences be A, B, C, and so on. Then by recursion I can get:

Either A or B.

Either C or Either A or B.

Either Either A or B, or Either C or Either A or B.

I get the idea. My third example's just the first two disjoined. The second example contains the first as its second disjunct. OK. Theaetetus, where's all this taking us?

THEAETETUS: Notice that recursion does not apply to the word, "that." It has to be attached to the beginning of a sentence. But the attachment creates a noun phrase. Therefore, the output of one use of the word, "that," cannot be the input for a second use of the word.

SOCRATES: I understand, because a "that" requires a sentence as input.

THEAETETUS: Another glue-word that's not an operator is our "prob." This word, "prob," is attached to a sentence as input, but the result of the attachment is the name of a number, not another sentence.

4. *Does applying the "if . . . then . . ." operator recursively yield propositions with sense?*

SOCRATES: What does recursion have to do with the question we're asking ourselves, which is: Does the PROBE of a conditional bet, which I consider fair, represent my degree of belief in a conditional proposition?

THEAETETUS: Is the "if" a dyadic operator?

SOCRATES: I think so. Let A and B stand for sentences. "A if B" is a sentence. And it seems recursive too. Putting the antecedent before the consequent, using the words, "if . . . then . . .," I can mark the order in which the "if's" are introduced:

If A then B.

If C then, If A then B.

If If A then B, then, If C then, If A then B.

My God! What might that last one mean?

THEAETETUS: That's what I'm asking.

SOCRATES: Since it seems grammatical, we might try hard to find some sense in it.

THEAETETUS: Or we might concede that being grammatical isn't always sufficient for making sense. There may be no proposition expressed by that grammatical form. In fact there may not be any such thing as a conditional proposition.

SOCRATES: But there are conditional *sentences.*

THEAETETUS: Yes. But they may not express conditional propositions.

5. *Should we construe prob(*|*) as a monadic, or as a dyadic, function?*

SOCRATES: Let's suppose they do. Would the PROBE of a conditional bet I consider fair represent my degree of belief in such a conditional proposition?

THEAETETUS: If that were true, all my work on probability would've been in vain. For the theory would be utterly trivial; it couldn't even describe your degrees of belief in the outcomes of throws of the dice, not even of one die.

SOCRATES: Why do you think that?

THEAETETUS: Let's look more closely at the symbol we're trying to interpret: "prob(A|B)." What's its structure? I see two possibilities. First, it might have the same structure as all the other "prob(*)" symbols we've seen. Within the parenthesis is a single proposition, and the whole symbol stands for a number from 0 to 1. If that's the case for the conditional probability symbol, then "A|B" stands for a single proposition, just as "A" does, and as "A&B" does. It's a compound proposition; so the "if" symbol, "|," must be an operator. The two propositions are glued together by what must be a dyadic operator to form a compound proposition, which then is inserted within the parentheses of another kind of glue word, prob(*), which is not an operator, because the result's not a proposition, but the name of a number. Am I clear?

SOCRATES: Yes. What's the other possibility?

THEAETETUS: The other's that prob(*) is being used ambiguously. Up to now, we've been treating it as a monadic glue-word, to be applied to a single proposition, whether of the simple or compound variety. But now we may be using it as a dyadic glue-word, prob(*,*), to be applied directly to two propositions that have not been joined together into a single compound proposition. In this case, the "if" symbol, "|," is not a dyadic operator at all. In fact it's nothing but a fancy way of writing a comma, that is, something that serves to separate the two propositions that prob(*,*) applies to, resulting in one number from 0 to 1, provided that the second proposition has a monadic probability greater than 0. Is this alternative clear?

SOCRATES: Yes.

6. *Theorems we need in order to prove that prob(*|*) is not a monadic function.*

THEAETETUS: Good, because I can prove that the first alternative's a disaster. I can, that is, if you agree to one assumption about the meaning of propositions compounded of three simpler propositions by an "if" operator. The assumption's that a sentence like ((C|B)|A), which we can read as, "if A then, if B (also) then C," is equivalent to (C|B&A), which we can read as "if both A and B, then C." In other words, we assume that the order of stating the two conditions, A and B, is immaterial to the proposition expressed. Does that seem reasonable?

SOCRATES: Let me think. "If it's a moonless night, then if it's cloudless also, I'll be able to see stars." You say this sentence expresses the same proposition as, "If it's a cloudless, moonless night, I'll be able to see stars."

THEAETETUS: That's correct.

SOCRATES: It's a reasonable assumption, I think. I mean that if our assuming the immateriality of order of "if's" were OK, this "exportation" would be reasonable also. . .

THEAETETUS: Convince yourself that the sentence, A, is equivalent to the compound: A&B, OR A&NOT-B. Look at a diagram.

SOCRATES: That's not necessary. I have it in these notes from the sleeper here.[18]

THEAETETUS: Then you'll agree their probabilities are equal, as theorem VI says of all

[18]See the second Appendix to Dialogue I.

equivalent forms.[19] Moreover, the two disjuncts, (A&B) and (A&NOT-B) are mutually exclusive, so that theorem IV lets us substitute a sum of their probabilities for the probability of the disjunction. The steps are:

> A is equivalent to (A&B)OR(A&NOT-B)
> prob(A) = prob((A&B)OR(A&NOT-B))
> prob(A) = prob(A&B)+prob(A&NOT-B)

The norm for conditional probability also permits us to derive these two equalities:

> prob(A&B) equals prob(B)prob(A|B) if prob(B)>0, and
> prob(A&NOT-B) equals prob(NOT-B)prob(A|NOT-B) if prob(NOT-B)>0.

You'll recognize them as two instances of the product law. So here's a theorem derived from my axiom and the norm for conditional probability:

> For any propositions, A and B, if your degree of belief in B is greater than 0, but less than 1, then
> prob(A) = prob(B)prob(A|B)+prob(NOT-B)prob(A|NOT-B).

Do you understand why I need that proviso on the theorem?

SOCRATES: Yes, of course; so that the conditional probability symbols are defined, since by definition they're fractions and mustn't have 0 denominators. This is all review for me, because I've used that theorem in calculating the probability of black on the second draw from a deck of cards.

THEAETETUS: Good. But I'm going to make it stronger so that I can make replacements of proposition letters and substitutions of equals for equals, without the worry that some symbols might not be defined. So you get to review the rule of replacement also. Here's a special case of the theorem:

> For any propositions, A and B, if your degrees of belief in (A&B) and in (A&NOT-B) are greater than 0, then,
> prob(A)=prob(B)prob(A |B)+prob(NOT-B)prob(A |NOT-B).

Do you see that this proviso includes the previous one?

SOCRATES: Yes. My degree of belief in a conjunction cannot be greater than 0 unless my degrees of belief in each of the conjuncts are greater than 0. Thus if my degree of belief in (A&B) is greater than 0, so's my degree of belief in B. If my degree of belief in (A&NOT-B) is greater than 0, so's my degree of belief in NOT-B. And since both are greater, neither equals 1.

7. *If A|B were a proposition and | a propositional operator, then probability theory would be trivial.*

THEAETETUS: Just so. Now here's the catch. Every theorem exhibits all the structure that

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[19]See the first Appendix to Dialogue I.

makes it true. If you added more structure to it, the addition would not make it false. So we may replace every A or every B in the theorem with symbols of another proposition with more internal structure, and the statement resulting from replacement of all the A's, say, by (B|A) will still be a theorem. Here's where we use the assumption that symbols like (B|A) and ((B|A)|B) stand for propositions. Let's replace the three A's in the theorem with (B|A).

SOCRATES: Aha! I was wondering when the lesson in operators would pay off. The rule of replacement replaces simple sentences with compound sentences created with operators.

THEAETETUS: Precisely. We conclude that the following's also a theorem:

> For any propositions, A and B, if your degrees of belief in (A&B) and in (A&NOT-B) are greater than 0, then,
> $\text{prob}(\underline{(B|A)})=\text{prob}(B)\text{prob}(\underline{(B|A)}|B)+\text{prob}(\text{NOT-}B)\text{prob}(\underline{(B|A)}|\text{NOT-}B)$.

I've underlined the replacements so you can compare the two forms, the theorem and this pseudo-theorem, and see they're the same except for the replacement.

SOCRATES: Yes, they are. I also see why you provided for a positive degree of belief in A as well.

THEAETETUS: To keep all the terms defined. The next step's to substitute prob(B|A&B), which equals 1, for prob((B|A)|B) and also prob(B|A&NOT-B), which equals 0, for prob((B|A)|NOT-B):

PSEUDO-THEOREM

> For any propositions, A and B, if your degrees of belief in (A&B) and in (A&NOT-B) are each greater than 0, then,
> $\text{prob}((B|A)) = \text{prob}(B)\underline{\text{prob}(B|A\&B)}+\text{prob}(\text{NOT-}B)\underline{\text{prob}(B|A\&\text{NOT-}B)}$
> $\text{prob}((B|A)) = \text{prob}(B)\underline{1}+\text{prob}(\text{NOT-}B)\underline{0}$
> $\text{prob}((B|A)) = \text{prob}(B)$.

That must be a theorem too, since it was derived from a theorem, provided our two assumptions were correct, namely, that "|" is an operator on propositions, making a compound proposition, and that ((B|A)|B) is the same proposition as (B|A&B). But it's absurd to suppose this equation a theorem.

SOCRATES: Why?

THEAETETUS: If it were, no theorems could describe rational degrees of belief in something as simple as the outcomes of a throw of the die. Do you see how?

SOCRATES: I'll take a stab at it. So what if I get it wrong; maybe I'll learn something. Right?

THEAETETUS: It's what you always told me, Socrates.

SOCRATES: Here goes: Let B stand for the proposition that the die will show a 2. And let A stand for the proposition that the die will show a 3. I believe the die's perfect and I'll throw it fairly. What's the probability of B, prob(B)?

THEAETETUS: 1/6.

SOCRATES: What's the probability of A, if B's true? That is, what's your degree of belief that it'll show a 3 if it shows a 2?

THEAETETUS: By the definition, that's the probability of both a 2 and a 3 at once, divided by 1/6. 0 divided by 1/6 is 0.

SOCRATES: But the "theorem," if it were a theorem, says it must remain at 1/6. That's absurd. The theorem's saying I should be open to what I believe to be impossible, namely, that the die show both a 2 and a 3. Ridiculous.

THEAETETUS: Ridiculous, yes. But you haven't given the reason why, because your example's irrelevant. Remember that proviso on the formula, "if your degrees of belief in A&B and in A&NOT-B are both positive"? Well, in your example, your degree of belief in A&B is 0. The pseudo-theorem doesn't even purport to apply to your case. It says, "for any A and B, if prob(A&B) and prob(A&NOT-B) are greater than 0, then they are mutually irrelevant: prob(B|A)=prob(B).

SOCRATES: Ah! Show me the right way to bring out the absurdity.

8. *Demonstrating the trivialization, but concluding that we've refuted its premise*.

THEAETETUS: Suppose three propositions, each incompatible with each other and your degree of belief in each greater than 0. Call them Q, R, and S.

SOCRATES: Easily done. Let Q be "the die will show 2 or 3," R be "the die will show 4 or 5," and S be "the die will show 1 or 6." No two can be true together, but I believe each to degree 1/3.

THEAETETUS: I'll now "prove" it's not so easily done; in fact it should be impossible, if that formula of irrelevance is a theorem.

SOCRATES: Really?

THEAETETUS: Yes. And it would be absurd to say such an easy thing's impossible.

SOCRATES: Yes, indeed.

THEAETETUS: Whatever implies an absurdity is itself absurd. Thus the formula's no theorem. Therefore it was derived from a false premise or with an invalid rule. But let me show you the derivation of the absurdity.

SOCRATES: OK.

THEAETETUS: There's no doubt that (Q OR R) is a proposition; so we replace A in the formula with (Q OR R). We replace B with Q. The result is:

> For any propositions, (Q OR R) and Q, if your degrees of belief in (Q OR R)&Q and in (Q OR R)&NOT-Q are each greater than 0, then,
>
> prob(Q|(Q OR R))=prob(Q).

In your example with the die, what are your degrees of belief?

SOCRATES: Time for three-circle diagrams:

Diagram 16: The left model has region numbers, the right has probabilities.

Using the left model, I find the regions you want to know my probabilities for:

(Q OR R) & Q	(Q OR R) & NOT-Q
(5-6-7-8 OR 3-4-7-8) & 5-6-7-8	(5-6-7-8 OR 3-4-7-8) & NOT 5-6-7-8
(3-4-5-6-7-8) & 5-6-7-8	3-4-5-6-7-8 & 1-2-3-4
5-6-7-8	3-4

Proceeding to the right model, my prob((Q OR R)&Q) is 1/3, for it's just prob(Q); my prob((Q OR R)&NOT-Q) is also 1/3. So now I do have a relevant example.

THEAETETUS: Yes, you do. Next, by definition,

prob(Q|(Q OR R))=prob((Q OR R)&Q)/prob(Q OR R)

You've verified by a diagram that,

prob((Q OR R)&Q)=prob(Q).

So:

prob(Q|(Q OR R))=prob(Q)/prob(Q OR R)

Substituting equals for equals in our pseudo-theorem, we derive:

> For any propositions, (Q OR R) and Q, if your degrees of belief in (Q OR R)&Q and in (Q OR R)&NOT-Q are each greater than 0, then,
>
> prob(Q)=prob(Q)prob(Q OR R)

It follows that prob(Q OR R) must be 1, and so prob(NOT(EITHER Q OR R))=0.

SOCRATES: Quite absurd. This leaves no room for S.

THEAETETUS: You remember S? We specified that it was incompatible with Q and with R. So it implies NOT(EITHER Q OR R). Therefore, by theorem V, its probability is at most the probability of NOT(EITHER Q OR R). We just proved it to be 0. Therefore, S's probability is 0.

SOCRATES: But it was part of our original supposition that my degree of belief in S was greater than 0. In my example it was 1/3.

THEAETETUS: Precisely. We've proved a contradiction follows from our pseudo-theorem by supposing that we can have positive degrees of belief in three mutually incompatible propositions.

9. *Extending the trivialization result to other cases.*

SOCRATES: Should we have had four or more instead, or somehow avoided incompatibilities?

THEAETETUS: Any larger number of propositions with different positive degrees of belief, whether mutually exclusive or not, is just as impossible. To see why, take any two whose probabilities are positive, different from one another, and sum to less than 1. Call them A and B. We can always find two such.

SOCRATES: Yes.

THEAETETUS: Do you see why, or are you just being agreeable?

SOCRATES: Yes.

THEAETETUS: We're imagining a set of beliefs more complex than the case we just considered, in which we had three positive, mutually exclusive beliefs. Let the more complex set be only minimally more complex. In our original set we had to have only four probability values, although we might have had more. But your own example has only four. They were 1 and 0, of course, and 1/3 for each of the three propositions, Q, R and S, and 2/3 for their denials and disjunctions of any two of them. A minimal increase in complexity is to have five probability values, 0 and 1 and three others for the propositions, their denials, and their disjunctions.

SOCRATES: I agree, and I see why among those three values, there'll be two summing to less than 1. For, among the supposed values, either one of them is 1/2 or none is. If one is, there may be three and, if so, the other two must be on each side of 1/2 and sum to 1, so that, if a proposition has one of them, its denial may have the other, as your axiom requires. In that case pick 1/2 and the lower one to be the probabilities of your A and B. . . .

THEAETETUS: That's it. If no probability is 1/2, there must be more than three, and we can pick two summing to less then 1 in several ways. For example, we may pick from .2, .4, .6, and .8 the combinations .2 and .4, or .2 and .6.

SOCRATES: Thank you for the help.

THEAETETUS: Having picked our A and B, we form the four compound propositions: A&B, NOT-A,&B, A&NOT-B, and NOT-A,&NOT-B. At least three of these must have positive probabilities, for our stipulations concerning A and B exclude all the ways two or more of them might have probabilities of 0. Let's see it's so:

compound	1	2	3	4	5	6	7	8	9	10	11
A&B	0	0	0				0	0	0		0
notA,&B	0			0	0		0	0		0	0
A&notB		0		0		0	0		0	0	0
notA&notB			0		0	0		0	0	0	0

Table 21: All the ways two or more compounds might have probabilities of 0.

Let's look at all eleven cases. Can you see how all six cases of two 0's have been excluded?

SOCRATES: I think so. Case 1 in the table assigns 0 probabilities to both A&B and NOT-A,&B. So their disjunction has a probability of 0. Yet the disjunction's equivalent to B, which we stipulated has a probability greater than 0. So case 1 contradicts a given. Is that the sort of reasoning you want?

THEAETETUS: Yes. We stipulated both A and B have positive probabilities. That eliminates alternatives 1 and 2, as well as the last five.

SOCRATES: The sum of our A's and B's probabilities is less than 1. We stipulated that also. But in alternative 3 their probabilities are behaving like those of contradictory propositions; so their probabilities sum to 1, as do those of contradictions. In alternative 4 their probabilities are behaving like equivalences; so their probabilities must be the same. We excluded same probabilities for A and B. The sum of their probabilities must be less than 1. But in alternative 5 A's sums to 1 by itself, and in alternative 6, B's does.

THEAETETUS: That does it. At least three of the four compounds have positive probabilities, and they're mutually exclusive. But that's precisely what the pseudo-theorem said was impossible.

SOCRATES: Let me review the argument. We imagined 5 numbers from 0 to 1 available for assignment as probability values in accordance with the axiom. After assuring ourselves that there would be two between 0 and 1 that summed to less than 1, we assigned them to propositions A and B. We then considered the four conjunctions of them and their negations. They're mutually exclusive, and there's no way two or more of them could have 0 probabilities. Therefore, at least three mutually exclusive propositions have positive probability. But the pseudo-theorem says that's impossible. So in this case it's denying a consequence of having 5 numbers from 0 to 1 ready for assignment as probabilities. By Beelzebub! It forbids all complexity whatsoever.

THEAETETUS: Yes. It would condemn the theory of probability to utter triviality, incapable of dealing with fields of propositions based on more than one idea.

SOCRATES: Clearly, we must reject the pseudo-theorem as itself absurd. So, rather than challenge the other premises from which you derive this absurdity, I agree we should reject the idea that the "if," the "|," is an operator on propositions.

THEAETETUS: We needn't assume that "|" stands for "if." The two may be different, in which case we only have to deny that "|" is an operator. "If" might be an operator on sentences and propositions, but "|" not.

SOCRATES: A nice distinction. But giving up the interpretation of "|" as an operator on propositions, yielding compound propositions, has a further consequence, whether or not "if" is the same as "|." We must conclude that conditional probability does not represent a degree of belief in any proposition. The "prob(*,*)" of conditional probability is a dyadic glue-word of the non-operator kind.

THEAETETUS: A similar point can be made of conditional bets. At the time they're agreed to, they're not then bets on any proposition. They'll become bets on a proposition, however, if and when the parties to the bet agree that the condition holds. [NOISE OUTSIDE. THEODORUS, SHARK, AND YOUNG SOCRATES BEGIN TO STIR FROM THEIR SLEEP.]

10. *So prob(*|*) is a dyadic function, coupled to a rule for changing one's degrees of belief.*

SOCRATES: Yes. Well, remember what the goal of our investigation was? We thought prob(A|B) is the degree of some mental state, and we wanted to find out what it was directed toward. Can you draw the conclusion we're looking for?

THEAETETUS: Back to our problem: Our PROBE and equal-length PROBE principles[20] led us to suspect that judgments of bets' fairness and disadvantage are based on comparisons

[20]See the first dialogue.

between their PROBEs and our own degrees of belief. Extending that idea to conditional bets requires that a person who judges a conditional bet fair must be comparing its PROBE to something. What is it?

SOCRATES: Yes; that's our problem. What's a conditional PROBE compared to?

THEAETETUS: No one thing. Since it *is* two PROBEs, it's compared to two things. It's nothing but the PROBEs of the combination of unconditional bets equivalent to a conditional bet. They' re compared to degrees of belief in one proposition each. I think I can rework all that I said about relevance to fit this scheme. . .

SOCRATES: You think timidly at a point where I sense a pay-off in boldness. You're put off by prob(A|B) having to be a dyadic operator, it seems.

THEAETETUS: Yes. A belief or a want is directed to a single object, a proposition which may be simple or compound, but which is one thing either way.

SOCRATES: But there are mental things which are one in their own right, but which are directed to two objects simultaneously. What of preference for one object over another? What of inferring one object from another? Can you think of something else mental, like preferring and inferring, for prob(A|B) to be?

THEAETETUS: It does seem to be some sort of comparison which prepares one for making a transition from one probability function to a successor probability function.

SOCRATES: Jargon. Tell me in plain words.

THEAETETUS: When you rolled the die for our drinks, your degree of belief that you'd beat the taverner went from 15/36 up to 22/36, then down to 12/36, then up to 1. Those changes can be thought of as selections of a single probability from four successive probability functions: prob'(*), prob"(*), prob'"(*), and prob""(*), with the same proposition in place of the asterisk each time. With other propositions in its place these functions yield other numbers. The symbol stands for your degrees of belief in all propositions at a particular moment. I'm afraid that's increased the jargon.

SOCRATES: But continue.

THEAETETUS: Note that *when* you were believing "I'll beat his roll" to degree 15/36, you were also believing "If Theaetetus beats his roll, then the chances are 22/35 that I will too." In other words, within the first probability function there were both these probabilities, an unconditional one and a conditional one: prob'(A)=15/36 and prob'(A|B)=22/36. Then you really learned that I beat him. At that point your degrees of belief changed to prob"(*). Your prob"(A)=22/36. You see? Your first probability function contained a way of making the transition to the second probability function. That's what conditional probability must be.

SOCRATES: So, when I believed I'd win the roll to degree 22/36, prob"(A), my eyes were still closed. But even while they were closed, I believed that if the taverner rolled a 4, my chances of beating him would be 12/36. That's to say, prob"(A|C)=12/36. When I opened my eyes and saw his 4, I obeyed a pre-established transition rule, changing my degree of belief again: prob'"(A)=12/36.

THEAETETUS: [TO SOME REVELERS] He's got it!

SOCRATES: And prob'"(*) contained the last transition rule; if I roll a 6, my chances of beating his roll, now known to be a 4, are 1: prob'"(A|D)=1. When I saw my 6, I followed this transition rule into prob""(*). My prob""(A)=1!

THEAETETUS: By George, you've got it!

SOCRATES: We've got it!

[REVELERS LEAVE SINGING "THE RAIN IN SPAIN STAYS MAINLY ON THE PLAIN."]

SOCRATES: We haven't got it yet. We can't go back to conditional propositions.

THEAETETUS: We're not. A transition rule is not monadic, because it mentions the exogenous cause of the transition, B, and its effect on my degree of belief in A.

SOCRATES: Is it self-psychology, if it's something mental?

THEAETETUS: It needn't be. Couldn't prob(A|B) describe something about my mind I'm unaware of? I think so.

SOCRATES: Then it has to describe more than a person's act of will to believe.

THEAETETUS: Right. It describes a deep propensity in a person for his beliefs to change as the rule says. A rule is more than a description of a decision.

SOCRATES: I understand. Then I may endogenously extend according to a rule what I learn exogenously, even if I can't tell when it governs me. Maybe I don't know whether it governs me.

THEAETETUS: Right. Your prob(*) can create propensities in you, hidden from you.

SOCRATES: Sometimes, however, I do forecast that number, usually because it ought to be my degree of belief, only rarely because I'm aware of my imperfections.

THEAETETUS: What're you driving at? When would I predict I'd act imperfectly?

SOCRATES: In your conditional bets with another, the both of you'll learn whether the proposition the bet was to be conditional on is true. But you'll've both learned it only, perhaps, by the time the bet's to be settled. You may not've learned it by the time the bet comes into force. For example, I may bet that I'm susceptible to deception: If you lie to me, probably I'll believe you. Obviously, I could never win my bet if I had to know the bet was in force at the time it came into force with your lying to me. I'd have to know you were lying to me, and then I couldn't believe you. So I only need to learn at the time of *settling* the bet that earlier you did lie to me. I win if I believed you then.

VOICE OUTSIDE: Over here, Anytus!

11. *We return to the notion of something normative for rationality, a rule of endogenous change.*

THEAETETUS: Let's distinguish prob(A|B) as prediction and as prescription, then. As a prediction or description of my propensities, it says *nothing* about my degree of belief in B, only that B's truth affects me. As prescription, we're imagining I know B at the time I give A its probability.

SOCRATES: Nicely distinguished. We can't follow a prescription unless we know that the condition holds for following it. And we can't bet on our own gullibility unless we *don't* know that the condition holds. The way around the dilemma is to distinquish between norms and descriptions as you did.

THEAETETUS: Thanks. We don't want the prediction to be prescriptive in our bet on our own gullibility anyway. It's just as well we can't know its condition's fulfilled.

SOCRATES: You say "know" the condition; do you mean being certain? Earlier you wanted to exclude certitude.

THEAETETUS: I want it now, Socrates. The equation for conditional probability is prob(B|B) = prob(B&B)/prob(B). Since B&B is just B, the fraction equals 1. If the rule's conditional on B, for me to obey it I must believe B to degree 1.

SOCRATES: But speak more carefully; the prescription binds us when we come to *feel* certain of the condition, not only when we've good warrant to be certain of it. To say we "know" is to suggest our certitude's warranted. It needn't be. What we become certain of needn't even be true; only that we think it is.

THEAETETUS: I'll be careful. Restrict prob(A|B)'s normative force to the degree of belief we should have *in A*, learned endogenously; our degree of belief in B is not prescribed.

SOCRATES: Correct. Let's agree that prob(A|B) exists in our minds unconsciously as a propensity or consciously as a prescription governing endogenous learning.

THEAETETUS: As analogous dyadic functions:

PRINCIPLE OF THE EXISTENCE OF prob(A|B) IN THE INTELLECT

PROB(A|B) IS THE DEGREE OF THE CAUSE OF:

FACTUAL:	A PERSON'S DEGREE OF BELIEF IN *A*, ACCORDING TO HIS PROPENSITIES, IF *B* AND ITS CONSEQUENCES COME TRUE AND AFFECT HIM, WHETHER OR NOT HE'S AWARE OF THEIR TRUTH.
NORMATIVE:	THE DEGREE OF BELIEF IN *A* THAT IS PRESCRIBED FOR THE PERSON, IF HE BECOMES CERTAIN THAT *B* IS TRUE, WHETHER OR NOT THAT CERTAINTY IS JUSTIFIED.

COINCIDENCE OF PREDICTION WITH PRESCRIPTION FAILS WHEN PREDICTION IS NOT BASED ON THE PERSON'S RATIONALITY.

I see a way of using diagrams to picture that a conditional probability of a proposition is a person's prescription of the degree of belief he ought to have in it upon becoming certain of the condition. We have to try for a more literal picturing of probabilities as areas, however.

Diagram 17: Prob'(*) and prob"(*) illustrate effect of "conditionalizing" on B. prob'(A)=.5 and prob'(B)=.7; but prob"(B)=1, so prob"(A)=3/7. Avoid thinking that possibilities are not represented on the second diagram. On the region interpretation of probability, they are there as regions of no area; see Appendix to Dialogue II. Up til now we've pictured possibilities of 0 area by just entering a 0.

SOCRATES: Please explain it.

THEAETETUS: The second field derives from the first by entering in the second the probabilities conditional on B that are derivable from the first. It's a description of a transition from not being certain of B to being certain of it. Then we see what happens to the probability of A. Let's call this transition "conditionalizing on B."

SOCRATES: OK. First I believe A to degree .5 and B to degree .7. These are values in prob'(*). Then I move to prob"(*): for some reason I change my mind about B to certitude. My belief in A thereby declines to 3/7.

THEAETETUS: Because B was negatively relevant to A in prob'(*).

SOCRATES: I see. In prob"(*) they're irrelevant, but skip that issue. You know, we haven't *proven* prob(A|B) is ever a norm; we've just asserted it is.

THEAETETUS: My current conditional probabilities are normative for my future unconditional probabilities, if I were to become certain of the conditions in the meantime. For a conditional bet I deem fair now ought to continue to seem fair if I become certain of the condition holding, while not learning anything else relevant to the subject of the bet.

SOCRATES: I don't know that a conditional bet I deem fair now will remain fair as my beliefs' degrees change, any more than I know that of my unconditional bets.

THEAETETUS: But *shouldn't* it remain fair as my belief in the condition it's conditional on changes in degree, while no other belief changes independently of it? Shouldn't there be at least that much constancy?

SOCRATES: Well, you haven't proved there should be.

THEAETETUS: I can.[21]

[Wise Conditionalizing on a Proposition]
[In Two Parts]

1. *Applying the idea of a dyadic function to objective chance.*

SOCRATES: But not now. We'd get sidetracked on diachronic issues, and we should finish up the synchronic ones, or sketch in other important features of prob(*), like your idea of its maturity. Do you recall my mentioning the idea of objective chance?

THEAETETUS: Oh, yes, I have thoughts about that too.

SOCRATES: It would help me to accept your subjectivist way of thinking of probability if you'd incorporate more objectivity than your observation that people with common wealth should agree with each other to avoid being victimized by biased books.

THEAETETUS: Can do. Suppose we symbolize the idea as you suggested to suggest a *prop*ensity: prop(*), the objective chance that a proposition's true, regardless of the degree of one's conviction about it. We can form a dyadic operator for conditional chance: prop(*|*).

SOCRATES: What does that mean? I presume it's defined as a quotient of unconditional chances, as prob(*|*) is a quotient of unconditional degrees of belief. The quotient's interpretation escapes me, however.

THEAETETUS: At the racetrack in Elis's stable, I noticed a hole in the ground. There was a chance before the race that, if she stepped in the hole, she'd sprain an ankle and not win the race. As the time of the race approached, the chance of her spraining an ankle decreased, don't you think?

SOCRATES: Because there was less time left for the accident to occur? Not just that; the chance was more a function of her distance from the hole. Was she within one step of it, or two

[21]Theaetetus knows a biased book argument for justifying the rule of conditionalizing, invented by David Lewis. It's presented in a dialogue on diachronic issues, one not given here.

steps of it, or three? The further she was from the hole, the less chance there was of her stepping into it.

THEAETETUS: Right, and the chance varied each time she took a step. So here's a way to interpret conditional objective chance. Before the race the

> prop(Elis sprains an ankle before the race | she needs to take *n* steps from where she's standing to step into the hole)

range from high to low as *n* ranges from low to high. These conditional chances make sense of the idea that her unconditional chance of spraining her ankle before the race *changed* as her position changed and as the time left for her to change position decreased.

SOCRATES: Objective propensities change too? At least there aren't various ones all coexisting, as is the case with different people's subjective probabilities.

THEAETETUS: Don't worry. There's just one world conforming to a single conjunction of physical laws, but the world is divided into times.

SOCRATES: Let's see what the quotient of propensities would be. The numerator would be the objective chance at any moment of a conjunction: She sprains an ankle sometime between that moment and the start of the race, and at one of the intervening moments she is *n* steps from the dangerous hole. The denominator is the unconditional chance of the second conjunct, her then being the *n* steps away. The unconditional chance that is the denominator depends on whether the time of the chance is or isn't the intervening time at which she is *n* steps away.

THEAETETUS: Yes; time does enter into the calculation of chance. We'll also need prime marks to distinguish prop(*) from the later prop'(*).

2. *The key to wisdom.*

SOCRATES: Another possibility dawns on me. We can mix prob(*|*) and prop(*) together. May I not have a degree of belief about what the objective chance of truth of a proposition may be?

THEAETETUS: Of course. We can say things like prob(prop(A)=*m*)=*n*. I may be dubious about a weather report, for example.

SOCRATES: No no, that's having a degree of belief about another's degree of belief. That's not what I mean. Think of the weatherman himself, not your reaction to his report. Doesn't he follow a rule of conforming his own degrees of belief to the propensities?

THEAETETUS: Yes, at least to the chances as they were when he last calculated them.

SOCRATES: Quite perceptive to distinguish the times. We shall have to mark our prob(*|*) as of a certain time, and our prop(*) as the last time before then at which we collected evidence of the chances. Can we do that?

THEAETETUS: I can't see any objection.

SOCRATES: So our weatherman might accept this principle. His procedure may be the rule to follow that leads to wisdom. It's this:

NORM OF WISDOM

(i) For any A, $prob_t(A|prop_{t-x}(A)=r) = r$.

I should believe A to degree *r* at a time, given that the chance of A being true (as of the last time before then for which I have evidence of the chance) is *r*.

THEAETETUS: Something like that should hold, by golly.

SOCRATES: If the chance now of a flipped coin coming up heads is 1/2, I should now believe to degree 1/2 that it will.

THEAETETUS: Of course. In that case the principle implies:

Prob(the coin lands heads|prop(the coin lands heads)=1/2) = 1/2.

The variation with time comes clearer with our example of Elis going lame. As the objective chance of that varied, so should our subjective degree of belief in it have varied.

SOCRATES: I wonder whether we've hit upon the key to wisdom. . . . Oh, no. Thunder and Damnation! Are our hopes to be dashed so quickly? Theaetetus, help me out of this difficulty.

THEAETETUS: What's the matter?

SOCRATES: In the example of the coin, the prop(A) = 1/2, if and only if the prop(A) = the prop(NOT-A). This truth I know independently of any experience of A. So I get a special case of our key to wisdom, namely, if (i) is the norm, then

(ii) For any A, prob(A|prop(A)=prop(NOT-A))=1/2.

THEAETETUS: Yes, that too we know independently of any knowledge of A.

SOCRATES: Similarly, I know another principle regardless of what A happens to be: If (i) is the norm, then

(iii) For any A, prob(A|prop(A)=2prop(NOT-A))=2/3.

THEAETETUS: Yes. I see no problem.

SOCRATES: May I not also substitute for the 1/2 in (ii) "prop(NOT-A)," when *r* = 1/2? I'd then get this:

(iv) For any A, prob(A|prop(A)=prop(NOT-A)) = prop(NOT-A).

THEAETETUS: I don't see what validates this step. (ii) does not say that the prop(A) = prop(NOT-A); when we look at the definition of the conditional probability on the left of the main equal sign in (ii) and (iv), we find in the denominator: prob(prop(A)=prop(NOT-A)), and that number need not be 1/2, although it'll be the same as prob(prop(A)=1/2), which also need not be 1/2.

SOCRATES: Let me try again. Since the 1/2 in (ii) is not due to any knowledge of A, but is simply a consequence of our accepting (i) and the condition that prop(A) = prop(NOT-A), it's true that the prop(NOT-A) = 1/2. Do you see the problem now, my boy?

THEAETETUS: I see an inference from (ii), which has the form X=Z, and (iv), which has the form X=Y, to the form Z=Y, namely,

for any A, prop(NOT-A)=1/2.

But (ii) and (iv) were supposed to be truths known independently of experience for any arbitrary A. Ah! Clearly we've gone wrong, if we can derive that 1/2 is the chance of truth of the denial of any arbitrary A.

SOCRATES: That is indeed the problem. To give the problem the starkness of out and out contradiction, consider substituting for the 2/3 in (iii) "2prop(NOT-A)" to yield:

(v) For any A, prob(A|prop(A)=2prop(NOT-A))=2prop(NOT-A).

What do you derive from (iii) and (v)?

THEAETETUS: By the same inference pattern as I used with (ii) and (iv) I derive that for any A, 2prop(NOT-A)=2/3.

Since (iii) and (v) are supposed to hold true of any arbitrary A, the denial of any arbitrary A is 1/3. But by our earlier reasoning it was to be 1/2! Contradiction!

SOCRATES: So shall we give up on our key to wisdom, which would disappoint me? Or shall we find an error somewhere in the derivation of the contradication? If we can find one, our key to wisdom will not've implied a contradiction.

THEAETETUS: Let's keep our key to wisdom. The errors were the substitutions in steps (iv) and (v). (i) and (ii) do not warrant (iv); (i) and (iii) do not warrant (v). The trick is that the A in (ii) and (iii) is an arbitrary A. Logically, it's quantified. We may not make a substitution into a formula in such a way as to let a quantifier capture a position it did not previously bind. Since the quantifier didn't bind the position of *r* in (i), we may not replace it or the numbers in its place in (ii) or (iii) with an expression that mentions A.

SOCRATES: That's jargon. Explain.

THEAETETUS: It may seem that, since "prop(NOT-A)" in (ii) occupies a position that *r* occupies in (i), we may also put it for the *r*-number in (ii) as well. But the quantifier, "for any A," has a scope that extends to the end of the sentence, but does not bind the position of the *r* in (ii). In (iv) the position becomes bound. This is invalid. The same reasoning invalidates the move from (iii) to (v).

SOCRATES: What you're saying reminds me of the words of a logician I read in my youth, "Variables free in the predicate must not be such as to be captured by quantifiers in the schema into which the predicate is substituted." You've shown how to avoid the contradication, and thanks to you we have a norm for wisdom!

THEAETETUS: I don't know about that. Don't we have to show that, if you follow this rule, you won't violate the axiom of probability and subject yourself to a biased book?

SOCRATES: Arrgghh! [STANDS UP AND COLLECTS ALL THE BETTING PARAPHERNALIA.]

SHARK: Oh, my head.

YOUNG SOCRATES: Where am I? [HE BUMPS INTO SOCRATES, AND THE BOOKS OF BETS, THE DICE, AND PLAYING CARDS GO FLYING. THE BOYS SCRAMBLE AROUND COLLECTING THEM. AT THE DOOR IS . . .]

MELETUS: Look at that, Anytus. See what I mean?

SOCRATES: Come along, boys. We go now to a house full of ladies, two old like Theodorus and me, but the rest young like you.

MELETUS: Oh!

SOCRATES: [PASSING MELETUS, AND LOOKING WIDE-EYED AT HIM] Boys, I promise you the girls' inter-discourse is at its best in the wee hours of the morning. You ain't heard nothin' yet!

Appendix to Dialogue II: *Theorems and Diagrams About Conditional Probability*

This appendix has three parts. In the first part we prove four theorems concerning the relationship of PROBEs to probs. In the second we extend the use of Venn diagrams to cover conditional probability. In the third part, we prove some theorems about conditional probability.

1. CHARACTERISTICS OF A COMBINATION BET, CONSISTING OF A BET ON "A&B" AND A BET AGAINST B (ON NOT-B)

KEY TO SYMBOLS: PROBE(A&B); prob(A&B); BET(A&B); STAKES(A&B); stand for PROBE of a bet on A&B; the degree of belief in A&B; a bet on A&B; and the total stakes in a bet on A&B.

PROBE(A|B), etc., stands for the PROBE of a bet on A, conditionally on B's acknowledged truth, etc. A conditional bet on A, given B, is a bet on A, provided B is recognized by bettors as true, and is no bet at all otherwise.

PROBE(A&B, not-B), etc., is the net PROBE of a combination of two bets, whose losses and gains are combined, etc.

<table>
<tr><td>1. A&B implies B</td><td>logic</td></tr>
<tr><td>2. prob(A&B) # prob(B)</td><td>1, theorem V</td></tr>
<tr><td>3. PROBE(A&B) is not disadvantageous <u>=</u>
PROBE(A&B) # prob(A&B)</td><td>PROBE principle</td></tr>
<tr><td>4. | Suppose: PROBE(A&B) not
| disadvantageous</td><td>hypothesis</td></tr>
<tr><td>5. | PROBE(A&B) # prob(A&B)</td><td>3,4 logic</td></tr>
<tr><td>6. | PROBE(A&B) # prob(B)</td><td>2,5, transitivity of #</td></tr>
<tr><td>7. | prob(B) = 1–prob(not B)</td><td>Theorem I</td></tr>
<tr><td>8. | PROBE(A&B) # 1–prob(notB)</td><td>6,7, substitution</td></tr>
<tr><td>9. | PROBE(A&B)–1 # –prob(notB)</td><td>8, algebra</td></tr>
<tr><td>10. | 1–PROBE(A&B) $ prob(notB)</td><td>9, algebra</td></tr>
<tr><td>11. | PROBE(notB) is not disadvan-
| tageous <u>=</u> prob(notB) $
| PROBE(notB)</td><td>PROBE principle</td></tr>
<tr><td>12. || Suppose: when PROBE(A&B) is
|| not disadvantageous, neither
|| is PROBE (notB)</td><td>hypothesis, combined with
hypothesis of step 4</td></tr>
<tr><td>13. || prob(notB) $ PROBE(notB)</td><td>11, 12, logic</td></tr>
<tr><td>14. || 1–PROBE(A&B) $ PROBE(notB)</td><td>10, 13, transitivity of $</td></tr>
<tr><td>15. || PROBE(notB) = 1–PROBE(B)</td><td>from definition of PROBE</td></tr>
<tr><td>16. || 1–PROBE(A&B) $ 1–PROBE(B)</td><td>14, 15, substitution</td></tr>
<tr><td>17. || –PROBE(A&B) $ –PROBE(B)</td><td>16, algebra</td></tr>
<tr><td>18. || PROBE(A&B) # PROBE(B)</td><td>17, algebra</td></tr>
</table>

NOTE: Step 12 states the condition under which step 18 holds true.

Theorem XVI: 19. If, when PROBE(A&B) is not disadvantageous, and neither is PROBE(notB), then PROBE(A&B) # PROBE(B)

Corollary: 20. If BET(notA) and BET(A&B) are not conjointly disadvantageous, then 0 <PROBE(A&B) # PROBE(B) # 1 — theorem XVI and definition of PROBE

Proof of equation of PROBE of combination bet, PROBE(A&B, not-B), with quotient of PROBES of bets on B and A&B:

1. Suppose [1–PROBE(notB)]STAKES(notB) = PROBE(A&B)STAKES(A&B) — 1st hypothesis; Condition II on the combination bet

2. Suppose PROBE(A&B) ... 0; STAKES(A&B) ... 0, — 2nd hypothesis

3. PROBE(B) / PROBE(A&B) x STAKES(notB) / STAKES (A&B) = 1 — 1, 2, algebra

4. PROBE(A&B,notB)= |net of both bets in the case B but not A| / sum of |net of both bets in the case B but not A| + net of both in case A&B, if denominator ... 0 — from definition of PROBE

5. net of both bets in case B but not A = –PROBE(A&B)STAKES(A&B)–PROBE(notB)STAKES(notB) — from table 17 in text

6. net of both bets in case B but not A = –[1–PROBE(notB\]STAKES(notB)–PROBE(notB)STAKES(notB) — from 5 and 1

7. net of both bets in case B but not A = –STAKES(notB) — from 6, algebra

8. net of both bets in case A&B = [1–PROBE(A&B)]STAKES(A&B)–PROBE(notB)STAKES(notB) — from table 17 in text

9. net of both bets in case A&B = STAKES(A&B)–[1–PROBE(notB)]STAKES(notB)–PROBE(notB)STAKES(notB) — from 8 and 1, algebra

10. net of both bets in case A&B = STAKES(A&B)–STAKES(notB)

11. $\text{PROBE}(A\&B,\text{notB}) = \dfrac{|-\text{STAKES}(\text{notB})|}{\text{STAKES}(A\&B)-\text{STAKES}(\text{notB})+|-\text{STAKES}(\text{notB})|}$ — from 4, 7, 10, substitution assuming 1 and 2

12. $\text{PROBE}(A\&B,\text{notB}) = \dfrac{\text{STAKES}(\text{notB})}{\text{STAKES}(A\&B)}$ — 11 simplified

13. $\dfrac{\text{PROBE}(B)}{\text{PROBE}(A\&B)} \times \text{PROBE}(A\&B,\text{notB})=1$ — 3, 12, substitution

14. $\text{PROBE}(B) > 0$ — 13, algebra

THEOREM XVII:

15. if 1, 2, then $\text{PROBE}(A\&B,\text{notB}) = \dfrac{\text{PROBE}(A\&B)}{\text{PROBE}(B)}$ — 13, 14, algebra, and collecting assumptions

RELATING PROBE OF COMBINATION BET THAT'S NOT DISADVANTAGEOUS TO PROBABILITIES:

16. Since we assume BET(A&B) not disadvantageous, PROBE(A&B) # prob(A&B) — by PROBE principle

17. PROBE(A&B,not-B)PROBE(B) # prob(A&B) — 15, 16, algebra

18. Since we assume PROBE(A&B) ... 0,

$\text{PROBE}(B)\ \# \ \dfrac{\text{PROBE}(A\&B)}{\text{PROBE}(A\&B,\text{not-}B)}$ — 17, algebra

19. 1–PROBE(notB) = PROBE(B) — from definition of PROBE

20. $1-\text{PROBE}(\text{notB})\ \# \ \dfrac{\text{prob}(A\&B)}{\text{PROBE}(A\&B,\text{not-}B)}$ — 18, 19, substitution

21. $\text{PROBE}(\text{notB})\ \$\ 1 - \dfrac{\text{prob}(A\&B)}{\text{PROBE}(A\&B,\text{not-}B)}$ — 20 algebra

22. Since we assume BET(notB) not disadvantageous when BET(A&B) is not, prob(notB) $ PROBE(notB) — PROBE principle

23. $\text{prob}(\text{notB})\ \$\ 1 - \dfrac{\text{prob}(A\&B)}{\text{PROBE}(A\&B,\text{not-}B)}$ — 21, 22 transitivity of $

24. prob(A&B)
1–prob(notB) # -------------------
PROBE(A&B, not-B) — 23 algebra

25. prob(B) = 1–prob(not B) — theorem I

prob(A&B)
26. prob(B) # -----------------
PROBE(A&B, not-B) — 24, 25 substitution

THEOREM XVIII:

27. if neither bet is disadvantageous in addition to satisfying conditions of theorem XVII, and prob(B) … 0, then

prob(A&B)
PROBE(A&B,not-B) # ---------
prob(B) — 26, algebra, & collecting accumulated assumptions

RELATING PROBE OF FAIR COMBINATION BET TO PROBABILITIES

THEOREM XIX: as above with PROBE designated FAIR and the relation is equality.

2. VENN DIAGRAMS FOR CONDITIONAL PROBABILITY

Time for a break! Let's go back to that easy stuff, diagrams. At the end of the second Appendix to Dialogue I, I promised some help with the idea of conditional probability. Here we go!

The probability of A given B we symbolize as prob(A|B). Read the "|" as "if" or "given." Since prob(A|B) = prob(A&B)/prob(B), if prob(B)>0, it is easy to compute a conditional probability from the information on a Venn diagram by the methods given in the 2nd Appendix to Dialogue I. In the following diagram, prob(A&B) = .2, and prob(B) = .5. So prob(A|B) = .4.

Diagram 18:

There's no area on this diagram that is prob(A|B). Alternatively, we can think of the way we read the conditional probability sign, namely, "given." Suppose we were given B. That is to suppose that prob(B) were 1. Then prob(A&B) would be in the same proportion to prob(B) as it is now:

Diagram 19a: The same as diagram 18.
(Supposing we're given B:)

Diagram 19b: The area of B is now the whole area.
So there is a sense in which an area represents conditional probability; prob(A|B) is represented by the area of A would have, once you're given B.

Several techniques demonstrated in the Appendix to Dialogue II carry over to conditional probabilities. For example, you may wonder if certain relations are theorematic for conditional probability. Venn diagrams are useful in exposing non-theorems. For example, you may wonder whether this is a theorem:

If prob(A&B)=prob(A)prob(B), then for any C, prob(A&B|C)=prob(A|C)prob(B|C)?

The suggestion is that, if two propositions are irrelevant to each other, then they are irrelevant no matter what else may be given. The suggestion is not a theorem, however, for the following diagram rejects it:

Diagram 20: Rejection of the formula as a theorem, since the formula is false of this model.
We make any if . . . then . . . sentence false by making its antecedent true and its consequent false. The diagram makes the antecedent of the statement true, for prob(A)=.3 and prob(B)=.3, and (.3)(.3)=.09, which is prob(A&B). Yet the diagram makes the consequent of the statement false. Prob(A|C)=8/15; prob(B|C)=1/2. (8/15)(1/2)=4/15, but prob(A&B|C)=1/6. Thus we reject the statement as a theorem. A theorem is supposed to be a universal truth. If we find one counterexample to a universal statement, we know it is not true.

For rare cases we can use Venn diagrams to demonstrate a theorem by the evident impossibility of constructing a counterexample. For example, is this a theorem?

If prob(B|A)=1, and any C is such that prob(A&C)>0, then prob(B|A&C)=1?

Diagram 21: The formula is very obviously not susceptible to rejection.
Making the antecedent of the statement true requires that two of the four subareas of A have a probability of 0. For prob(B|A)=1 just in case prob(A&B)=prob(A), and so the two subareas of A outside B must have 0 probability. The remaining subarea of A&C must have non-zero probability to fulfil the second antecedent. The consequent of the conditional is now forced to be true. The statement is a theorem. But what of

If prob(B|A)=1 and prob(C|B)=1, then prob(C|A)=1?

This also is a theorem, as the evident impossibility of a counterexample shows. However, no simple transitivity condition holds when the probabilities are less than certain.

Sometimes Venn diagrams are auxiliaries to a proof of a theorem with many conditions. In such cases it is necessary to show that the conditions are mutually consistent, for anything follows from a set of mutually inconsistent conditions. One Venn diagram in which all the conditions come out true suffices to demonstrate that the conditions are consistent. Two theorems from another Dialogue will illustrate this.

THEOREM XXXIV. If X and Y are mutually exclusive, prob(X)>0, prob(Y)>0, and prob(A|X) = prob(A|Y), then prob(A|(X OR Y)) = prob(A|X) = prob(A|Y).

The theorem is proved in the text of a Dialogue not in this set. This diagram demonstrates that the four conditions can be simultaneously fulfilled without inconsistency:

Diagram 22:

THEOREM XXXV: If X and Y are mutually exclusive, prob(X)>0, prob(Y)>0, and prob(A|X) ... prob(A|Y), then prob(A|X) < prob(A|(X OR Y)) < prob(A|Y).

This theorem is also proved in the text of a Dialogue not in this set. The two diagrams show the consistency of the four conditions, the first one with prob(A|X) > prob(A|Y) and the second one with prob(A|X) < prob(A|Y)

Diagram 23:

3. THEOREMS INVOLVING CONDITIONAL PROBABILITY

This theorem is cited in the Dialogue to show that prob(*|*) is an additive function:

THEOREM XX, that conditional probability obeys the axiom of probability:

If A_i are *n* mutually exclusive and jointly exhaustive propositions within B, and prob(B)>0, then

$$\Sigma_1^n \text{prob}(A_i|B) = 1$$

1. For each A_i and A_j, $i \neq j$, prob(A_i&A_j&B)=0
2. prob((A_1or . . . or A_n)&B)=prob(B)
3. prob(A_1&B)+ . . . +prob(A_n&B)=prob(B) — from 1 and 2 and theorem III
4. prob(A_1|B)+ . . . +prob(A_n|B)=1 — dividing both sides of 3 by prob(B)

THEOREM XXI, Bayes's Theorem:

If prob(A)>0 and prob(B)>0, then
prob(A|B) = [prob(A)prob(B|A)]/prob(B)

1. prob(A|B) = prob(A&B)/prob(B), if prob(B)>0
2. prob(A&B) = prob(A)prob(B|A), if prob(A)>0
3. prob(A|B) = [prob(A)prob(B|A)]/prob(B), if prob(B)>0 & prob(A)>0

Since B=(A&B, OR NOT-A,&B), and the disjuncts are mutually exclusive,
4. prob(A|B) = [prob(A)prob(B|A)]/[prob(A&B)+prob(NOT-A&B)]
Substitutions based on step 2 can now be made in the denominator to yield another form of Bayes's theorem:
5. prob(A|B) =[prob(A)prob(B|A)]/[prob(A)prob(B|A) + prob(NOT-A)prob(B|NOT-A)]

THEOREM XXII, Equivalence of the two definitions of d-rel(A|B) given in the dialogue:

First definition of d-rel(A|B) is, if 1>prob(B)>0,
1. prob(A|B) – prob(A|NOT-B)

2. prob(A&B)/prob(B) – prob(A&,NOT-B)/prob(NOT-B)

3.[prob(NOT-B)prob(A&B)/prob(NOT-B)prob(B)]
– [prob(B)prob(A&, NOT-B)/prob(B)prob(NOT-B)]

4. [prob(NOT-B)prob(A&B) – prob(B)prob(A&,NOT-B)]/prob(NOT-B)prob(B)

These two theorems will allow substitutions in the previous line:

prob(NOT-B)=1–prob(B)
prob(A&,NOT-B) = prob(A) – prob(A&B)

5. prob(A&B)[1 – prob(B)] – prob(B)[prob(A) – prob(A&B)] / prob(NOT-B)prob(B)

6. [prob(A&B)–prob(A&B)prob(B)–prob(A)prob(B)+prob(A&B)prob(B)]/prob(NOT-B)prob(B)

7. prob(A&B) – prob(A)prob(B) / prob(NOT-B)prob(B),
which is the second definition of d-rel(A|B). The numerator is the definition of s-rel(A|B).

All steps are algebraic and valid in either direction. Thus the definitions are equivalent.

The following theorems are all proved in the text:

The general product law for conditional probability from pages 94f:
THEOREM XXIII. If for each A_i prob(A_i)>0, excepting perhaps A_n, then
$\text{prob}(\boldsymbol{\&}_1^n A_i) = \text{prob}(A_1)\text{prob}(A_2|A_1)\text{prob}(A_3|A_1\&A_2) \ldots \text{prob}(A_n|A_1\& \ldots \&A_{n-1})$

If A is s-irrelevant to B, then B is s-irrelevant to A, proved on page 97:
THEOREM XXIV. If either prob(B) = prob(B|A) or prob(A)= 0, then prob(A|B) = prob(A)

The product law for mutually irrelevant conjuncts, proved on page 97:
THEOREM XXV. If prob(B|A) = prob(B), then prob(A&B) = prob(A)prob(B).

Four corollaries of theorem XXVII are listed in the text.

If A is s-irrelevant to B, then NOT-A is s-irrelevant to B, proved on pages 98f:
THEOREM XXVI.
If either prob(B) = prob(B|A) or prob(A) = 0, then prob(B) = prob(B|NOT-A)

If A is s-relevant to B, then B is s-relevant to A. This symmetry of s-relevance is proved on page 102:
THEOREM XXVII. If prob(B|A) > prob(B), then prob(A|B) > prob(A)

Like theorem XXIII for irrelevance, we can also prove that if A is s-relevant to B, then NOT-A is oppositely s-relevant to B:
THEOREM XXVIII. If prob prob(B|A) ... prob(B), then prob(B) ... prob(B|NOT-A)
The proof parallels the text's proof of theorem XXVI, first alternative:
1. prob(B) < prob(B&A)/prob(A)
2. prob(B) = prob(B&A) + prob(B&NOT-A)
3. prob(A) = 1 ! prob(NOT-A)
4. [prob(B&A) + prob(B&NOT-A][1 ! prob(NOT-A)] < prob(B&A)
5. prob(B&A) ! prob(B&A)prob(NOT-A) + prob(B&NOT-A) ! prob(B&NOT-A)prob(NOT-A) < prob(B&A)
6. !prob(B&A)prob(NOT-A) ! prob(B&NOT-A)prob(NOT-A) < !prob(B&NOT-A)
7. prob(B&A)prob(NOT-A) + prob(B&NOT-A)prob(NOT-A) > prob(B&NOT-A)

8. prob(NOT-A)[prob(B&A) + prob(B&NOT-A)] > prob(B&NOT-A)
9. prob(B&A) + prob(B&NOT-A) > prob(B&NOT-A)/prob(NOT-A)
10. prob(B) > prob(B|NOT-A)
11. Redo steps 1 and 4-10 with the reverse inequality signs.

Notes to Dialogue II

THE COHERENCE OF A COMMUNITY OF BELIEVERS:

1.

In the opening exchange Theaetetus refers to David Lewis's argument that propositions do not exhaust the objects of knowledge. See his "Attitudes *De Dicto* and *De Se*," in his *Philosophical Papers*, vol.1 (Oxford, 1983). This is just a tease; no more in the dialogue is about this subject.

For the three-shell game, also called the old Army game, see John Fisher, *Never Give a Sucker an Even Break* (NY: Pantheon, 1976), pp. 120ff.

Socrates's biased book exploits the irrationality of taking a consequence to be less probable than the condition it is a consequence of.

2.

Connoisseurs of vituperation will recognize the borrowings from P. G. Wodehouse, Alfred Hitchcock, George Bernard Shaw, Oscar Wilde, Samuel Johnson, Groucho Marx, and Shakespeare's Falstaff, Prince Hal, and Kent, mostly in this section.

3.

A common objection to personalist theories of probability is here refuted. Clark Glymour, *Theory and Evidence* (Princeton, 1980) pp. 74f, states the objection: "What we want is an explanation of scientific argument; what the Bayesian gives us is a theory of learning, indeed a theory of personal learning. But arguments are more or less impersonal; I make an argument to persuade anyone informed of the premisses, and in doing so I am not reporting any bit of autobiography. To ascribe to me degrees of belief that make my slide from my premisses to my conclusion a plausible one fails to explain anything not only because the ascription may be arbitrary, but also because, even if it is a correct assignment of my degrees of belief, it does not explain why what I am doing is *arguing*—why, that is, what I say should have the least influence on others, or why I might hope that it should." The dialogue addresses a part of this issue.

WHEN A BIASED BOOK IS THE LESSER EVIL:

1.

For the rules of craps, see John Scarne, *Scarne's New Complete Guide to Gambling* (1974). The probability of winning he figures this way (p. 278): Using the number 1980 to avoid fractional wins, in 1980 initial throws, 330 will be wins by virtue of rolling a 7 and 110 will be wins by virtue of rolling an 11. An initial roll of a 4 will occur 165 times but only wins 1/3 of the time, so there are 55 wins that way; the same calculation applies to an initial roll of a 10. An initial roll of a 5 occurs 220 times but only wins 2/5 of the time, so there are 88 wins that way; the same calculation applies to rolling a 9. An initial roll of a 6 will occur 275 times and will win 5/11 of the time, so there are 125 wins that way; the same calculation applies to rolling an 8. Adding up all the eight distinct ways of winning, there are 976 ways of winning out of 1980 initial rolls of the dice, or 488/495 probability of winning.

2.

Table 11 is based on Glenn Shafer, *A Mathematical Theory of Evidence*. Its source: G. R.

Iverson, W. H. Longcor, F. Mosteller, J. P. Gilbert, and C. Yontz, "Bias and Runs in Dice Throwing and Recording: A Few Million Throws" *Psychometrica* 36 (1971) 1-19. In this section I am glossing over the reasons for having one's subjective probabilities match objective chances or empirically determined frequencies.

I tie wisdom, over and above rationality, to following David Lewis's Principal Principle, that wise people conform their degrees of credence with known degrees of objective chance. Frank Ramsey suggested the same principle in his “Truth and Probability,” pages 195f of *The Foundations of Mathematics*. The principle will come up for discussion at the end of this dialogue.

3.

For the crooked dice the equiprobable outcomes, listed as sums, are 2,3,7; 3,4,8; 7,8,12.

CONDITIONAL BETS AND COMBINATION BETS:

6

The "synchronic" biased book argument for the equivalence of a conditional bet with a certain combination bet is due to Bruno de Finetti. See his "Foresight: . . ." in Kyburg and Smokler, p. 69.

A PERSON'S FAIR CONDITIONAL PROBE EQUALS HIS CONDITIONAL PROBABILITY

2.

For a brief history of the concept of conditional probability, its symbolization and its definition, see Glenn Shafer, "Bayes's Two Arguments for the Rule of Conditioning" *The Annals of Statistics* 10 (1982) 1075-1089.

THE MIND DETECTS DEGREES OF SYNCHRONIC AND DIACHRONIC RELEVANCE BETWEEN PROPOSITIONS

1.

I prefer the terminology of "relevance" to the standard terminology of "dependence." It is more accurate. It comes from Hans Reichenbach.

2.

The Greeks probably had games of chance with cards, but the modern deck originated in China in the twelfth century, reaching Europe three hundred years later. So says Richard Epstein, *The Theory of Gambling and Statistical Logic* (Academic Press, 1977) p. 158.

4.

The needed theorems are proved in the appendix to Dialogue I, steps 90-100.

5.

I get my measure of synchronic relevance from Rudolf Carnap.

6.

Theaetetus would like to show us all modern probability calculus based on the strong irrelevance he calls interchangeability (of A with NON-A in the irrelevance definitions). But we won't let him. What he calls interchangeability is de Finetti's exchangeability, introduced in 1931. See Kyburg and Smokler's "Introduction" to their *Studies in Subjective Probability* (Krieger Publishing, 1964) for an explanation of exchangeability and the theory that can be built up from it.

11.

By this point in the dialogue, Socrates is using conditional probability as a rule for revising his degrees of belief in response to sure evidence. This reading of conditional probability is called "Bayes's Rule" so-called after the Reverend Thomas Bayes, who died in 1763. His articles, published after his death, contained the rule. See the article by Glenn Shafer cited above.

12.

Wesley Salmon provides a model in which positive relevance is turned to negative relevance in his "Confirmation" *Scientific American* (1973).

13.

The example is adapted from one given by de Finetti.

The weighted average is also called the "expectation" or "expected value," usually when monetary values are averaged, using the probabilities of getting them as weights.

14.

Diachronic relevance is taken from Richard Jeffrey, *The Logic of Decision*, section 11.4. We will use it again in a Dialogue on diachronic issues (not included here), when we discuss learning without certitude.

CONDITIONAL PROBABILITIES CANNOT BE DEGREES OF BELIEVING A PROPOSITION

1.

The argument that a conditional probability is not the probability of the so-called material conditional, defined as this disjunction, is found in Jeffrey's *Formal Logic*, 2nd ed.

2.

The argument that a conditional probability is not the probability of any conditional is from David Lewis's "Probabilities of Conditionals and Conditional Probabilities" reprinted in *Ifs*, 129-147. For further refinements, see his "Probabilities of Conditionals and Conditional Probabilities II" *Philosophical Review* 95 (1986) 581-589

3.

Mon-adic, one; poly-adic, many. The polyadic are either dy-adic, two; tri-adic, three; tetradic, four; pentadic, five; etc. Medadic is the zero case.

6.

Van Fraassen identifies the assumption of the immateriality of order as the most challengeable point in this demonstration. See "Probabilities of Conditionals" in W. L. Harper & C. A.

Hooker, eds., *Foundations of Probability Theory, Statistical Inference, and Statistical Theories of Science* vol 1 (Reidel, 1976) 261-308. We return to this matter in the Dialogue on diachronic issues, mentioned earlier but not included in this set of three.

10.

See Lewis Carroll's "What the Tortoise Said to Achilles" *Mind* 4 (1895) 278-280, for the importance of distinguishing something dyadic, like a rule, from something monadic, like a belief.

The transition rule must be descriptive of the propensities created by prob(*) rather than descriptive of decisions or choices, a point stressed by Ramsey, "Truth and Probability," pp. 194ff.

The bet on one's gullibility is based on an example created by Richmond Thomason, as reported in Bas Van Fraassen, "Belief and the Will" *Journal of Philosophy* 81 (1984) p. 246.

11.

For this kind of diagram, as well as the area interpretation of probability it displays (discussed in the second Appendix to Dialogue I) see Lindley, *Making Decisions*.

WISE CONDITIONALIZING ON A PROPOSITION

1.

The example of a time-dependent objective chance parallels David Lewis's example of a time-dependent chance of finding one's way out of a labyrinth presented in his "A Subjectivist's Guide to Objective Probability" in his *Collected Papers*, vol. II.

2.

The formula is a simplified version of David Lewis's Principal Principle, from the same article. Ramsey also suggests the principle in his "Truth and Probability," and says it is derivable from the very concept of degree of belief. But I fail to see what argument he had in mind.

The contradiction is a form of Miller's paradox; see Richard Jeffrey, "Review of Eight Discussion Notes," *Journal of Symbolic Logic*, 35 (1970) 124-127. I follow the presentation and solution of Colin Howson and Peter Urbach, *Scientific Reasoning: The Bayesian Approach* (Open Court, 1989) 267-270.

The logician whom Socrates read in his youth was Willard V. Quine, *Methods of Logic*, 3rd ed. (Holt, Rinehart, and Winston, 1972). The quotation is Quine's second restriction on substitution on page148.

The proof that one's failure to follow the rule of conditionalizing makes one open to a biased book, suggested here by Theaetetus, was created by David Lewis, but reported by Paul Teller, "Conditionalization and Observation," *Synthese* 26 (1973) 218-258. I have a dialogue that presents the proof, not given in this set.