Transition to Quantified Predicate Logic

Predicates

You may remember (but *of course* you do!) during the first class period, I introduced the notion of validity with an argument much like (with the same logical form as) the following:

All humans are mortal.
Jones is human.
So, Jones is mortal.

I used this example because I thought it was easy to "see" how the truth of the premises guarantees the truth of the conclusion. (I may have used a "Venn diagram" to visually represent how the information in the conclusion is already contained in the premises--if the class of H's is contained inside the class of M's, and j is included in the class of H's, then j is included inside the class of M's.) But, after using this as an example of a valid argument, I then said that we would have to return later in the course to arguments of that sort. That's what we're doing now.

So let's assume that you see that the argument is valid. You should also be able to see that we have no way to represent the validity of this argument in the truth functional propositional logic we have been studying thus far. That is, you should recognize that there are no truth functional connectives in any of the three statements of this argument. Each statement is truth-functionally atomic. Were we to represent this argument using capital letters to stand for truth functionally atomic statements, we would end up with something like this:

A
B
So, C.

And this, of course, is not a valid truth-functional argument.

I go back to this example to show the limits of how we have been translating truth functionally atomic statements. If we translate distinct atomic statements with distinct capital letters, each of the statements in this argument get translated with a different capital letters. This manner of translating these statements fails to capture the obvious fact that these statements are alike in certain ways. The last two statements both talk Jones, but say different things about her (that she is human, and that she is mortal). The first statement doesn't talk about Jones, but it talks about the two different things (being human and being mortal) that are said about Jones in the other two statements. So each statement has some internal "parts" in common with the others. But since we have used a single symbol (a capital letter) for truth functionally atomic statements, we had no way to indicate any "parts" within these truth-functionally atomic statements.

The first statement (the first premise) contains the word "all." Capturing the word "all" (and its good friend, "some") brings us into the realm of quantified logic. Let me come back to "quantified statements" like this in a moment or two. For now, examine the second and third statements:

Jones is human.
Jones is mortal.

These statements obviously have something in common--they both talk about Jones--and something not in common--they say different things about Jones. This suggests that we need a new way of representing truth-functionally atomic statements, a way that uses 2 symbols rather than 1, so that we can capture the way in which these two statements are the same (and so they will have one part in common), as well as the way in which they are different (and so the other part of each will be different).

One last point before actually describing how we will translate these statements: note that what they have in common is that they each share the same grammatical subject (they each talk about the same thing), while where they differ is with respect to their grammatical predicates (they each say different things about their respective subjects). So, we will want a way to capture the difference between the grammatical subject of a statement, and its grammatical predicate.

To that end, we will let capital letters stand for grammatical predicates and lower case letters stand for grammatical subjects. Each of the above statements is about the same subject, so it will contain the same lower case letter. But they contain different predicates (they say different things about Jones), so they will contain different capital letters.

Finally, the order in which we will write these two letters is different than in conventional English. We will start with the capital letter, standing for the predicate, which will then be followed by a lower case letter, standing for the subject. So

Jones is human.

becomes

Hj (think: "Being H is true of j," or "j has the property H.")

and

Jones is mortal.

becomes

Mj (think: “M is true of j” or “j is M.”)

Before moving on, let’s get some terminology straight. We are using capital letters to represent “**general terms**,” terms that describe something or put it in some *general category*. And we using small letters to represent “**singular terms**,” terms, that is, that pick out some *specific* person or thing (or refer non-specifically to a member of a specified group).

Note that “Hj” and “Mj” are still truth functionally atomic. Neither contain any of the five truth functional connectives. All we have done at this point is complicate how we translate truth functionally atomic (non-quantified, we’ll get to quantified shortly) statements: Instead of capital letters, when the statement has a subject/predicate form, we use a capital letter and one (or more--keep reading!) small letter.

Relations

Both of the above statements describe a single specific thing: they say of that specific thing that it has some property, or that it is a member of some general category or class of things. But in addition to talking about the properties an individual thing might have, sometimes we want to talk about how two or more specific things are *related* to one another. So consider statements such as:

Romeo loves Juliet.

9 is greater than 5.

Susan is the brother of James.

These statements all contain two singular terms--i.e., they talk about two specific things, and say how those things are related. Thus, we will also want to let letter capital letters stand for general *relations* that specific things can stand in to one another. So we could translate these as follows:

Lrj	(the “__ loves __” relation holds between “r” and “j.”)
Gnf	(the “ __is greater than __” relation stands between “n” and “f.”)
Bsj	(the “__is the brother of __” relation stands between “s” and “j.”)

Note that the order in which the singular terms (the lower case letters) occur matters: "Lrj" ("Romeo loves Juliet.") say something different from "Ljr" ("Juliet loves Romeo").

Though we won't see any examples in the text, we should note that in addition to "two place" relations like " __ loves __", there are three (and more) place relations as well. Consider:

Jones gave this box to Smith.

This statement says that three individuals stand in a certain complex relation: namely, the first thing gives the second thing to the third thing. Letting "G" stand for this 3-place relation, we would translate this as:

Gjtb (read: j gave t to b.)

With this, we now have a way of capturing the subject/predicate form implicit in many atomic statements. Our Rules for WFF's will be modified to include *a capital letter followed by one or more lower cases letters* as a well formed formula.

Universal Quantifiers

But we have still not said anything about the first premise of our original argument, "All humans are mortal." Translating this statement will still take us a couple of steps. The first of those steps is to introduce what is called "the universal quantifier," which we can represent "(x)." Intuitively, we can read this as "for all x."

"For all x"   is written   "(x)"

It says that something (some wff that immediately follows the quantifier) is true for all values of x. "x" is being used here as a "variable." It stands for some un-specified thing.

We will use the universal quantifier to translate statements about "all" or "everything." So, suppose I wanted to say "Everything is pink." Think about how you would say this, in English, if you had to begin the sentence with "for all x." (Our universally quantified symbolic statements will always begin with a universal quantifier, so we need to think of re-phrasing the English sentence in way that begins with "For all x, ...") You would say something like, "For all x, x is pink." We just said that "For all x" can be translated as "(x)," so where does that leave us with the rest of this statement, namely "x is pink"? Following what we did above with statements like “Jones is mortal,” we can translate “x is pink” as “Px.” And with this we can now translate “Everything is pink” as:

(x)Px (i.e., “For all x, x is P.”)

This says that “Px” is true for all values of “x,” i.e., that, take anything you like, ***it*** (***that thing***, call it ***x***) has the property P. If “P” stands for “is pink,” it says that “x is pink” is true for all values of x. Take anything you like, it is pink. I.e., everything is pink.

With this, we must now make a distinction between lower case letters. Lower case letters from “a” through “w” are used as ***constants***. You can view these as “names.” Constants are expressions that refer to a specific individual (like “Jones,” or “this table” or “the best logician in the class.” In addition to constants, we use the lower case letters “x,” “y,” and “z” as ***variables***. Variables do not stand for specific individuals, but function more like the pronoun “it.”

So, note that

Pj

is a complete statement: it asserts that something specific thing, “j” (maybe “Jones”) has the property “P.” It has a truth value (even though we don’ know what it is). But consider:

Px

This is not a complete statement. “x” does not stand for a specific individual, but functions more like the pronoun “it” in English. So, “Px” would be translated as something like

it is P

But what is this “it?” Note that “it” here has no “context.” There is nothing to tell us what “it” stands for. That is why we called wffs like “Px” *incomplete* statements. Without being “completed,” it has no truth value, as we have no context for understanding what this “it” is that is being talked about. But if I preceded “Px” with the universal quantifier, thus:

(x)Px

then I would have a complete statement. This says, "For all x, x is P," or, if you like, "Take anything you like (call it x), it is P." So the quantifier tells us what this "it" is (what the "x" is) that we are talking about when we say "Px."

To repeat: both "Px" and "(x)Px" are wffs. But the first is not a complete statement. By itself, we don't know what it's talking about, and so can't know if its true or false. But the second wff is a statement or assertion. Prefixing the universal quantifier in front of "Px" gives us a complete statement, because it tells us what this "it" is: "Take any thing you like, *it* has property P."

Going back to our example, what if I wanted to say that "Everything is both pink and square"? (Don't bother with the silliness of my examples!) Again, how would I say this, in English, if I had to begin with "for all x"? I would say that "For all x, x is pink and x is square." And I could now write that:

(x)(Px • Sx)

(I could also read this: "Everything is such that it is both pink and square," or as "Take anything you like, call it x, it is true both that x has property P and that x has property S.")

Likewise, if I wanted to claim that everything is both human and mortal, I would write:

(x)(Hx • Mx)

This says that "(Hx • Mx)" is true no matter what "x" stands for, or "for all values of x." So it says that everything is human and mortal.

But, now let's look at our initial premise. We just saw that "(x)(Hx • Mx)" says that everything is human and mortal. Hmmm... How do I say "All humans are mortal"?

I could just answer you, but think a bit first about what the statement says and does not say: it does *not* say that everything is both human and mortal. What it does say is that *all humans* are mortal. That is, it says that all the things which are human things are likewise mortal things. How do we say this, in English, if we must begin with the phrase, "for all x?" We would say something like, "For all x, if x is human, then x is mortal." And so our original premise becomes:

(x)(Hx ⊃ Mx)

I.e., for all x, if x is H, then x is M.

This gives us a general way of saying "All A's are B's." Such statements say that anything which is in the class of A's is also in the class of B's, i.e.,

$(x)(Ax \supset Bx)$ (Take anything you like, if its an A, then its a B.)

Understanding that statements like "All A's are Bs" involve quantified conditionals takes a bit of getting used to. This comes with practice! So,

Everything is both A and B (or "All are both A and B")

is translated,

$(x)(Ax \bullet Bx)$ (take anything you like, it is A and it is B).

But

All A's are B's

is translated

$(x)(Ax \supset Bx)$

With this, we can translate our original argument:

All humans are mortal.
Jones is human.
So, Jones is moral.

becomes

$(x)(Hx \supset Mx)$
Hj

∴ Mj

Proving that this is valid will require new rules of inference, which we start studying in 7.2. But before that, we still have one more quantifier to introduce.

Existential Quantifiers

We have already introduced the "universal quantifier" into our language. It is a variable ("x," "y," or "z") that occurs between enclosing parentheses. It is our way of capturing statements about "everything." But there are times when we want to talk not about "everything" but about "something." We already looked at the (rather silly) claim that everything is pink. This claim is false if there is even one thing which is not pink. But suppose we wanted to say merely that there is something that is pink? The statement,

Something is pink.

will be true as long as there is even one thing that is pink. There may be more. It may even be true that everything is pink. (After all, if everything is pink, it follows that something is pink!) But, in general, "Something is P" will be taken to mean that "There is at least one thing such that it is P." (The claim "Something is P" is false only if *everything* is such that it isn't P.)

To translate statements like "something is pink," we now introduce the ***existential quantifier***. This is normally written with an upside down "E" preceding a variable, and enclosed within parentheses. So,

(∃x)

can be read "For some x" (or, "There is at least one x such that ...."). And the claim that "Something is pink" can now be written:

(∃x)Px

(Read: There is an x such that x is P.)

The text contains a number of practical "hints" for doing translations, and you should look at these. But an issue that causes problems for some students is recognizing the difference between how we translate “All A’s are B’s” and “Some A’s are B’s.” These statements look like they differ only in the quantifier they begin with. One talks about “all x,” and the other talks about “some x.” But their internal structure is different. We have already noted that

All A’s are B’s.

must be translated as

$(x)(Ax \supset Bx)$ (Everything is such that, if its A, then its B.)

But

Some A’s are B’s

is different. It says that there is some A which is also B. So, it says that there is something that is both A and B, thus:

$(\exists x)(Ax \bullet Bx)$ (There is something that is both A and B.)

So, “All A’s are B’s” (and all other statements of this same form) is a universally quantified conditional , while “Some A’s are B’s (again, and all other statements of the same form) is an existentially quantified conjunction.

Practice, practice, practice!