First Lecture on Truth Functional Logic

—**Logic**: the study of the proper forms of reasoning. "*Analysis and appraisal of arguments*."

—**Premises vs. conclusion**: the premises purport to show that the conclusion is true.

—**Inductive vs deductive**: Inductive arguments attempt to show that the premises make it probable that the conclusion is true. Deductive arguments attempt to show that the truth of the premises guarantees the truth of the conclusion. We will be concerned with deductive arguments in this course.

—**Validity**: The name we give to "successful" deductive arguments. In a valid argument, it is logically impossible for the conclusion to be false if the premises are all true. That is, to assert the joint truth of all the premises and the falsity of the conclusion would be to assert a logical contradiction, something of the form "P and not-P."

—**Soundness**: A valid argument with all true premises is known as a "sound" argument. But truth or falsity (in this case, of the premises of an argument) is not a matter of logic (unless the premises are either contradictions or tautologies), and so "soundness" is not our main concern in a logic course. (It is up to domains other than logic to determine the truth or falsity of factual propositions.) In logic, we are concerned with the strength of the reasoning, i.e., with whether or not the truth of the premises do indeed establish the truth of the conclusion. It is the role of science (observation and/or common sense) to tell us whether or not the premises are actually true.

Truth Functional Logic:

Review from last time:

KB weighs 400 lbs and it is not the case that KB weighs 400 lbs.

This statement, we noted, is a *contradiction*. Contradictions literally both assert and deny the truth of the very same claim. *Statements*, for us, are always either true or false, never both. Given this, contradictions are necessarily false. It is “logically impossible” for them to be true. The sole concept in this course is that contradictions are *always* false, as is any statement that implies the truth of any contradiction. Got it? Cool! The rest (and there is so much more!) is simply repeated applications of this one core principle.

When we say that in a valid argument that the truth of the premises guarantees the truth of the conclusion, what we mean is that the truth of the conclusion follows from the truth of (all) the premises *on pain of contradiction*. That is, it would be a contradiction to assert the truth of the premises and the falsity of the conclusion. If this (the truth of the premises together with the falsity of the conclusion) isn’t a contradiction, then it’s not a valid argument. So validity is a very strong feature of arguments. It is not a matter of extremely probability. If it is even possible for the premises to be true and the conclusion false, then it’s not a valid argument—it is *invalid*. It is also not a matter of degrees. An argument is either valid or invalid. There is no such thing as “almost valid” or “a little valid.”

Validity, I noted, is a *formal* feature of arguments. It is has nothing to do with the specific contents of the statements involved, but rather something to do with their internal structure. So we will look at the structure of statements, i.e., their "forms."

Note that in our example, the statement "KB weighs 400 lbs" occurs twice. So, the original statement is known as a *compound* statement, because it contains other statements as parts. But "KB weighs 400 lbs" does not contain any additional statements as parts, and so it is not compound, but rather *atomic*.

One more important concept: our original statement is not just compound (i.e., it has statements as parts), it is what logicians call a "truth functional compound." What this means is that the truths or falsity of the compound statement is entirely determined by the truth or falsity of the components statements, along with the logical symbols that compose the entire compound. The truth value of the compound is a (mathematical) *function* of the truth values of its components.

Consider the second part of our original statement: "It is not the case that KB weighs 400 lbs". This is what we call a "negation." "It is not the case" is just a long winded way of saying "Not" (or "It is not true that"). So, negations are truth functional compounds. If we know the truth or falsity of the original statement, given that we know how the word "not" works, we know the truth or falsity of the compound, of the negation.

Note we keep talking about the "truth or falsity" of statements. For simplicity, we will shorten this to the "truth value" of statements. There are two "truth values," *true* and *false*. (Soon, we will represent these two values as "1" and "0.") Remember that every statement is either true or false, never both, never neither. So every statement has a unique truth value. (Much of the time, of course, we do not know the truth value of a given statement. But we will just assume that it still *has* a truth value—that it is either true or false—even if we don't know which.)

So, negations are truth functional compounds. We know how the word "not" operates, so, for any given statement, if that statement is true, its negation is false. If it is false, then the negation is true. Simple, huh? We can represent this with a table, known as a "truth table."

We will use capital letters to represent truth functionally atomic statements. We will list all of the atomic components of a compound, and list all of the possible combinations of truth values to its components. In this case, this is simple. We have only one component, and it can either be true or false. So, the truth table for negation has two lines or rows, one where the component is false, and another where it is true. We then add a column (to the right) that contains the truth value of the compound—in this case, a negation, which is determined by the truth values of its components, given to the left. So, here is the truth table for negation.

Negation, "Not P," "~P"

P	~P
0	1
1	0

"~", called the "squiggle" or "tilde" is our symbol for negation. This table tales us that for any statement (proposition) "P," if P is false, then ~P is true, and if P is true, then ~P is false.

Now, and important point: The table above perfectly represents our understanding of how English (our natural language) works. It tells us nothing we don't already know. But what we are actually doing in this table is *stipulating* how "~" works in the "artificial language" that we are constructing. We are literally *creating* a language here. There are rules, rules of grammar if you like, for what a proposition must look like. We'll get to those soon. We want this artificial language to mirror the logic of how we actually speak, so that is why we fill out this table in the way we do. But what the table literally does is tell us that, in the language we are creating, the truth value of ~P is the opposite of the truth value of P, and this is true no matter what "P" stands for, whether itself atomic or compound.

But now return one last time to our original statement: "KB weighs 400 lbs and it is not the case that KB weighs 400 lbs." What we have just looked at is the second part of this, i.e., "It is not the case that KB weighs 400 lbs." But this is only one part of the complete statement. The statement as a whole is a compound of that statement and "KB weighs 400 lbs." So, the contradiction is, if you will, multiply compound. It contains components some of which (in this case, the second part) are themselves truth functionally compound. Let's use "K" for "KB weighs 400 lbs." So in this case what we have, known as a *conjunction*, is the claim that "K and not K." It is a truth functional compound where the components are "K" and "not K." In a conjunction, the components (in this case "K" and "not K" or "~K") are known as *conjuncts*.

We can now introduce the truth table for conjunctions:

Conjunction, P and Q, or (P • Q)

P	Q	(P • Q)
0	0	0
0	1	0
1	0	0
1	1	1

We are using "•" as our symbol for conjunctions. It functions as does the English word "and." It is called "dot." So, the conjunction could be read as "P dot Q."

Note that our table is a bit more complicated than above. It is has two distinct components, unlike negation, which has only one. Since each component can have one of two different truth values, we must list all four possible combinations. That is what we have done on the left side of the table. The components can either be, false-false, false-true, true-false, or true-true. *What the table tells us is that a conjunction is true if, but only if, each of its conjuncts is true. Otherwise, it is false.* This is something that you need to internalize, i.e., memorize so completely that you don't have to think about it. If even one conjunct is false (and so obviously if both are), the conjunction is false. The conjunction is true only if and only if both conjuncts are true.

Note that the table lists all possible combinations of truth values to the components of the compound. It must do this if the table is to tell us, in all cases, what the truth value of (P • Q) will be given any possible combinations of truth values to its components. So, in a negation, there is one distinct component, and two possible truth value assignments to it. In a conjunction, there are two components, and so there are four possible unique assignments of truth values to its components. Can anyone guess how many unique truth values assignments we would need for a compound with *three* distinct components? 8. In general, if “n” is the number of components, 2^n is the number of distinct truth values assignments to those components. (You don’t need to worry about the math here. Just understand that for every additional distinct component, you double the total number of different possible truth value assignments to the components of a truth functional compound.)

Note also that the distinct assignments are actually listed in a kind of order. We start with (all) 0’s and end with (all) 1’s. They are listed in numerical order counting in “base 2.” Don’t worry if you don’t remember this from some math class. You will have to learn to “fill in” complicated truth tables, but the tables themselves (and so this order), will always be provided for you.

Note also that we have used parentheses around “P • Q” In this text, i.e., in the artificial language we are creating here, these are not optional. There are strict rules for when they must and must not be used. These rules are known as the rules for “Well formed formulas.” In general, these groups of symbols we are constructing are known as “formulas.” But not any arbitrary group of symbols is well formed. Here is our initial set of rules for well formed formulas (WFFs):

<u>Well Formed Formulas</u>

1. Any capital letter is a wff.
2. The result of prefixing any wff with “~” is a wff.

3. The result of joining any two wffs with "•", "∨", "⊃" or "≡" and enclosing the result in parentheses is a wff.

Note that "~P" doesn't need or use parentheses. "~P" is a wff, "~(P)" is not. See rule 2.

So, ~(P • Q) is a wff. So is (~P • Q) (but note that these two wffs say different things!) while ~P • Q is not. Without parentheses, we wouldn't know if the "~" was being applied to the entire conjunction (Not Both-P-and-Q), or being applied only to the first conjunct (Both not-P-and-Q).

There are three additional symbols here that we haven't yet discussed, "∨" "⊃" and "≡", known as "wedge," "horseshoe," and "triple bar." These are used for three additional truth functional connectives that we will now introduce.

Our third compound is known as a disjunction, and it is used to represent the logic of the word "or" in English, as in "Either it will rain today or KB will eat a bug." The word "or" in natural language is ambiguous. Sometimes when we use the word "or" we mean to include the possibility that both parts—both *disjuncts*—are true, but sometimes we mean to exclude the possibility that both disjuncts are true (as in "Either it will rain today or it will rain tomorrow"—the implication seems to be that it won't rain both days). The first is known as the "inclusive" use of the word "or," while the second is known as the "exclusive" use. These two uses of the word "or" are described by different truth tables. So we will arbitrarily use wedge, ∨, to represent the inclusive use. This is just an arbitrary choice and has no lasting importance, as there will be ways to capture what we mean by either the inclusive or exclusive senses of the word "or." So, here is the truth table for disjunction:

Disjunction, P or Q, (P ∨ Q)

P	Q	(P ∨ Q)
0	0	0
0	1	1
1	0	1
1	1	1

What this tells us is that a disjunction is false if and only if both of its disjuncts are false. Otherwise, it is true.

Our fourth type of truth functional compounds are formally known as "conditionals," i.e., "if/then" statements. Example, "If it rains today, then KB will eat a bug." These statements say that if the first part is true, then the second part is also true. Unlike conjunctions and disjunctions, the order of these two parts matters, and so they have different names. The "if" clause is known as the "antecedent" of the conditional, and the "then" clause is known as the "consequent." So, in my example, "It rains today" is the antecedent, and "KB will eat a bug" is the consequent.

Understanding how to create a truth table for conditionals is bit more complicated. Clearly if it *does* rain today and I do *not* eat a bug, then the conditional is false. But what if it doesn't rain today? How does this affect the truth or falsity of the conditional? This get complicated in ways that are philosophically more complicated than we can talk about now. So what we will do is simply stipulate the *weakest* possible understand of "if/then" in the truth table for conditionals. Later (in another course), there are ways of complicating our symbolism to capture other things that we sometimes mean with "if/then" statements. But for now, we will say that a conditional is false if but only if the antecedent is true and the consequent is false. It is true otherwise.

We use the symbol "⊃", or "horseshoe" for conditional statements. So, here is the truth table for conditionals:

Conditionals, If P then Q, (P ⊃ Q)

P	Q	(P ⊃ Q)
0	0	1
0	1	1
1	0	0
1	1	1

Internalizing this truth table will take more time than the previous 3, and that is because it is more arbitrary. This causes students problems. At this stage, I can only say, "Trust me." The table may not seem to capture what we typically mean by "if/then." That is true! But that is because what we mean by "if/then" is often far more subtle and complicated than it initially seems. So the truth table we introduce one use of such statements, the least philosophically interesting. If, after philosophical analysis, we think we need different kinds of truth tables to capture what we really meant, there are ways to do this (in further courses).

So, again, a conditional, as we are representing it here, simply says that it is not the case both that the antecedent is true and the consequent is false. Alternately, it says that either the antecedent is false or the consequent is true (or both). You will get better at doing the exercises to come the sooner you start thinking of conditionals as just another way of saying one of these two things.

One further point before introducing our fifth and final truth functional compound. I just noted that if a conditional is true this means that it is not the case both that its antecedent is true and its consequent is false. Likewise, saying that a conditional is true means that either the antecedent is false or the consequent is true. Look at the truth table above if you don't see this. So, (P ⊃ Q) says the same things as (is logically equivalent to) ~(P • ~Q), and also the same thing as (~P ∨ Q). What this means is that with the five connectives we will introduce, there are alternate ways of saying the same things. Some compounds are *logically equivalent* to others, meaning that they come out true or false on exactly the same assignment of truth values to their components.

This is relevant in understand our final truth functional compound, "if and only if," known as the *biconditional*. As the words suggest, "P if and only if Q" is essentially just a short had way of saying, "P if Q and Q if P are both true."

So here is the truth table for biconditionals:

Biconditionals, P if and only if (iff) Q, (P ≡ Q)

P	Q	(P ≡ Q)
0	0	1
0	1	0
1	0	0
1	1	1

So a biconditional is true if and only if both components have the same truth value. It is false otherwise.

Finally, I noted at the beginning that contradictions were logically impossible. We can use a truth table to establish this. Our initial contradiction was “KB weighs 400 lbs and it is not the case that KB weighs 400 lbs.” Since “KB weighs 400 lbs” is truth functionally atomic (i.e., it does not contain other statements as truth functional components), we will represent it with a capital letter, in this case, “K.” So, the statement says that “K and not K” which we represent as “(K • ~K).” We can demonstrate its logical impossibility with a slightly more complicated version of a truth table created above:

K	~K	(K • ~K)
0	1	0
1	0	0

The first two columns simply list the possible truth values of K and ~K, while the third column gives us the truth value for the compound, (K • ~K) given the truth or falsity of K on that row.

Consider what this shows us. K is either true or false. The first row considers the possibility where K is false. In that case, ~K is true, but the conjunction (the final column on the right) is false because at least one of its conjuncts (the first one) is false. On the second row, we consider the possibility that K is true. In that case, ~K is false, and once again, the conjunction is false because one of its conjuncts (this time, the second one) is false. So, the compound statement "(K • ~K)" is false on all possible truth value assignments. It is *logically impossible* because it is false in all possible circumstances. It is not false because of "the way the world is" (i.e., because of the truth or falsity of K), but in virtue of its form alone. No matter *what* statement "K" stands for, (K • ~K) is false. It is a logical contradiction, and so is false in virtue of the rules of logic, and not because of any facts about the nature of the world.

Translating English into Symbolic Form

There is "syntax" for this new language we are introducing. These are rules for what counts as a "well formed formula," or a "wff." Those rules were provided above.

We will spend a lot of time in this course "translating" English statements into this artificial language we are creating. Look closely at the text for help with this. Let me note a couple pointers.

Note that "**~(P • Q)**" says something different (and is true and false under different circumstances) than "**(~P • Q)**." The first says that it is not the case that *both* P snd Q are true, while the second says that *both* not-P *and* Q are true. So the first— **~(P • Q)** —is the negation of a conjunction (Not *both* P and Q), and the second is a conjunction where the first conjunct is a negation (Both not P and Q). Visually, these are similar and easy to confuse, but they say very different things. So you need to be careful!

Likewise, these are different:

(P • (Q ⊃ R)) P, and if Q then R

This is a conjunction with a conditional as the second conjunct.

((P • Q) ⊃ R) If both P and Q, then R

This is a conditional with a conjunction as its antecedent.

Some Hints for translations (from the text)

Put "(" whenever you see "both" (what follow "both" is always a conjunction), "either" (what follows "either" is always a disjunction), or "if" (what follows "if" is always the antecedent of a conditional).

Use parentheses to group together parts of an English sentence on either side of a comma:

If A, then B and C.	(A ⊃ (B • C))
If A and B, then C.	((A • B) ⊃ C)

Use capital letters to stand for atomic but *complete* statements. E.g., "Gensler is happy." should be just "G" (or any capital letter), **not** something like "(G • H)." But "Gensler is a professor and a logician" is a *compound* statement: it contains other statements as proper parts, so it would not be translated with a single capital letter, but as a conjunction, say, "(P • L)."

You must pay close attention to such subtle differences.

(We will be doing *lots* of translation in this course. I have provided a link to a guide to all the common forms you will need to learn. It may look intimidating, but only the *fist page* is relevant to what we are doing in Chapter 6. Here is the link: http://homepages.wmich.edu/~baldner/transguide.pdf .)

<u>Truth Evaluations</u> (from Chapter 6.3)

If you know (or are given) the truth value of the atomic components, you can infer or calculate the truth value of any compounds containing them.

Suppose that P=1, Q=0, and R=0.

Consider: ((P ⊃ Q) ≡ ~R)

First, replace the letters with the given truth values:

((1 ⊃ 0) ≡ ~0)

Then, in a step-by-step fashion, calculate the truth value of the whole, starting with the smallest parts. First, replace "~0" with its truth value—1, then "(1 ⊃ 0)" with its truth value—0, then calculate the truth value of the biconditional. Thus, a completed exercise would look like this:

((P ⊃ Q) ≡ ~R)

((1 ⊃ 0) ≡ **~0**)

((**1 ⊃ 0)** ≡ 1)

(0 ⊃ 1)

1 So, on the given assignment, the truth value of the compound is "true."

(I have also provided a link with more discussion of how to do the kinds of "Truth Evaluation" Exercises you will find in 6.3 and 6.4. Here is the link: http://homepages.wmich.edu/~baldner/truthevaluationinstructions.pdf .)