Special Note: Filter Design Methods

Spectral Power Responses

For $$T(jw) = |T(jw)| \cdot \exp(\angle T(jw))$$

What is $$T(-jw)$$

Magnitude (must be the same!):

$$|T(jw)| = K \cdot \frac{\prod_{i=1}^{Q} \sqrt{1 + \left(\frac{w}{z_i}\right)^2}}{w^R \cdot \prod_{m=1}^{M} \sqrt{1 + \left(\frac{w}{p_m}\right)^2} \cdot \prod_{n=1}^{N} \sqrt{\left(1 - \left(\frac{w}{w_n}\right)^2\right)^2 + \left(2 \cdot \frac{\zeta_n \cdot w}{w_n}\right)^2}}$$

Phase:

$$\angle T(jw) = \sum_{i=1}^{Q} a\tan\left(\frac{w}{z_i}\right) - R \cdot \frac{\pi}{2} - \sum_{m=1}^{M} a\tan\left(\frac{w}{p_m}\right) - \sum_{n=1}^{N} a\tan\left(\frac{2 \cdot \frac{\zeta_n \cdot w}{w_n}}{1 - \left(\frac{w}{w_n}\right)^2}\right)$$

$$\angle T(-jw) = \sum_{i=1}^{Q} a\tan\left(\frac{-w}{z_i}\right) + R \cdot \frac{\pi}{2} - \sum_{m=1}^{M} a\tan\left(\frac{-w}{p_m}\right) - \sum_{n=1}^{N} a\tan\left(\frac{-2 \cdot \frac{\zeta_n \cdot w}{w_n}}{1 - \left(\frac{-w}{w_n}\right)^2}\right)$$

Therefore $$|T(-jw)| = |T(jw)| \text{ and } (\angle T(-jw)) + -(\angle T(jw)) = 0$$

Then, what is $$T(jw) \cdot T(-jw) = |T(jw)| \cdot \exp(\angle T(jw)) \cdot |T(jw)| \cdot \exp(-\angle T(jw))$$

or $$T(jw) \cdot T(-jw) = |T(jw)|^2$$

This is a power term (notice the square), so we use 10*log to create decibels. It is the same result!! Note, this works for (s,-s) too!!

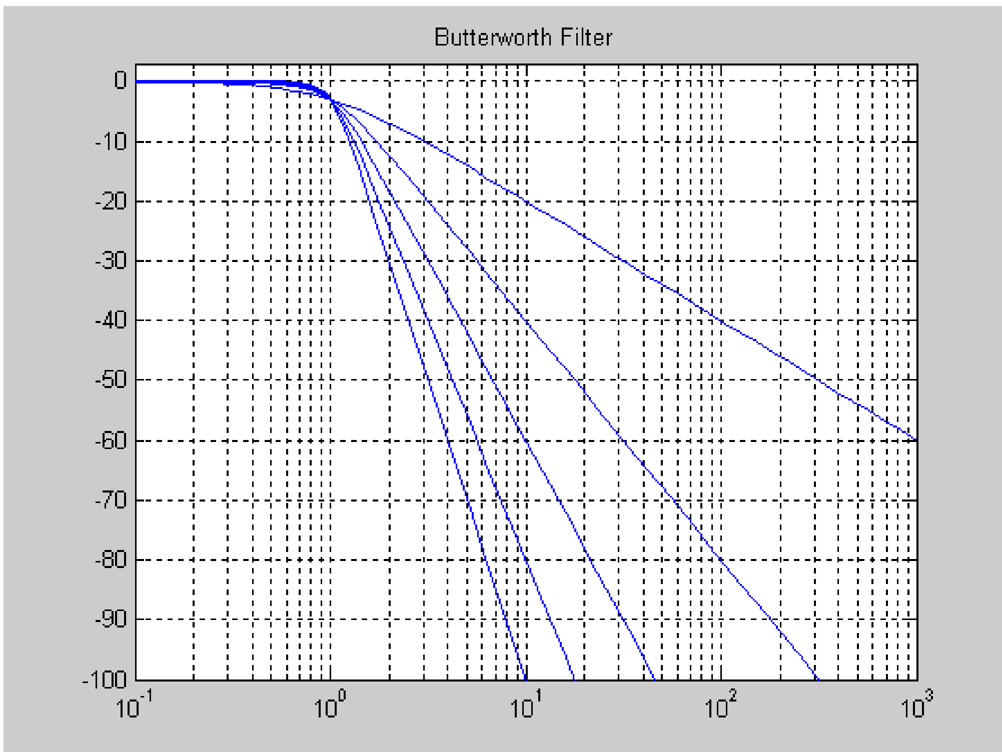
$$T(s) \cdot T(-s) = |T(s)|^2$$

This is the generic form for defining the magnitude response of a filter! Why …
The poles and zeros are symmetric about the jw axis of the s-plane! Therefore, the LHP and RHP elements can be separated into T(s) and T(s) and guarantee marginal stability!!

The Butterworth Lowpass Filter

$$T_n(jw)\cdot T_n(-jw)=\frac{1}{1+\left(\frac{w}{w_0}\right)^{2n}}$$

$$T_n(s)\cdot T_n(-s)=\frac{1}{1+\left(\frac{s}{j\cdot w_0}\right)^{2n}}=\frac{1}{1+(-j)^{2n}\cdot\left(\frac{s}{w_0}\right)^{2n}}=\frac{1}{1+(-1)^n\left(\frac{s}{w_0}\right)^{2n}}$$



Characteristic Eq. $$1+(-1)^n\left(\frac{s}{w_0}\right)^{2n}=\Delta(s)\cdot\Delta(-s)=0$$

Frequency normalized $$\Delta(s)\cdot\Delta(-s)=1+(-1)^n s^{2n}=0$$

Reference: M.E. Van Valkenburg, Analog Filter Design, Oxford Univ. Press, 1982
ISBN: 0-19-510734-9

Solving for the Butterworth Filter poles:

Filter in jw $$T_n(jw)\cdot T_n(-jw)=\frac{1}{1+\left(w/w_0\right)^{2n}}$$

Laplace $$T_n(s)\cdot T_n(-s)=\frac{1}{1+\left(s/j\cdot w_0\right)^{2n}}=\frac{1}{1+(-j)^{2n}\cdot\left(s/w_0\right)^{2n}}=\frac{1}{1+(-1)^n\left(s/w_0\right)^{2n}}$$

Characteristic Eq. $$1+(-1)^n\left(s/w_0\right)^{2n}=\Delta(s)\cdot\Delta(-s)=0$$

Normalize $$\Delta(s)\cdot\Delta(-s)=1+(-1)^n s^{2n}=0$$

For n odd:

$$\Delta(s)\cdot\Delta(-s)=1-s^{2n}=\left(1+s^n\right)\cdot\left(1+(-s)^n\right)=0$$

Roots at $$s^{2n}=1=\exp(j2\cdot m\cdot\pi)\ \rightarrow\ s=\exp\left(\frac{jm\cdot\pi}{n}\right)$$

Let $\Delta(s)$ be the LHP poles and $\Delta(-s)$ be the RHP poles

For n even:

$$\Delta(s)\cdot\Delta(-s)=1+s^{2n}=\left(1+js^n\right)\cdot\left(1-js^n\right)=0$$

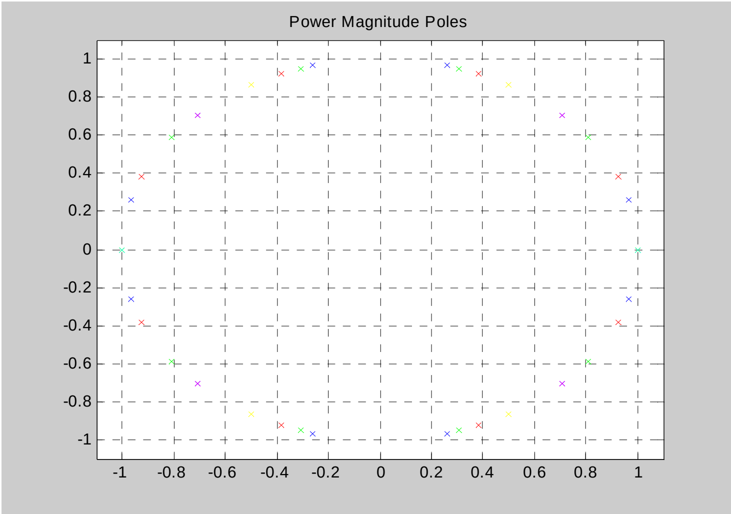
Roots at $$s^{2n}=-1=\exp(j2\cdot m\cdot\pi+j\pi)\ \rightarrow\ s=\exp\left(\frac{j2\cdot m\cdot\pi+j\pi}{2\cdot n}\right)$$

Let $\Delta(s)$ be the LHP poles and $\Delta(-s)$ be the RHP poles

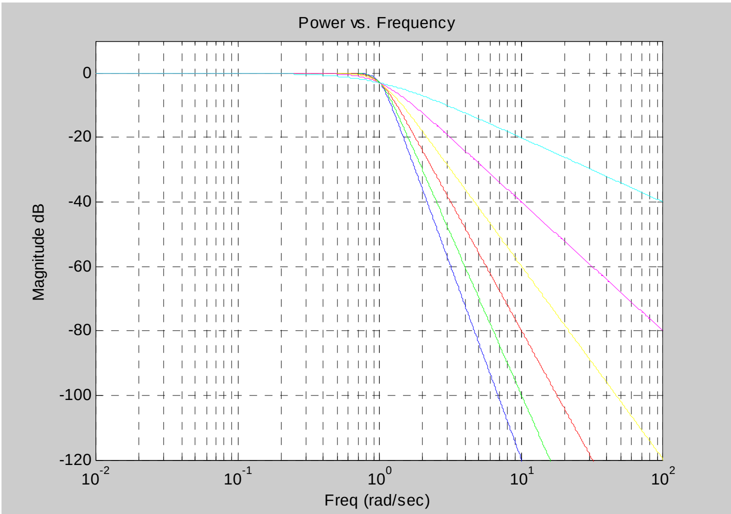
Matlab Code

```
% BW Filter generation demonstration
%

close all
clear all

Rin=1;
Rload=1;
Rmatch=1;

PBfreq=1;

PiW=logspace(log10(PBfreq)-2,log10(PBfreq)+2,1024);
colorseq=['b' 'g' 'r' 'y' 'm' 'c'];
ii=0;

PolesRange=6:-1:1

for BWn=PolesRange
    ii=mod(ii,6)+1;

    denP=roots([((-1/(PBfreq^2))^(BWn)) zeros(1,2*BWn-1) 1])
    [Y,I] = sort(real(denP));
    denPsort=denP(I)
    den=poly(denPsort(1:BWn));

    figure(1)
    plot(real(denP),imag(denP),sprintf('%cx',colorseq(ii)) );
    title('Power Magnitude Poles')
    grid on;
    hold on;

    num = [PBfreq^(BWn)];

    zpi=abs(roots(num));
    ppi=abs(roots(den));
    BWsys=tf(num,den)
    [PiMAG, PiPHASE]=bode(BWsys,PiW);

    figure(2)
    semilogx(PiW, dBv(squeeze(PiMAG)),colorseq(ii) );
    grid on;
    hold on;
    title('Power vs. Frequency')
    xlabel('Freq (rad/sec)');
    ylabel('Magnitude dB');
    plotv=axis;
    axis([plotv(1) plotv(2) -120 10])

    figure(3)
    semilogx(PiW, (squeeze(PiPHASE)),colorseq(ii) );
    grid on;
    hold on;
    title('Phase vs. Frequency')
    xlabel('Freq (rad/sec)');
    ylabel('Phase');
    axis([plotv(1) plotv(2) -max(PolesRange)*90 15])

    pause
end
```

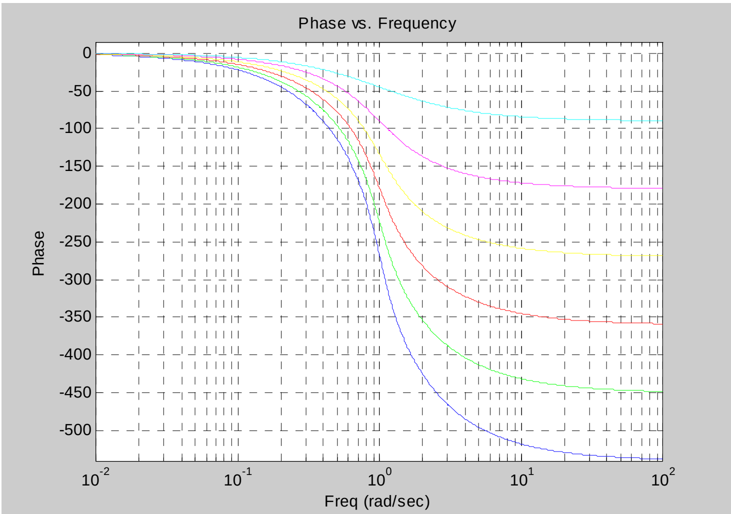
Results



Power Magnitude Poles
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1 -0.8 -0.6 -0.4 -0.2 0 0.2 0.4 0.6 0.8 1



Power vs. Frequency
Magnitude dB
0
-20
-40
-60
-80
-100
-120
10^-2 10^-1 10^0 10^1 10^2
Freq (rad/sec)



Phase vs. Frequency
Phase
0
-50
-100
-150
-200
-250
-300
-350
-400
-450
-500
10^-2 10^-1 10^0 10^1 10^2
Freq (rad/sec)

What if we want to change the frequency …

$$T_n(s)\cdot T_n(-s)=\frac{1}{1+\left(\frac{s}{j\cdot w_0}\right)^{2n}}=\frac{1}{1+(-1)^n\left(\frac{s}{w_0}\right)^{2n}}$$

Just change the natural frequency, $w_0 = 2\pi f_0$; the center frequency is simply scaled!

$$T_n(s)\cdot T_n(-s)=\frac{w_0{}^{2n}}{w_0{}^{2n}+(-1)^n(s)^{2n}}$$

Design approach:

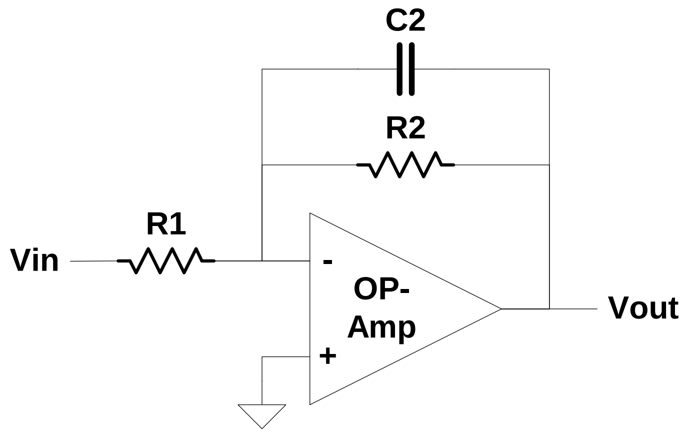
1. Determine the order of the filter you want. What attenuation do you need at the $10\cdot w_0$ point?
   (There are plenty of curves, like those above, if the value you need comes before t 10x the cutoff frequency.)

2. Generate the Butterworth Coefficients on the unit circle for w=1

3. Scale the poles by the desired frequency (remember that w=1 is in radians/sec, therefore multiply by $w_0 = 2\pi f_0$.

Active Audio Frequency Filters

An active lowpass filter implementation of a 1st order Butterworth filter



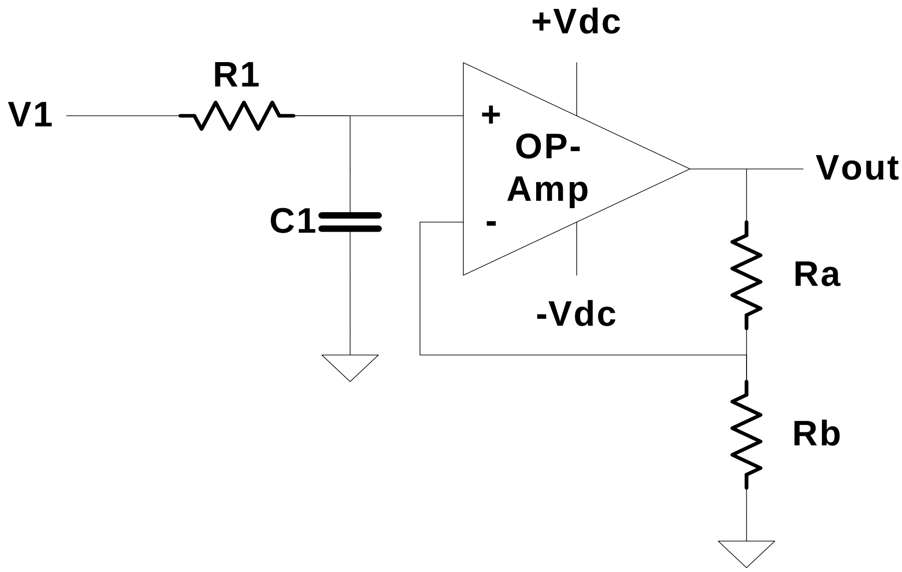
The transfer function for this circuit is

$$\frac{Vout(s)}{Vin(s)} = -\frac{R_2}{R_1 \cdot (1 + sR_2C_2)}$$

$$MaxGain = G = -\frac{R2}{R1} \qquad w_0 = \frac{1}{C_2 \cdot R_2}$$

To tune the circuit ….

An alternate approach:



$$\frac{Vout(s)}{Vin(s)} = \frac{R_a + R_b}{R_b} \cdot \frac{1}{1 + sR_1C_1}$$

$$MaxGain = G = \frac{R_a + R_b}{R_b} \qquad w_0 = \frac{1}{R_1C_1}$$

To tune the circuit ….

A Second Order Butterworth Lowpass Filter

Let's do the math for the second order system, for n=2 and $w_0 = 1$.

$$T_2(s) \cdot T_2(-s) = \frac{1}{1 + \left(s / j \cdot w_0 \right)^{2 \cdot 2}} = \frac{1}{1 + (-1)^2 \left(s / w_0 \right)^{2 \cdot 2}}$$

$$T_2(s) \cdot T_2(-s) = \frac{1}{1 + s^4} = \frac{1}{\left(s^2 + \sqrt{2} \cdot s + 1\right) \cdot \left(s^2 - \sqrt{2} \cdot s + 1\right)}$$

$$T_2(s) = \frac{1}{\left(s^2 + \sqrt{2} \cdot s + 1\right)} \quad \textit{and} \quad T_2(-s) = \frac{1}{\left(s^2 - \sqrt{2} \cdot s + 1\right)}$$

For $T_2(s)$

$$T_2(s) = \frac{1}{s^2 + \sqrt{2} \cdot s + 1} = \frac{1}{s^2 + 2 \cdot \zeta \cdot s + 1}$$

A second order underdamped system with $2 \cdot \zeta = \sqrt{2}$ or $\zeta = 1/\sqrt{2} = 0.707$

$$s_1, s_2 = -\zeta \pm \sqrt{\zeta^2 - 1} = -1/\sqrt{2} \pm j\, 1/\sqrt{2}$$

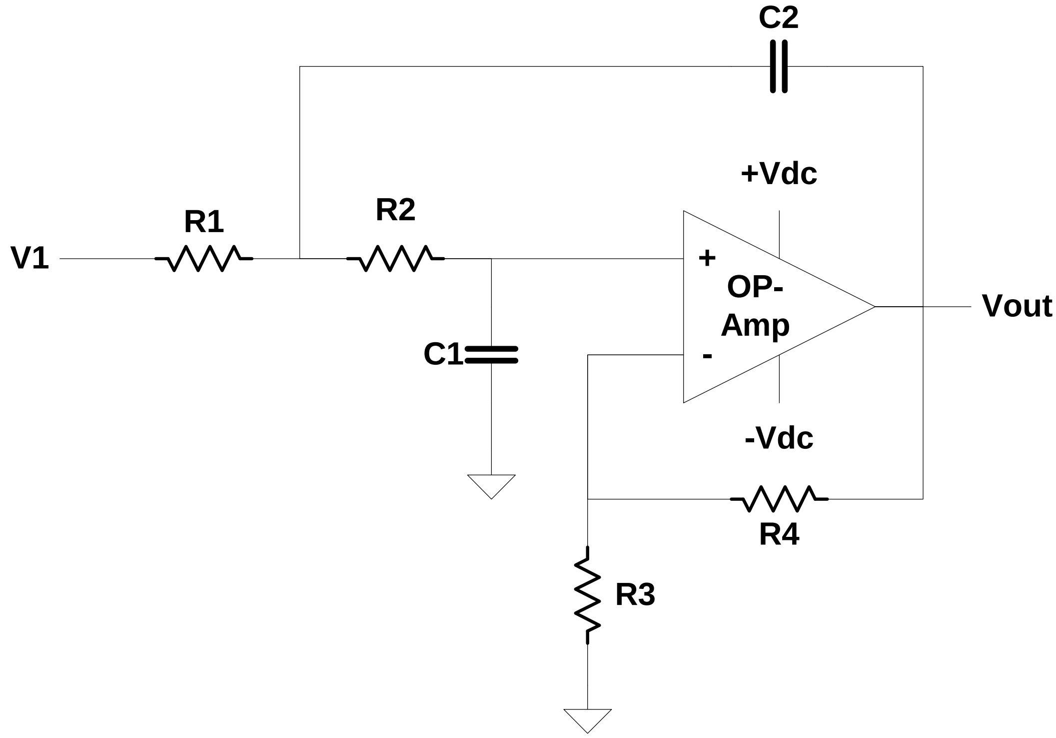
After frequency scaling

$$s_1, s_2 = -\zeta \cdot w_0 \pm w_0 \cdot \sqrt{\zeta^2 - 1} = -w_0/\sqrt{2} \pm j\, w_0/\sqrt{2}$$

How to make a second order LPF for audio ….

Sallen-Key Circuit Lowpass Filter

An active lowpass filter implementation of a unity gain Friend Circuit, also referred to as a Sallen-Key circuit as described in: Walter G. Jung, IC OP-Amp Cookbook, Howard W. Sams Co. Inc, Indianapoli, IN, 1974.



The transfer function for this circuit is (a generic second order filter equation is also shown)

$$\frac{Vout(s)}{V1(s)} = \frac{\left(\frac{R3+R4}{R3}\right)\cdot\frac{1}{C1\cdot C2\cdot R1\cdot R2}}{s^2 + s\cdot\left(\frac{1}{C2\cdot R1}+\frac{1}{C2\cdot R2}-\frac{R4/R3}{C1\cdot R2}\right)+\frac{1}{C1\cdot C2\cdot R1\cdot R2}} = \frac{K\cdot w^2}{s^2+2\cdot\zeta\cdot w\cdot s+w^2}$$

$$\frac{Vout(s)}{V1(s)} = \left(\frac{R3+R4}{R3}\right)\cdot\frac{1}{1+s\cdot\left(C1\cdot R2+C1\cdot R1-R4/R3\cdot C2\cdot R1\right)+s^2\cdot\left(C1\cdot C2\cdot R1\cdot R2\right)}$$

Letting $C1 = C2 = C$ and $R1 = R2 = R$ and $K = \frac{R3+R4}{R3}$

$$MaxGain = K = \frac{R3+R4}{R3} \qquad w = \frac{1}{C\cdot R} \qquad \zeta = \frac{3-G}{2}$$

Function Derivation

The circuit derivation assumes a perfect op-amp, with infinite gain, infinite input impedance, and zero output impedance, non-limiting power supplies and voltage drops, and no frequency response considerations.

The circuit derivation follows:

$$V2 \cdot \left(\frac{1}{R1} + \frac{1}{R2} + s \cdot C2 \right) = \frac{V1}{R1} + \frac{Vp}{R2} + Vo \cdot s \cdot C2$$

$$Vp \cdot \left(\frac{1}{R2} + s \cdot C1 \right) = \frac{V2}{R2}$$

$$Vn = \frac{R3}{R3 + R4} \cdot Vo$$

Letting $Vp = Vn$

$$Vo \cdot \left(\frac{R3}{R3 + R4} \right) \cdot \left(\frac{1}{R2} + s \cdot C1 \right) = \frac{V2}{R2}$$

$$V2 = \frac{R3 \cdot (1 + s \cdot C1 \cdot R2)}{R3 + R4} \cdot Vo$$

$$V2 \cdot \left(\frac{1}{R1} + \frac{1}{R2} + s \cdot C2 \right) = \frac{V1}{R1} + \frac{V2}{R2 \cdot (1 + s \cdot C1 \cdot R2)} + Vo \cdot s \cdot C2$$

$$V2 \cdot \left(\frac{1}{R1} + \frac{s \cdot C1}{1 + s \cdot C1 \cdot R2} + s \cdot C2 \right) = \frac{V1}{R1} + Vo \cdot s \cdot C2$$

$$Vo \cdot \frac{R3 \cdot (1 + s \cdot C1 \cdot R2)}{R3 + R4} \cdot \left(\frac{1}{R1} + \frac{s \cdot C1}{1 + s \cdot C1 \cdot R2} + s \cdot C2 \right) - Vo \cdot s \cdot C2 = \frac{V1}{R1}$$

$$Vo \cdot \left(\frac{R3}{R3 + R4} \right) \cdot \left(\frac{1 + s \cdot C1 \cdot R2 + s \cdot C1 \cdot R1 + s \cdot C2 \cdot R1 + s^2 \cdot C1 \cdot C2 \cdot R1 \cdot R2}{R1} \right) - Vo \cdot s \cdot C2 = \frac{V1}{R1}$$

$$Vo \cdot \left(\frac{R3}{R3 + R4} \right) \cdot \left(\frac{1 + s \cdot C1 \cdot R2 + s \cdot C1 \cdot R1 - s \cdot C2 \cdot R1 \cdot \left(\frac{R4}{R3} \right) + s^2 \cdot C1 \cdot C2 \cdot R1 \cdot R2}{R1} \right) = \frac{V1}{R1}$$

$$Vo = \left(\frac{R3 + R4}{R3}\right) \cdot \left(\frac{1}{1 + s \cdot C1 \cdot R2 + s \cdot C1 \cdot R1 - s \cdot C2 \cdot R1 \cdot \left(\frac{R4}{R3}\right) + s^2 \cdot C1 \cdot C2 \cdot R1 \cdot R2}\right) \cdot V1$$

$$\frac{Vo}{V1} = \frac{\left(\frac{R3 + R4}{R3}\right) \cdot \frac{1}{C1 \cdot C2 \cdot R1 \cdot R2}}{s^2 + s \cdot \left(\frac{1}{C2 \cdot R1} + \frac{1}{C2 \cdot R2} - \frac{R4/R3}{C1 \cdot R2}\right) + \frac{1}{C1 \cdot C2 \cdot R1 \cdot R2}}$$

Letting $C1 = C2 = C$ and $R1 = R2 = R$ and $G = \frac{R3 + R4}{R3}$

$$\frac{Vo}{V1} = \frac{G \cdot \frac{1}{(C \cdot R)^2}}{s^2 + s \cdot \left(\frac{3 - G}{C \cdot R}\right) + \frac{1}{(C \cdot R)^2}}$$

Resulting in

$$MaxGain = G = \frac{R3 + R4}{R3} \qquad w = \frac{1}{C \cdot R}$$

And $\zeta = \frac{3 - G}{2}$

Note that for a stable system $1 \le G < 3$

Implying that $0 \le R4 < 2 \cdot R3$

Multiple Feedback (MFB) Circuit Lowpass Filter

An active lowpass filter implementation of a multiple feedback circuit (MFB), that is may also be referred to as a derivative of the Sallen-Key Filter.

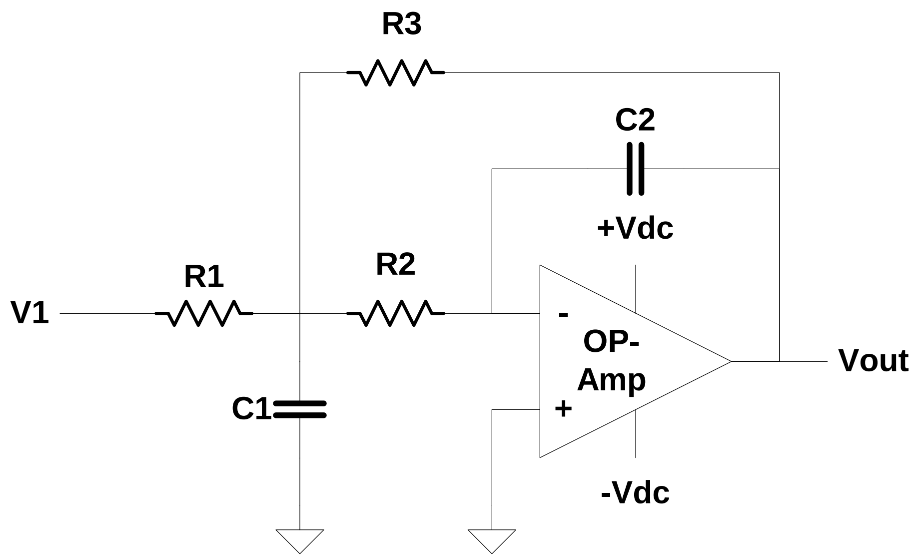


Figure 1. MFB Lowpass Filter

The transfer function for this circuit is

$$\frac{Vo}{V1} = -\left(\frac{R3}{R1}\right)\cdot\frac{1/C1\cdot C2\cdot R2\cdot R3}{s^2 + s\cdot\left(1/C1\cdot R1 + 1/C1\cdot R2 + 1/C1\cdot R3\right) + 1/C1\cdot C2\cdot R2\cdot R3}$$

$$\frac{Vo}{V1} = -\left(\frac{R3}{R1}\right)\cdot\frac{1}{1 + s\cdot\left(C2\cdot R2\cdot R3/R1 + C2\cdot R3 + C2\cdot R2\right) + s^2\cdot\left(C1\cdot C2\cdot R2\cdot R3\right)}$$

Resulting in

$$MaxGain = K = -\frac{R3}{R1} \qquad w = \sqrt{1/C1\cdot C2\cdot R2\cdot R3}$$

Letting $C2 = C$, $C1 = n\cdot C$, $R1 = R3 = R$, $R2 = m\cdot R$, and $G = -1$

$$\frac{Vo}{V1} = \frac{-1}{1 + s\cdot\left(C\cdot R\cdot(1 + 2\cdot m)\right) + s^2\cdot\left(n\cdot m\cdot R^2\cdot C^2\right)} = \frac{K\cdot w^2}{s^2 + 2\cdot\zeta\cdot w\cdot s + w^2}$$

$$MaxGain = K = -1 \qquad w = \frac{1}{R\cdot C\cdot\sqrt{n\cdot m}} \qquad \zeta = \frac{1 + 2\cdot m}{2\cdot\sqrt{n\cdot m}}$$

Function Derivation

The circuit derivation assumes a perfect op-amp, with infinite gain, infinite input impedance, and zero output impedance, non-limiting power supplies and voltage drops, and no frequency response considerations.

The circuit derivation follows:

$$V2 \cdot \left(\frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} + s \cdot C1 \right) = \frac{V1}{R1} + \frac{Vo}{R3}$$

$$s \cdot C2 \cdot Vo + \frac{V2}{R2} = 0$$

Combining $$Vo \cdot (-s \cdot C2 \cdot R2) \cdot \left(\frac{1}{R1} + \frac{1}{R2} + \frac{1}{R3} + s \cdot C1 \right) = \frac{V1}{R1} + \frac{Vo}{R3}$$

$$Vo \cdot (-s \cdot C2 \cdot R2) \cdot \left(\frac{R1 \cdot R2 + R1 \cdot R3 + R2 \cdot R3 + s \cdot C1 \cdot R1 \cdot R2 \cdot R3}{R1 \cdot R2 \cdot R3} \right) - \frac{Vo}{R3} = \frac{V1}{R1}$$

$$Vo \cdot \left(\frac{R1 + s \cdot C2 \cdot (R1 \cdot R2 + R1 \cdot R3 + R2 \cdot R3) + s^2 \cdot C1 \cdot C2 \cdot R1 \cdot R2 \cdot R3}{R1 \cdot R3} \right) = -\frac{V1}{R1}$$

$$\frac{Vo}{V1} = -\frac{R3}{R1 + s \cdot C2 \cdot (R1 \cdot R2 + R1 \cdot R3 + R2 \cdot R3) + s^2 \cdot C1 \cdot C2 \cdot R1 \cdot R2 \cdot R3}$$

$$\frac{Vo}{V1} = -\left(\frac{R3}{R1} \right) \cdot \frac{1/C1 \cdot C2 \cdot R2 \cdot R3}{s^2 + s \cdot \left(1/C1 \cdot R1 + 1/C1 \cdot R2 + 1/C1 \cdot R3 \right) + 1/C1 \cdot C2 \cdot R2 \cdot R3}$$

Resulting in

$$MaxGain = G = -\frac{R3}{R1} \qquad w = \sqrt{1/C1 \cdot C2 \cdot R2 \cdot R3}$$

And $$\zeta \cdot w = \frac{1}{2} \cdot \left(1/C1 \cdot R1 + 1/C1 \cdot R2 + 1/C1 \cdot R3 \right)$$

Higher Order Butterworth Lowpass Filters

Take multiple stages and cascade them!

Remember to determine the pole locations that each stage of the filter requires.

As a rule-of-thumb, you should select the order for the stages of your filter. If you look at the output of each stage, it will be the product of the transfer functions to that location! So, possible use those with damping factors closest to one before the smaller ones ….

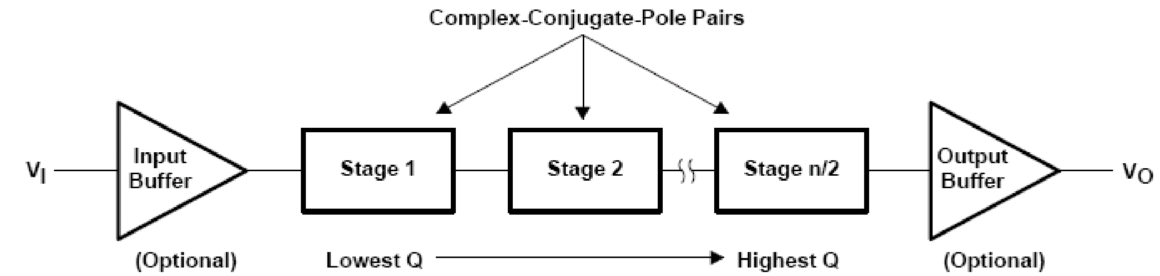


Figure 3. Building Even-Order Filters by Cascading Second-Order Stages

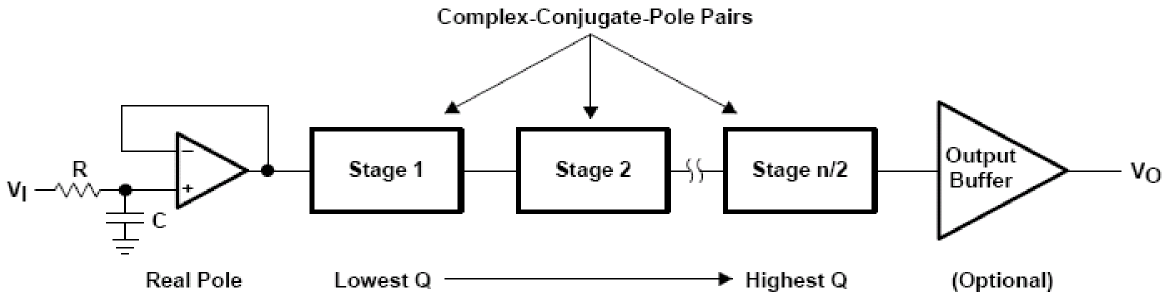


Figure 4. Building Odd-Order Filters by Cascading Second-Order Stages and Adding a Single Real Pole

Jim Karki, Texas Instruments, Active Low-Pass Filter Design, Application Report, SLOA049B, September 2002.

Note:

1. Real elements may not exactly match the values you select.

2. Components have a tolerance, they are within +/- some %!

3. If possible use cheaper components and one (or two) that are adjustable (potentiometers).

What do RF designers do?

Why might it be different?

Low Pass Filter, 3rd Order

The classic 3rd order LC Ladder Low Pass Filter.

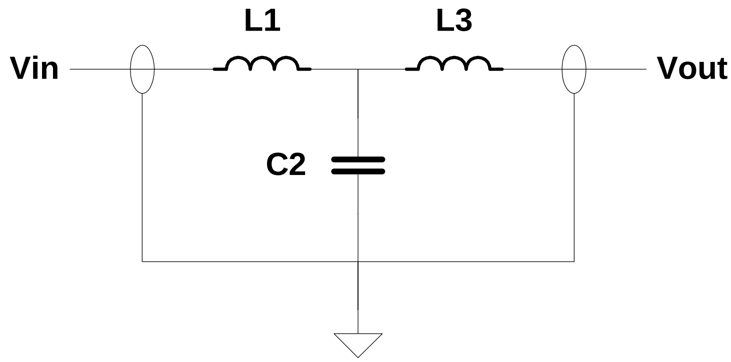


Figure 2. LC Ladder 3rd Order Low Pass Filter

The circuit derivation assumes a source and load resistance. The source resistance is placed prior to the input voltage and the load is placed on the output.

For $Rs = RL = R$ and $L1 = L2 = L$:

$$\frac{Vout}{Vin} = \frac{1/2}{\left(1 + s \cdot \frac{L}{R}\right) \cdot \left(1 + s \cdot \frac{C \cdot R}{2} + s^2 \cdot \frac{C \cdot L}{2}\right)}$$

Theoretical Derivation

The circuit derivation assumes a source and load resistance. The source resistance is placed prior to the input voltage and the load is placed on the output.

The circuit node equations follow:

$$Vout \cdot \left(\frac{1}{RL} + \frac{1}{s \cdot L3} \right) = V2 \cdot \left(\frac{1}{s \cdot L3} \right)$$

$$V2 \cdot \left(\frac{1}{Rs + s \cdot L1} + s \cdot C2 + \frac{1}{s \cdot L3} \right) = Vout \cdot \left(\frac{1}{s \cdot L3} \right) + Vin \cdot \left(\frac{1}{Rs + s \cdot L1} \right)$$

Solving for V2 and substituting:

$$V2 = Vout \cdot \left(\frac{RL + s \cdot L3}{RL} \right)$$

$$Vout \cdot \left(\frac{RL + s \cdot L3}{RL} \right) \cdot \left(\frac{1}{Rs + s \cdot L1} + s \cdot C2 + \frac{1}{s \cdot L3} \right) = Vout \cdot \left(\frac{1}{s \cdot L3} \right) + Vin \cdot \left(\frac{1}{Rs + s \cdot L1} \right)$$

$$Vout \cdot \left(\frac{RL + s \cdot L3}{RL} \right) \cdot \left(\frac{Rs + s \cdot L1 + s \cdot L3 + s^2 \cdot C2 \cdot L3 \cdot (Rs + s \cdot L1)}{(Rs + s \cdot L1) \cdot s \cdot L3} \right) - Vout \cdot \left(\frac{1}{s \cdot L3} \right) = Vin \cdot \left(\frac{1}{Rs + s \cdot L1} \right)$$

$$Vout \cdot \left(\frac{(RL + s \cdot L3) \cdot (Rs + s \cdot L1 + s \cdot L3 + s^2 \cdot C2 \cdot L3 \cdot (Rs + s \cdot L1)) - (Rs + s \cdot L1) \cdot RL}{s \cdot L3 \cdot RL} \right) = Vin$$

$$\frac{Vout}{Vin} = \frac{s \cdot L3 \cdot RL}{(RL + s \cdot L3) \cdot (Rs + s \cdot L1 + s \cdot L3 + s^2 \cdot C2 \cdot L3 \cdot (Rs + s \cdot L1)) - (Rs + s \cdot L1) \cdot RL}$$

$$\frac{Vout}{Vin} = \frac{s \cdot L3 \cdot RL}{RL \cdot (s \cdot L3 + s^2 \cdot C2 \cdot L3 \cdot (Rs + s \cdot L1)) + s \cdot L3 \cdot (Rs + s \cdot L1 + s \cdot L3 + s^2 \cdot C2 \cdot L3 \cdot (Rs + s \cdot L1))}$$

$$\frac{Vout}{Vin} = \frac{RL}{RL + s \cdot C2 \cdot RL \cdot Rs + s^2 \cdot C2 \cdot L1 \cdot RL + Rs + s \cdot L1 + s \cdot L3 + s^2 \cdot C2 \cdot L3 \cdot (Rs + s \cdot L1)}$$

$$\frac{Vout}{Vin} = \frac{RL}{RL + Rs + s \cdot (L1 + L3 + C2 \cdot RL \cdot Rs) + s^2 \cdot (C2 \cdot L1 \cdot RL + C2 \cdot L3 \cdot Rs) + s^3 \cdot C2 \cdot L3 \cdot L1}$$

$$\frac{Vout}{Vin} = \left(\frac{RL}{RL+Rs}\right)\frac{1}{1+s\cdot\left(\frac{L1+L3}{RL+Rs}+C2\cdot\frac{RL\cdot Rs}{RL+Rs}\right)+s^2\cdot\frac{(C2\cdot L1\cdot RL+C2\cdot L3\cdot Rs)}{RL+Rs}+s^3\cdot\frac{C2\cdot L3\cdot L1}{RL+Rs}}$$

For $Rs = RL = R$:

$$\frac{Vout}{Vin} = \frac{R}{2\cdot R+s\cdot\left(L1+L3+C2\cdot R^2\right)+s^2\cdot C2\cdot(L1+L3)\cdot R+s^3\cdot C2\cdot L3\cdot L1}$$

or

$$\frac{Vout}{Vin} = \frac{1}{2+s\cdot\left(\frac{L1}{R}+\frac{L3}{R}+C2\cdot R\right)+s^2\cdot C2\cdot R\cdot\left(\frac{L1}{R}+\frac{L3}{R}\right)+s^3\cdot C2\cdot R\cdot\frac{L1}{R}\cdot\frac{L3}{R}}$$

For $L1 = L2 = L$:

$$\frac{Vout}{Vin} = \frac{RL}{RL+Rs+s\cdot(2\cdot L+C\cdot RL\cdot Rs)+s^2\cdot C\cdot L\cdot(RL+Rs)+s^3\cdot C\cdot L^2}$$

$$\frac{Vout}{Vin} = \frac{RL/_{RL+Rs}}{1+s\cdot\left(2\cdot\frac{L}{RL+Rs}+C\cdot\frac{RL\cdot Rs}{RL+Rs}\right)+s^2\cdot C\cdot L+s^3\cdot\frac{C\cdot L^2}{RL+Rs}}$$

For $Rs = RL = R$ and $L1 = L2 = L$:

$$\frac{Vout}{Vin} = \frac{1/2}{1+s\cdot\left(\frac{L}{R}+C\cdot\frac{R}{2}\right)+s^2\cdot C\cdot L+s^3\cdot\frac{C\cdot L^2}{2R}} = \frac{1/2}{\left(1+s\cdot\frac{L}{R}\right)\cdot\left(1+s\cdot\frac{C\cdot R}{2}+s^2\cdot\frac{C\cdot L}{2}\right)}$$

Theoretical Derivation Pi-Filter

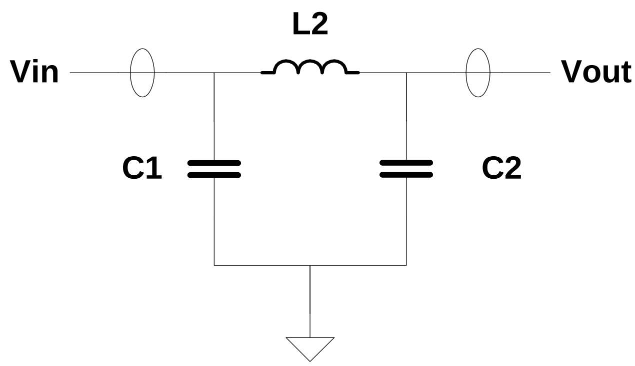


Figure 3. LC Ladder 3rd Order Low Pass Filter

The circuit derivation assumes a source and load resistance. The source resistance is placed prior to the input voltage and the load is placed on the output.

For $Rs = RL = R$ and $C1 = C3 = C$:

$$\frac{Vout}{Vin} = \left(\frac{1}{2}\right) \cdot \frac{1}{(1 + s \cdot C \cdot R) \cdot \left(1 + s \cdot \frac{L}{2R} + s^2 \cdot \frac{L \cdot C}{2}\right)}$$

Theoretical Derivation Pi-Filter

The circuit derivation assumes a source and load resistance. The source resistance is placed prior to the input voltage and the load is placed on the output.

The circuit node equations follow:

$$Vout \cdot \left(\frac{1}{RL} + \frac{1}{s \cdot L2} + s \cdot C3 \right) = V2 \cdot \left(\frac{1}{s \cdot L2} \right)$$

$$V2 \cdot \left(\frac{1}{Rs} + s \cdot C1 + \frac{1}{s \cdot L2} \right) = Vout \cdot \left(\frac{1}{s \cdot L2} \right) + Vin \cdot \left(\frac{1}{Rs} \right)$$

Solving for V2 and substituting:

$$V2 = Vout \cdot \left(\frac{RL + s \cdot L2 + s^2 \cdot L2 \cdot C3 \cdot RL}{RL} \right)$$

$$Vout \cdot \left(\frac{RL + s \cdot L2 + s^2 \cdot L2 \cdot C3 \cdot RL}{RL} \right) \cdot \left(\frac{1}{Rs} + s \cdot C1 + \frac{1}{s \cdot L2} \right) = Vout \cdot \left(\frac{1}{s \cdot L2} \right) + Vin \cdot \left(\frac{1}{Rs} \right)$$

$$Vout \cdot \left(\frac{RL + s \cdot L2 + s^2 \cdot L2 \cdot C3 \cdot RL}{RL} \right) \cdot \left(\frac{Rs + s \cdot L2 + s^2 \cdot C1 \cdot L2 \cdot Rs}{s \cdot L2 \cdot Rs} \right) - Vout \cdot \left(\frac{1}{s \cdot L2} \right) = Vin \cdot \left(\frac{1}{Rs} \right)$$

$$Vout \cdot \left(\frac{\begin{array}{l} RL \cdot Rs + s \cdot (L2 \cdot Rs + L2 \cdot RL) + s^2 \cdot (L2 \cdot C3 \cdot RL \cdot Rs + L2 \cdot C1 \cdot Rs \cdot RL + L2^2) \\ + s^3 \cdot (L2^2 \cdot C3 \cdot RL + L2^2 \cdot C1 \cdot Rs) + s^4 \cdot L2^2 \cdot C1 \cdot C3 \cdot RL \cdot Rs - RL \cdot Rs \end{array}}{s \cdot L2 \cdot Rs \cdot RL} \right) = Vin \cdot \left(\frac{1}{Rs} \right)$$

$$Vout \cdot \left(\frac{\begin{array}{l} (Rs + RL) + s \cdot (C3 \cdot RL \cdot Rs + C1 \cdot Rs \cdot RL + L2) \\ + s^2 \cdot (L2 \cdot C3 \cdot RL + L2 \cdot C1 \cdot Rs) + s^3 \cdot L2 \cdot C1 \cdot C3 \cdot RL \cdot Rs \end{array}}{Rs \cdot RL} \right) = Vin \cdot \left(\frac{1}{Rs} \right)$$

$$\frac{Vout}{Vin} = \frac{RL}{\begin{array}{l} (Rs + RL) + s \cdot (C3 \cdot RL \cdot Rs + C1 \cdot Rs \cdot RL + L2) \\ + s^2 \cdot (L2 \cdot C3 \cdot RL + L2 \cdot C1 \cdot Rs) + s^3 \cdot L2 \cdot C1 \cdot C3 \cdot RL \cdot Rs \end{array}}$$

$$\frac{Vout}{Vin} = \left(\frac{RL}{Rs+RL}\right) \cdot \frac{1}{1 + s \cdot (C3 + C1)\frac{Rs \cdot RL}{(Rs+RL)} + s\frac{L2}{(Rs+RL)} + s^2 \cdot \frac{(L2 \cdot C3 \cdot RL + L2 \cdot C1 \cdot Rs)}{(Rs+RL)} + s^3 \cdot L2 \cdot C1 \cdot C3 \cdot \frac{Rs \cdot RL}{(Rs+RL)}}$$

For $Rs = RL = R$:

$$\frac{Vout}{Vin} = \left(\frac{1}{2}\right) \cdot \frac{1}{1 + s \cdot \left(R \cdot C3 + R \cdot C1 + \frac{L2}{2R}\right) + s^2 \cdot L2 \cdot (C3 + C1) + s^3 \cdot L2 \cdot C1 \cdot C3 \cdot \frac{R}{2}}$$

For $C1 = C3 = C$:

$$\frac{Vout}{Vin} = \left(\frac{RL}{Rs+RL}\right) \cdot \frac{1}{1 + s \cdot \frac{2C \cdot Rs \cdot RL + L2}{(Rs+RL)} + s^2 \cdot L2 \cdot C + s^3 \cdot L2 \cdot C \cdot C \cdot \frac{Rs \cdot RL}{(Rs+RL)}}$$

For $Rs = RL = R$ and $C1 = C3 = C$:

$$\frac{Vout}{Vin} = \left(\frac{1}{2}\right) \cdot \frac{1}{1 + s \cdot \left(C \cdot R + \frac{L}{2R}\right) + s^2 \cdot L \cdot C + s^3 \cdot L \cdot C^2 \cdot \frac{R}{2}}$$

$$\frac{Vout}{Vin} = \left(\frac{1}{2}\right) \cdot \frac{1}{(1 + s \cdot C \cdot R) \cdot \left(1 + s \cdot \frac{L}{2R} + s^2 \cdot \frac{L \cdot C}{2}\right)}$$

1.5 MHz Low Pass Filter, 7th Order, Coilcraft P7LP155

The 7th order elliptical LC Ladder Low Pass Filter.

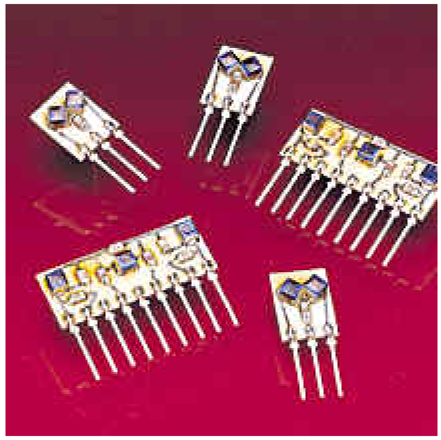
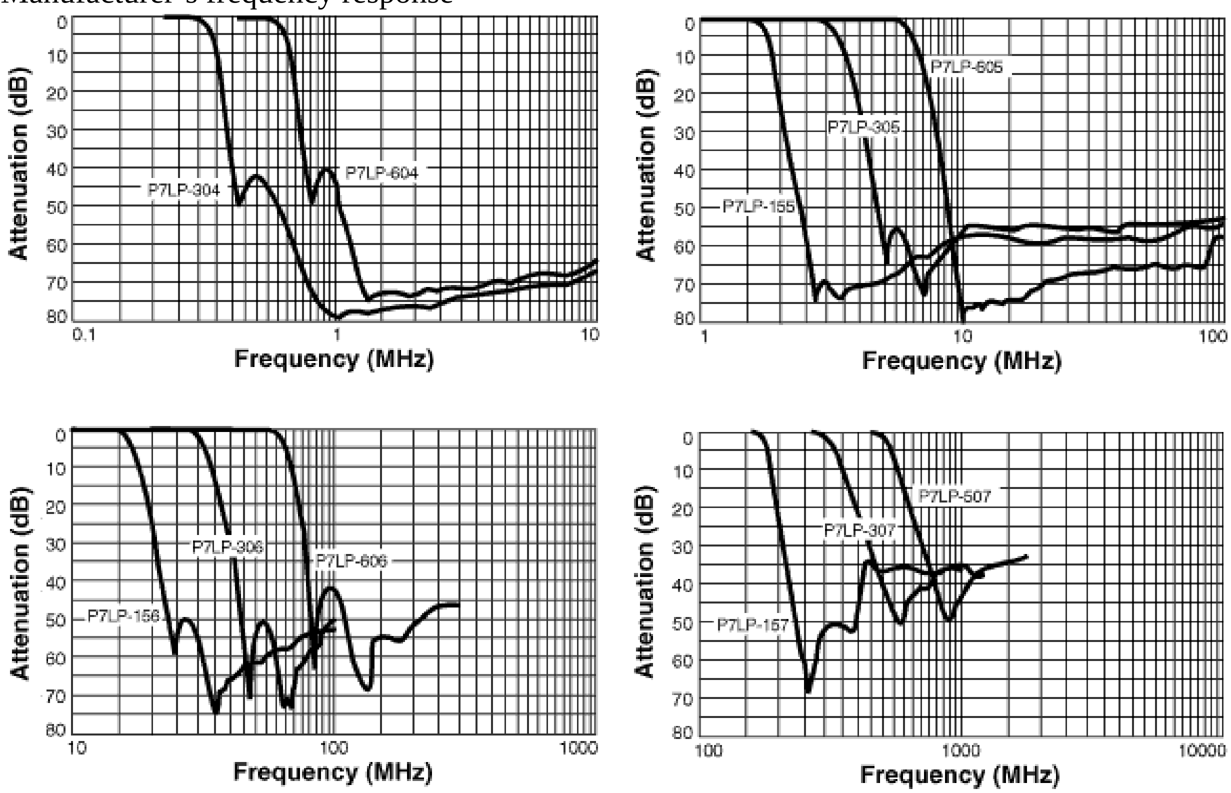


Figure 4. Coilcraft LC Ladder Low Pass Filter

Manufacturer's frequency response



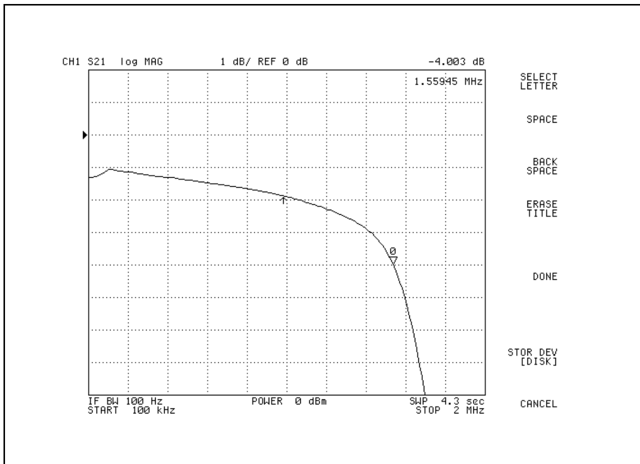
Coilcraft LC Ladder Low Pass Filter

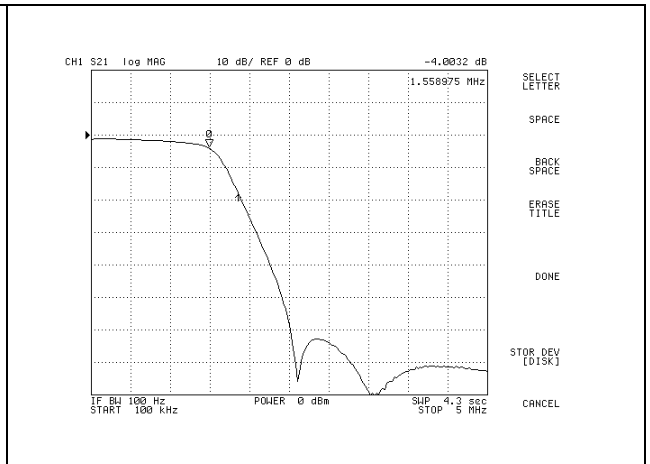
Test Analysis

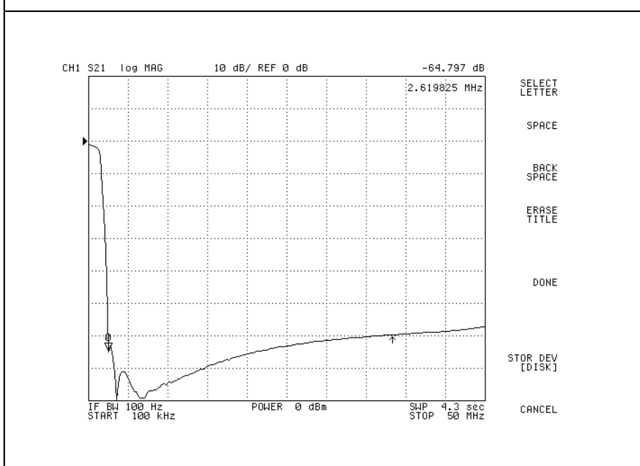
A test circuit was built and tested using the network analyzer.

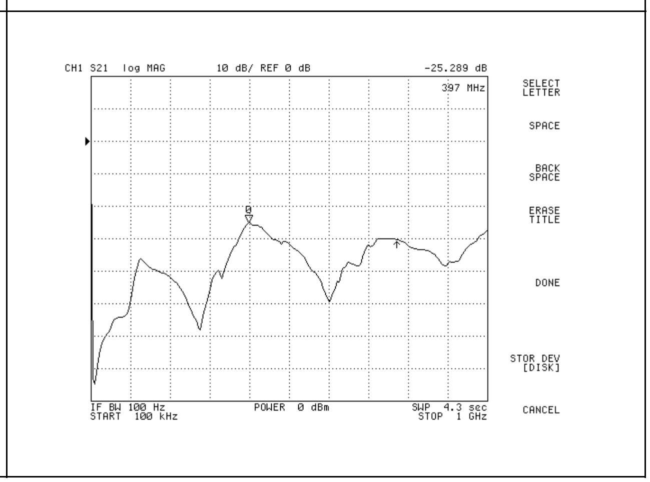
R is assumed to be 50 ohms, L1 and L3 were variable inductors in the range of 0.578 to 0.95 uH and C2 was 4 parallel 100pF (101K) capacitors or 400 pF.

Network Analyzer Measurements









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[2] M.E. Van Valkenburg, Analog Filter Design, Oxford, 1982. ISBN: 0-19-510734-9.

[3] http://www.circuitsage.com/filter.html

[4] http://focus.ti.com/analog/docs/techdocs.tsp?contentType=8&familyId=78&navSection=app_notes

TI Application Notes:

- Slod006b
- Sloa093

TI Application Notes on Filtering

- Active Filter Design Techniques SLOA088
- Analysis of the Sallen-Key Architecture (Rev. B) SLOA024
- FilterPro MFB and Sallen-Key Low-Pass Filter Design Program SBFA001A
- Active Low-Pass Filter Design (Rev. A) SLOA049
- Using the Texas Instruments Filter Design Database SLOA062
- Filter Design in Thirty Seconds SLOA093
- Filter Design on a Budget SLOA065
- More Filter Design on a Budget SLOA096