Logical Equivalence, Tautologies, and Contradictions

To say that two propositions are "logically equivalent" is to say that they are true or false in exactly the same circumstances. We can show this by the use of truth tables. To say that two propositions are true in the same circumstances is just to say that they have the same truth-value (i.e., truth or falsity) for any given assignment of truth values to their atomic components. When we create a truth table, we are simply evaluating a statement (or a group of statements) for all such possible assignments.

As an example, let's start with the example from class. I said that "unless" could be translated as either "if not" or "or," because (~Q ⊃ P) was logically equivalent to (P v Q). Let's construct a truth table to prove this.

First, we list on the top line of the table all of the atomic components (in this case, "P" and "Q"), and then each of the two statements we are evaluating for logical equivalence (in this case "(~Q ⊃ P)" and (P v Q)). Thus,

P	**Q**	**(~**	**Q**	**⊃**	**P)**	**(P**	**v**	**Q)**

Then, we fill in the all the possible truth value assignment to the atomic components, as listed on the left (left of the double lines) side of the table. Thus,

P	Q	(~	Q	⊃	P)	(P	v	Q)
0	**0**							
0	**1**							
1	**0**							
1	**1**							

Note again that this list contains *all* the possible truth value assignments for a combination of two atomic propositions. We now move from the left side of the table to the right. For each "P" and "Q" on the right, we write the truth value assignment we find on the left. So, for the top line, we put a "0" under both "P" and "Q" on the right, while on the second line we put a "0" under each "P" and a "1" beneath each "1." Thus:

P	Q	(~	Q	⊃	P)	(P	v	Q)
0	0		**0**		**0**	**0**		**0**
0	1		**1**		**0**	**0**		**1**
1	0		**0**		**1**	**1**		**0**
1	1		**1**		**1**	**1**		**1**

There are two propositions on the right part of the table, (~Q ⊃ P) and (P v Q). Before we can evaluate the truth value of (~Q ⊃ P), we must determine the value of ~Q for each line of the table. So, we do that next, entering this value for each line directly underneath the "~" sign. Thus:

P	Q	(~	Q	⊃	P)	(P	v	Q)
0	0	**1**	0		0	0		0
0	1	**0**	1		0	0		1
1	0	**1**	0		1	1		0
1	1	**0**	1		1	1		1

Now that we know the value of ~Q, we can determine the value of (~Q ⊃ P). Thus:

P	Q	(~	Q	⊃	P)	(P	v	Q)
0	0	1	0	**0**	0	0		0
0	1	0	1	**1**	0	0		1
1	0	1	0	**1**	1	1		0
1	1	0	1	**1**	1	1		1

Note that in this last move, we are applying the truth table for "⊃" to the values under "P" and under the "~" column (to the left of the "Q" column) because we want to know the value of "(~Q ⊃ P)." This gives us the truth value of (~Q ⊃ P) for all possible truth value assignments to all of its atomic components.

Our last step is to calculate the truth value of (P v Q) by filling in the truth values underneath the "v" column. This gives us:

P	Q	(~	Q	⊃	P)	(P	v	Q)
0	0	1	0	**0**	0	0	**0**	0
0	1	0	1	**1**	0	0	**1**	1
1	0	1	0	**1**	1	1	**1**	0
1	1	0	1	**1**	1	1	**1**	1

Now we need only examine the results. We have considered the truth value of each statement for every possible combination of truth values to their atomic components. We find that for each assignment of truth values to these components (i.e., on each line), each compound proposition has the same truth value. (Look at the two columns in bold.) This means that the two propositions are true and false in exactly the same circumstances, i.e., that they are logically equivalent.

Just for fun, let's consider another logically equivalent pair of statements, "(P ⊃ Q)" and "(~Q ⊃ ~P)."

P	Q	(P	⊃	Q)	(~	Q	⊃	~	P)
0	0	0	**1**	0	1	0	**1**	1	0
0	1	0	**1**	1	0	1	**1**	1	0
1	0	1	**0**	0	1	0	**0**	0	1
1	1	1	**1**	1	0	1	**1**	0	1

We see that the two statements are true and false under exactly the same circumstances, and so that they are logically equivalent. We will make use of fact later in this chapter, when we will introduce a "rule of inference" that from any conditional statement, we can infer a second conditional that we create by reversing the order of the antecedent and the consequent and adding a tilde (a squiggle) in front of each. But more of this later.

Logical equivalence is a *relation* between two or more statement, and as we have just shown, we can use truth tables to determine whether or not two truth-functionally compound statements stand in this relation to one another: if two statements are true and false on exactly the same assignments of truth values to their atomic components (i.e., on exactly the same lines of a truth table), then they are logically equivalent; otherwise not.

We can also use truth tables to demonstrate and illustrate two different *properties* that statement can have: statements are said to be "logically true," or "tautologies" if they are true for all assignments of truth values to their atomic components (i.e., on all lines of their truth tables), and they are said to be "logically false," or "contradictions" if they are false for all assignments of truth values to their atomic components (i.e., on al lines of their truth tables). Let us look at an example of each:

Consider "(P ∨ ~P)." This says that 'P' is either true or false, and is a tautology. Here is its truth table:

P	(P	∨	~	P)
0	0	1	1	0
1	1	1	0	1

Whether "P" is true or false, "(P ∨ ~P)" is true. It is "logically true," because it is true simply in virtue or its form, regardless of what statement 'P' stands for.

Likewise, consider “(P · ~P).”

P	(P	·	~	P)
0	0	0	1	0
1	1	0	0	1

It is false whether “P” is true or false. It is “logically false” because it is false simply in virtue of its form, regardless of what statement ‘P’ stands for.