Conditional Risk-Return Relationship in a Time-Varying Beta Model

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Abstract

We investigate the asymmetric risk-return relationship in a time-varying beta CAPM. A state space model is established and estimated by the Adaptive Least Squares with Kalman foundations proposed by McCulloch (2006). Using S&P 500 daily data from 1987:11-2003:12, we find a positive risk-return relationship in the up market (positive market excess returns) and a negative relationship in the down market (negative market excess returns). This supports the argument by Pettengill, Sundaram and Mathur (1995), who use a constant beta model. However, our model outperforms theirs by eliminating the unexplained returns and improving the accuracy of the estimated risk price.

Key words: CAPM, time-varying beta, Adaptive Least Squares, Kalman Filter, asymmetric risk-return relationship

JEL Classifications: G12, C11

I. Introduction

Since Sharpe (1964) and Lintner (1965) proposed the Capital Asset Pricing Model (CAPM) to describe the risk-return relationship, substantial empirical work has been conducted to investigate the validity of the model. Many empirical studies, by using the three-step approach proposed by Fama and MacBeth (1973), show that the Sharpe-Lintner CAPM (SLC) provides an inadequate explanation of the risk-return relationship due to the lack of evidence that supports a statistically significant relationship between risk and return (e.g., Fama and French 1992; He and Ng 1994). This unsuccessful empirical performance of the SLC causes people to cast doubts on the model.

Pettengill, Sundaram, and Mathur (PSM) (1995) argue that the validity of the SLC is not directly examined by the Fama-MacBeth methodology. The SLC postulates a positive relationship between beta and expected return. To test for this relationship, empirical studies use observable *realized* returns in place of unobservable *expected* returns. PSM show that this procedure is biased against finding a significant relationship between beta and expected returns because the relationship between beta and realized returns is conditional on the market return. In up markets high beta securities should be rewarded for bearing risk with higher returns than low beta securities, but in down markets high-risk, high-beta securities experience lower returns than low beta securities. Thus, standard tests are biased against finding a relationship between beta and returns because these tests mix periods where the relationship between beta and returns is direct (up markets) with periods where the relationship between beta and returns is inverse (down markets). To solve this problem, PSM partition the data into up market and down market periods based on the sign of the realized market excess return. Their empirical results confirm a significant direct relationship between beta and returns in up markets and a significant inverse relationship between beta and returns in down markets.

Another criticism of the Fama-MacBeth methodology is their OLS regressions that assume a constant beta risk. Studies such as Harvey (1989) and Ferson and Harvey (1991, 1993) suggest that a constant beta estimated by OLS may not capture the dynamics of beta. Jagannathan and Wang (1996) and Lettau and Ludvigson (2001) show that conditional CAPM with a time-varying beta outperforms the unconditional CAPM with a constant beta. Adrian and

Franzoni (2004, 2005) also argue that an econometric model that fails to mimic the investors' learning process of time-evolving beta may lead to inaccurate estimates of beta.

The purpose of this paper is to check the robustness of PSM's argument by incorporating a time-varying beta, in order to see whether PSM's success is based on an incorrectly specified constant beta.[1] That is, we re-examine the asymmetric risk-return relationships in the up and down markets with a time-varying beta model. The Adaptive Least Squares with Kalman foundations (ALSKF) proposed by McCulloch (2006) is used to estimate the time-varying beta model. The ALSKF provides a better way of estimating time-varying coefficients and proxying investors' time-evolving expectations by incorporating the learning process. This methodology nests the Kalman solution of the elementary local level model and uses a simple way to setup a rigorous initial condition.

Using daily data of the stocks listed in the S&P 500 from November 1987 to December 2003, our results support PSM's argument: when the market excess return is positive, there exists a significant and positive risk-return relationship; when the market excess return is negative, there exists a significant and negative risk-return relationship.

Not only confirming PSM's argument, this paper also improves PSM's model on two aspects. First, the estimated intercepts, which represent the unexplained returns in the model, are found to be significantly different from zero in both the up and down markets using the PSM model. In contrast, in the time-varying beta model estimated by the ALSKF, neither of the estimated intercepts is significantly different from zero. Second, the magnitude of the risk-loading estimated by our model is closer to the realized market excess return than that estimated by the PSM model. These results indicate that the ALSKF successfully improves the accuracy of the estimation of beta risk by mimicking the investors' learning process on the unobservable beta that the OLS cannot account for.

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[1] Many studies apply the PSM methodology to other markets. These include: Fletcher (1997, 2000) and Hung et al. (2004) for the UK market; Isakov (1999) for the Swiss market; Lam (2001) and Ho et al. (2006) for the Hong Kong market; Elsas et al. (2003) for the German market; Hodoshima et al. (2000) for the Japanese market; Faff (2001) for the Australian market; and Sandoval and Saens (2004) for four Latin American markets. The overwhelming preponderance of these studies supports the PSM conclusion.

The remainder of the paper is organized as follows. The next section introduces our time-varying beta CAPM under the up and down market conditions. The ALSKF methodology is also outlined in this section. Section III shows the empirical results. The last section concludes the paper.

II. Models and Methodology

A. The risk-return relationship based on the up and down market conditions

The Sharpe-Lintner CAPM (SLC) shows that the expected excess return of asset *i*, represented by the expected return [$E(R_i)$] minus the risk-free rate (R_f), equals the beta risk of *i* times the expected market excess return:

$$E(R_i) - R_f = \beta_i \, [\, E(R_m) - R_f \,], \tag{1}$$

where β_i measures the systematic risk for *i*, and is equal to the covariance between the return of *i* and the market return divided by the variance of the market return:

$$\beta_i \;=\; \frac{\text{cov}(\,R_i,\,R_m)}{\sigma^2_{R_m}}. \tag{2}$$

Equation (1) shows a positive risk-return tradeoff because the expected market excess return should be positive (otherwise no one will buy risky assets). However, empirical research uses the realized returns to proxy the expected returns:

$$R_{i,t} - R_{f,t} = \alpha_i + \beta_i \, (R_{m,t} - R_{f,t}) + \nu_{i,t} \,. \tag{3}$$

Pettengill, Sundaram and Mathur (PSM) (1995) argue that the use of the realized returns instead of the unobservable expected returns could be the reason why the SLC fails in empirical tests. Even though investors expect that on average the market return is greater than the risk-free rate, they must perceive a non-zero probability that the realized market return will be smaller than the risk-free rate. If this were not the case, no one would hold the risk-free assets.

From (3), it can be seen that for assets with a positive β, when the realized market return is greater than the risk-free rate, the realized return of an asset with a higher β should exceed that of an asset with a lower β. That is, there is a positive relationship between realized return and risk. In contrast, when the realized market return is less than the risk-free rate, the realized returns of such assets are negative. Under this situation, the realized return of an asset with a higher β is less than that of an asset with a lower β. That is, there is a negative relationship between realized return and risk. Therefore, PSM argue that if the realized returns are used, there exist a positive risk-return relationship when the market excess return is positive and a negative relationship when the market excess return is negative.

By revising the Fama-MacBeth methodology, PSM uses the following three steps to test the risk-return relationship. Each one of the steps is conducted in a separate sample period with five years of data in each period. In the first step, equation (3) is estimated across time ($t = T_0$, $T_0+1, \ldots, T_1$) to yield the estimate of beta for each individual stock in the first sample period, and portfolios are formed based on the ranking of $\hat{\beta}_i$. The second step involves estimations of each portfolio's beta in the next sample period:

$$R_{p,t} - R_{f,t} = \alpha_p + \beta_p\,(R_{m,t} - R_{f,t}) + \nu_{p,t}\,, \tag{4}$$

for $t = T_1+1,\ T_1+2, \ldots, T_2$.[2] Finally, the third step estimates the risk-return relationship by running a cross-sectional regression at each t and calculating the time series average of the estimated coefficients. Specifically, portfolio returns at each t from the third sample period are regressed on the portfolio betas estimated in the second step, with a dummy added to separate the risk-return relationship in the up market from that in the down market:

$$R_{p,t} - R_{f,t} = \gamma_{1,t}\cdot\delta_t + \gamma_{2,t}\cdot(1-\delta_t) + \gamma_{3,t}\cdot\delta_t\cdot\hat{\beta}_p + \gamma_{4,t}\cdot(1-\delta_t)\cdot\hat{\beta}_p + \varepsilon_{p,t}\,, \tag{5}$$

for $p = 1, 2, \ldots, N$, where N is the number of portfolios; $\hat{\beta}_p$ is estimated from (4); and $\delta_t = 1$ if $R_{m,t} > R_{f,t}$ (an up market) and $\delta_t = 0$ if $R_{m,t} < R_{f,t}$ (a down market). Equation (5) is estimated across N portfolios and the averages of $\hat{\gamma}_{j,t}$'s across time ($t = T_2+1$, $T_2+2, \ldots, T_3$), i.e.,

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[2] The purpose of the first step is to reduce the "errors-in-the-variables problem," because the portfolio betas are supposed to be more precise estimates of true betas than the individual stock betas. The second step estimates the portfolio betas in a fresh, subsequent period in order to minimize the "regression problem" that positive and negative sampling errors are bunched within portfolios. See Fama and MacBeth (1973) for a detailed discussion.

$\bar{\hat{\gamma}}_j = \frac{1}{m_j}\sum_t \hat{\gamma}_{j,t}$ (j = 1, 2, 3, 4; m_1=m_3 is the number of up-market days and m_2=m_4 is the number of down-market days), are used to test the risk-return relationship. The standard deviation is $\hat{\sigma}_j = \sqrt{\frac{1}{m_j - 1}\sum_t (\hat{\gamma}_{j,t} - \bar{\hat{\gamma}}_j)^2}$, and the simple t-tests can be used to test the following hypotheses:[3]

H_0: $\bar{\gamma}_3 = 0$, H_1: $\bar{\gamma}_3 > 0$; and

H_0: $\bar{\gamma}_4 = 0$, H_1: $\bar{\gamma}_4 < 0$.

A systematic conditional relationship between beta and realized returns is supported if both null hypotheses are rejected in favor of the alternatives.

Note that equation (5) is different from the original PSM model in the specification of the constant terms. There is a single constant term in PSM's original specification, but they do not report the estimate of it because they only focus on the asymmetric risk-return relationship. Another test that is important to justify the validity of the Sharpe-Lintner CAPM is that the constant term should be zero. This test, however, is ignored by PSM.[4] We address this test as well and calculate the constant terms separately in the up and down markets. As will be discussed later, this specification shows the advantage of our model over the PSM model.

Before we introduce the time-varying beta model, it is worthwhile to point out that in the third step [equation (5)], the realized return is regressed on past beta ($\hat{\beta}_p$ is estimated in the second sample period). This is because the CAPM was initially developed in the spirit of Markowitz (1959), who suggests that the model be treated as a normative model to help people

[3] Petersen (2005) claims that the Fama-MacBeth methodology is superior to a pool time-series/cross-section estimation. In finance application, residuals of a given year may be correlated across firms (cross-sectional dependence), which is called the time effect. He argues that the Fama-MacBeth methodology is designed to address the time effect and shows that the standard errors in the Fama-MacBeth methodology are unbiased in the presence of the time effect.

[4] Note that PSM regress security returns on excess market returns while we regress excess security returns on excess market returns. Thus, the intercept estimated by PSM ought to be equal to the risk-free rate. This may be another reason why they do not do a zero-null hypothesis test on the constant term.

make better decisions, rather than being used as a positive model. Therefore, the model makes sense as a normative theory only if there is some relationship between future returns and the estimated risk based on current information. Our time-varying beta model will make this concept even clearer because it updates the daily information when estimating the market beta.

B. A time-varying β CAPM based on the Adaptive Least Squares with Kalman foundations

The OLS regressions used by PSM assume that beta is constant over time. This assumption has been challenged by many studies (e.g., Harvey 1989; Ferson and Harvey 1991, 1993; Jagannathan and Wang 1996). To estimate a time-varying beta, studies have tried different modeling strategies. For example, Jagannathan and Wang (1996) and Lettau and Ludivigson (2001) treat beta as a function of several state variables in a conditional CAPM. Engle, Bollerslev, and Wooldridge (1988) model the movements of beta in a GARCH model. Adrian and Franzoni (2004, 2005) suggest a time-varying parameter linear regression model and use the Kalman filter to estimate the model. Since the purpose of this paper is to test PSM's asymmetric risk-return relationship in a time-varying beta setup, we adopt Adrian and Franzoni's approach because a time-varying parameter linear regression model allows us to use the same regression models as those in PSM, with the only difference being the time-varying betas as opposed to the constant betas in PSM. No other variables such as GARCH terms or state variables are needed.

In Adrian and Franzoni's time-varying parameter linear regression model, however, to make the estimation tractable, the covariance matrix of the random coefficients is assumed to be constant. McCulloch (2006) considers a more general case where the covariance matrix is time-varying, in an attempt to better proxy agents' expectation through learning from prediction errors. However, the extended Kalman filter would make the estimation of the model intractable. Therefore, he proposes the Adaptive Least Squares with Kalman Foundations (ALSKF) to estimate a time-varying parameter model. A simple but reasonable initialization of the filter is another attractive feature of this methodology. In addition, the time-varying Kalman gain (proxying for agents' learning process) can be estimated by Maximum Likelihood.[5] Therefore, we apply McCulloch's methodology to the estimation of time-varying beta in a conditional CAPM. For the details on the derivations of the ALSKF, see McCulloh (2006).

We summarize the ALSKF as follows. Consider the following state-space form of a time-varying beta model (the portfolio index subscripts are omitted for a clear presentation):

$$r_t = \boldsymbol{x_t}\boldsymbol{\lambda_t} + \nu_t, \qquad \nu_t \sim \mathrm{N}(0, \sigma_\nu^2), \tag{6}$$

$$\boldsymbol{\lambda_t} = \boldsymbol{\lambda_{t\text{-}1}} + \boldsymbol{\eta_t}, \qquad \boldsymbol{\eta_t} \sim \mathrm{N}(0, \boldsymbol{Q_t}), \tag{7}$$

where $r_t \equiv R_{p,t} - R_{f,t}$ denotes the excess return of portfolio *p* at time *t*; $\boldsymbol{x_t} = (1, r_{m,t})'$ is a 1×2 row vector in which $r_{m,t} \equiv R_{m,t} - R_{f,t}$ denotes the excess market return; $\boldsymbol{\lambda_t} = (\alpha_{p,t}, \beta_{p,t})$ is a 2×1 column vector, where $\beta_{p,t}$ represents the β risk of portfolio *p* at *t* and $\alpha_{p,t}$ is a time-varying intercept; and $\boldsymbol{Q_t}$ is a 2×2 covariance matrix. The idiosyncratic shocks to portfolio *p*, ν_t and $\boldsymbol{\eta_t}$, are independent of each other and uncorrelated with shocks to other portfolios. The conditional distribution of the unobservable $\boldsymbol{\lambda_t}$ conditional on observed excess returns up to time *t* is:

$$\boldsymbol{\lambda_t} \mid r_t, r_{t-1}, \cdots, r_0 \sim \mathrm{N}(\boldsymbol{b_t}, \boldsymbol{P_t}). \tag{8}$$

Rather than assuming $\boldsymbol{Q_t}$ as a constant matrix as usually being done in practice, McCulloh (2006) assumes that $\boldsymbol{Q_t}$ is time varying and directly proportional to $\boldsymbol{P_{t\text{-}1}}$:

$$\boldsymbol{Q_t} = \rho T_{t\text{-}1}\boldsymbol{P_{t\text{-}1}}, \tag{9}$$

where ρ is the signal/noise ratio (a parameter to be estimated), which is an index of the uncertainty of the transition error ($\boldsymbol{Q_t}$) to the measurement error per effective observation at time *t*-1 ($T_{t-1}\boldsymbol{P_{t\text{-}1}}$); and T_t, which measures the effective sample size, is derived based on the Kalman solution of the local level model:

$$T_t = (1+\rho T_{t\text{-}1})^{-1} T_{t\text{-}1} + 1, \text{ and } T_0 = 0. \tag{10}$$

Then the ALSKF can be expressed as:

$$\boldsymbol{b_t} = \boldsymbol{W_t^{-1}}\boldsymbol{z_t}, \tag{11}$$

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[5] A constant covariance matrix would lead to a constant gain coefficient, which is not desirable. See McCulloch (2006) for details.

$$\boldsymbol{P_t} = \sigma_{\varepsilon}^2 \boldsymbol{W_t^{-1}} \,, \tag{12}$$

where

$$\boldsymbol{z_t} = (1+\rho T_t)^{-1}\boldsymbol{z_{t-1}} + \boldsymbol{x'_t} r_t \,, \tag{13}$$

$$\boldsymbol{W_t} = (1+\rho T_t)^{-1}\boldsymbol{W_{t-1}} + \boldsymbol{x'_t x_t} \,. \tag{14}$$

To initialize the filter, since there is a diffuse prior about the coefficients at time 0, all the eigenvalues of the covariance matrix $\boldsymbol{P_0}$ would be infinite, which implies that the elements in $\boldsymbol{P_0^{-1}}$ are all zeros. Therefore, McCulloh argues that it is reasonable to initialize equations (13) and (14) with zeros: $\boldsymbol{z_0} = \boldsymbol{0}_{2\text{x}1}$, and $\boldsymbol{W_0} = \boldsymbol{0}_{2\text{x}2}$.[6] The log-likelihood for the corresponding ALSKF can be determined by:

$$r_t \mid r_{t-1}, \cdots, r_0 \sim \text{N}(\boldsymbol{x_t b_{t-1}}, \sigma_{\nu}^2 s_t^2) \,, \tag{15}$$

where $s_t^2 = (1+\rho T_t)\boldsymbol{x_t W_{t-1}^{-1} x'_t} + 1$.

The estimate of interest is $\boldsymbol{b_t}$. Note that $\boldsymbol{b_t}$ is the conditional expectation of the unobservable $\boldsymbol{\lambda_t}$ conditional on observed excess returns up to time *t*, i.e., $\boldsymbol{b_t} \equiv \boldsymbol{\lambda_{t|t}}$. In addition, since $\boldsymbol{\lambda_t}$ follows a random walk, the expected $\boldsymbol{\lambda_{t+1}}$ conditional on the information available at *t* is $\boldsymbol{\lambda_{t+1|t}} = \boldsymbol{\lambda_{t|t}}$. Therefore, the estimate of the expectation of beta risk at *t*+1 based on the observation of excess return at *t* ($\hat{\beta}_{p,t+1|t}$) can be obtained from the ALSKF.

C. The test of risk-return relationship based on the up and down market conditions in a time-varying β CAPM

Recall that the first step of the three-step approach in PSM is to estimate β across time for each individual stock in the first sample period and then to form portfolios on the basis of the ranked $\hat{\beta}_i$. The purpose of this step is to pool stocks with similar β's into a portfolio, in an attempt to obtain more accurate estimates of portfolio β's for the testing of risk-return relation in

[6] On the contrary, previous ALS studies arbitrarily set the initial values of the parameters. The ALSKF circumvents this problem and provides a simple but rigorous initialization.

the next two steps. However, if β is modeled as time-varying and portfolios are formed based on the ranking of betas, we would have to re-form portfolios at each time t. Then the portfolio returns calculated in PSM and those in our time-varying beta model would be based on different sets of stocks. Therefore, to rule out the effect of different portfolio formations, we replace the first step by forming portfolios based on industry classifications, which is a common practice in the literature.[7]

With a time-varying β, the second step of the test becomes:

$$R_{p,t} - R_{f,t} = \alpha_{p,t} + \beta_{p,t}\,(R_{m,t} - R_{f,t}) + \nu_{p,t}\text{, with} \tag{4'}$$

$$\alpha_{p,t} = \alpha_{p,t-1} + \eta^{\alpha}_{p,t}\text{, and } \ \beta_{p,t} = \beta_{p,t-1} + \eta^{\beta}_{p,t}\,,$$

for $t = T_1+1, T_1+2, \ldots, T_2, T_2+1, \ldots, T_3-1$. The forecasts of betas ($\hat{\beta}_{p,t+1|t}$) are estimated by the ALSKF, and these forecasted betas from $t = T_2+1$ to $t = T_3$ are used in the third step:

$$R_{p,t} - R_{f,t} = \gamma_{1,t}\cdot\delta_t + \gamma_{2,t}\cdot(1-\delta_t) + \gamma_{3,t}\cdot\delta_t\cdot\hat{\beta}_{p,t|t-1} + \gamma_{4,t}\cdot(1-\delta_t)\cdot\hat{\beta}_{p,t|t-1} + \varepsilon_{p,t}\,. \tag{5'}$$

Again, Equation (5') is estimated across N portfolios and the averages of $\hat{\gamma}_{j,t}$'s across time ($t = T_2+1, T_2+2, \ldots, T_3$) are used to test the risk-return relationship.

III. Estimation Results

A. Data

The data used in this paper, from CRSP, are daily returns of the stocks in the S&P 500 as of December 31, 2003 that have complete return data from November 1987 to December 2003. There are a total of 358 stocks and 4079 observations for each stock.[8] As is a common practice in the literature, we start the sample from November 1987 to avoid the concern that the enormous

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[7] Fraser, Hamelink, Hoesli, and McGregor (2004) and Galagedera and Faff (2004) also use industry-sorted portfolios to test the PSM model.

[8] We recognize the possibility that using larger and more mature firms in the S&P 500 may bias the results, and the fact that eliminating firms without complete data reduces the sample size. However, the current data selection criteria allow us to track the industrial classification codes for the stocks more easily and precisely. Using stocks with the same sample size also greatly simplifies the already-complicated Kalman filter estimations.

daily movements during October 1987 may dominate our inferences.[9]

These 385 stocks are classified into ten industries based on their Global Industry Classification Standard (GICS) codes, which includes energy, material, industrials, consumer discretionary, consumer staples, health care, financials, information technology, telecom services, and utilities. The portfolio returns are value-weighted. Table 1 reports the means and the standard deviations of these ten portfolio returns.

The market return is defined as the value-weighted return on all NYSE, AMEX, and NASDAQ stocks. The risk-free rate is measured as the daily one-month U.S. Treasury bill rate from the FRED database published by the Federal Reserve Bank of St. Louis. Table 2 reports the means and the standard deviations of the market excess returns in the up market, the down market, and the whole sample period. Note that the first five years of data are used to estimate portfolio betas [Equations (4) and (4')] for the estimation of the risk-return relationship [Equations (5) and (5')]. Therefore, the relevant data period in Table 2 is from November 1992 to December 2003.

B. Results

The main purpose of this paper is to test the robustness of PSM's asymmetric risk-return relationship by using a more reasonable (time-varying beta) model. However, PSM's first step (portfolio formation) of the three-step approach is replaced by industry classification in our model. To make sure that this change is not the reason that may alter PSM's conclusion, before estimating the time-varying beta model, we first compare the PSM models with two alternative first-steps: forming portfolios based on the ranking of betas and forming portfolios based on the industry classifications. The second and third steps and the sample periods used are identical. Following PSM, we use a five-year interval in each step. The results are shown in Table 3.

Panel (A) of Table 3 shows the results with ten beta-ranked portfolios. Panel (B) shows the results from the model that skips the first step of PSM and simply uses the industry

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[9] For example, Fraser et al. (2004) find that the betas estimated prior to October 1987 are different from the beta estimated immediately after the crash. They argue that adding the one single October 1987 observation to the estimation completely changed the forecast of the beta risk.

classifications to form ten portfolios for the next two steps. The qualitative results are the same across these two panels. All the estimated coefficients have correct signs and the null hypotheses are rejected at the 5% level. Therefore, it is shown that replacing the beta-ranked portfolios by the industry-sorted portfolios does not alter PSM's conclusion that there exists an asymmetric risk-return relationship.

Another noticeable result in Table 3 is that the estimated coefficients of the constant terms are all significantly different from zero at the 5% level. Market risk premium tends to overstate the portfolio excess returns in the up market and understate the portfolio excess returns in the down market. This is a violation of the CAPM and is not addressed in PSM. Later we will show that this problem is solved in our time-varying beta model.

We proceed to the estimation of our time-varying beta model dependent on the up/down market regimes. The results are shown in Panel (A) of Table 4. Since there are only two steps [(4') and (5'), the second and the third steps in PSM], the sample period used in each step are different from those in Table 3. The first five years of data (11/1987-10/1992) are used in (4') immediately to obtain the first beta, and this beta is used in (5') to obtain the first risk-return relationship. One data point is added to the sample in each of the subsequent estimations. Time series average in (5') runs from 11/1992 to 12/2003. For comparison, Panel (B) of Table 4 repeats the industry-sorted PSM model [Panel (B) of Table 3] with the new sample periods. Since there are only two steps, three constant betas are obtained from 11/1987-10/1992, 11/1992-10/1997, and 11/1997-10/2002. Time series average in (5) also runs from 11/1992 to 12/2003.

Panel (A) shows that $\bar{\hat{\gamma}}_3$ is positive and significantly different from zero at the 5% level, which indicates a positive risk-return relationship in the up market. The estimated daily risk price paid for per unit of β risk in the up market equals 0.743%. On the other hand, the negative and statistically significant $\bar{\hat{\gamma}}_4$ indicates a negative risk-return relationship in the down market. Portfolios incur, on average, a reduction of 0.723% in daily returns per unit of β risk in the down markets. Therefore, the existence of an asymmetric risk-return relationship suggested by PSM is confirmed by the time-varying beta model, which is better in describing the agent's learning process.

The time-varying beta model not only supports the asymmetric risk-return relationship, but also shows improvements of PSM's constant-beta model in two aspects. This can be seen by comparing the results in Panel (A) to those from the constant-beta PSM model shown in Panel (B). First, as mentioned before, in addition to the estimated slopes, the significance of the estimated intercepts is also an important criterion to judge an asset pricing model because the estimated intercepts represent the unexplained returns. A well-specified asset pricing model should have estimated intercepts that are insignificantly different from zero. In Panel (B), the estimated intercepts are both significantly different from zero at the 5% level. Market risk premium tends to overstate the portfolio excess returns by 0.239% per day in the up market and understate the portfolio excess returns by 0.215% per day in the down market. This is a violation of the CAPM and is not addressed in PSM.[10] On the other hand, both intercepts in Panel (A) are statistically insignificant. Therefore the time-varying beta model improves the PSM model by eliminating the unexplained returns.

The second advantage of our time-varying beta model over the PSM model is on the estimation of the per-unit risk price. According to the Sharpe-Lintner CAPM, the estimated coefficients on beta indicate the risk prices compensated for holding per unit of beta risk. Since the market beta is the only risk measure here, the price paid for the beta risk should be equal to the market excess return. Therefore, we would expect that in the test period, $\bar{\hat{\gamma}}_3$ and $\bar{\hat{\gamma}}_4$ are close to the market excess returns in the up and down markets, respectively. Table 2 shows that during the test period, the average daily market excess return is 0.739% in the up market and -0.760% in the down market. In Panel (A), the estimated daily risk prices in the up and down markets are 0.743% and -0.723%, respectively, which are very close to the realized market excess returns. In contrast, the risk prices estimated by the PSM model in Panel (B) are 0.893% and

[10] As mentioned earlier, PSM have a single constant term and do not distinguish the intercepts in the up and down markets. Using our data to estimate such a single-constant setup shows an insignificant constant term. However, by theory the intercept should be zero no matter whether it is an up market or a down market. Therefore, our specification proves the advantage of our time-varying beta model over the PSM model.

-0.823% in the up and down markets, respectively, which are both less accurate than those obtained in the time-varying beta model. Therefore, we conclude that it is the time-varying beta estimated by the ALSKF that helps to improve the accuracy of the estimation of the risk-return relationship.

More specifically, Figure 1 plots the estimated betas in both the constant-beta model and the time-varying-beta model and reports the time-series averages of these betas for those ten industries. It can be seen that the estimated betas in the time-varying-beta model are far from constant. More importantly, in eight of the ten portfolios, the constant-beta model tends to overestimate beta in most of the time periods. Since $\bar{\hat{\gamma}}_3$ is positive and $\bar{\hat{\gamma}}_4$ is negative, the intercept in the constant-beta model is biased downward in the up market and biased upward in the down market. This is why the intercept is significantly negative in the up market and significantly positive in the down market. Using the time-varying-beta eliminates the negative intercept in the up market and the positive intercept in the down market.

IV. Conclusion

Empirical studies using the Fama-MacBeth (1973) methodology show that the Sharpe-Lintner CAPM (SLC) provides an inadequate explanation of the risk-return relationship. Pettengill, Sundaram, and Mathur (PSM) (1995) argue that the validity of the SLC is not directly tested by the Fama-MacBeth methodology because the realized returns rather than the expected returns are used in the test. To provide a valid test, PSM partition the data into up market and down market periods based on the sign of the realized market excess returns. A positive risk-return relationship should exist when the realized market excess return is positive and an inverse relationship should exist when the realized market excess return is negative. Their empirical results confirm these relationships.

However, it is well known in the literature that the beta risk tends to be time varying. The revised Fama-MacBeth methodology used by PSM involves OLS regressions that assume a constant beta risk. This paper re-examines the asymmetric risk-return relationship in the up and down markets by using a time-varying beta model, in an attempt to check the robustness of

PSM's argument and see whether PSM's success is based on an incorrectly specified constant beta. The Adaptive Least Squares with Kalman foundations proposed by McCulloch (2006) is used to estimate the time-varying beta model.

Using the daily return from the S&P 500 stocks, we find evidence supporting the asymmetric risk-return relationship proposed by PSM. Furthermore, our model also improves PSM's constant beta model in that first, our time-varying beta model eliminates the unexplained returns in the constant beta model. Second, our time-varying beta model obtains a more accurate estimate of the per unit risk price. Both indicate that the Adaptive Least Squares with Kalman foundations successfully improves the accuracy of the estimation of the beta risk by mimicking the investors' learning process on the unobservable beta.

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Table 1

Means and standard deviations of the returns (in %) for ten industry-sorted portfolios

Industry	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
Mean	0.063	0.069	0.079	0.085	0.085	0.088	0.092	0.111	0.060	0.058
S.D.	1.299	1.287	1.194	1.212	1.099	1.280	1.316	1.800	1.405	0.966

The stocks are classified into ten industry sectors by the Global Industry Classification Standard (GICS) code. The industries are: (1) energy, (2) material, (3) industrials, (4) consumer discretionary, (5) consumer staples, (6) health care, (7) financials, (8) information technology, (9) telecom services, and (10) utilities. The numbers of firms in each industry are, respectively, 18, 26, 48, 65, 33, 29, 56, 43, 8, and 32. The returns are value-weighted in each portfolio. The sample period is from 12/02/1987 to 12/31/2003, a total of 4079 trading days.

Table 2

Basic Statistics of the Market Excess Returns (in %)

	The whole period	The up market	The down market
Number of days	2814	1485	1329
Mean	0.031	0.739	-0.760
S.D.	1.064	0.734	0.781

The market excess return is the market return minus the risk-free rate. The market return is defined as the value-weighted return on all NYSE, AMEX, and NASDAQ stocks. The risk-free rate is measured as the daily one-month U.S. Treasury bill rate. If the market excess return is positive, the market is defined as an up market. If the market excess return is negative, the market is defined as a down market. The sample period is from November 1992 to December 2003.

Table 3

Estimation results of the PSM models

	(A) Ranked betas	(B) Industry classifications
$\bar{\hat{\gamma}}_1$	-0.430* (-5.424)	-0.433* (-5.189)
$\bar{\hat{\gamma}}_2$	0.415* (5.327)	0.411* (5.089)
$\bar{\hat{\gamma}}_3$	1.316* (16.272)	1.257* (15.864)
$\bar{\hat{\gamma}}_4$	-1.249* (-16.936)	-1.167* (-16.157)

This table reports the test statistics for two alternative first-steps (portfolio formation) in the PSM model. In Panel (A), the portfolios are formed based on the ranking of the estimated beta for each individual stock. In Panel (B), the portfolios are formed based on the industry classifications. The numbers in parentheses are t-statistics. The asterisk indicates statistical significance at the 5% level.

Each step uses five years of data. The data from the period 11/1987-10/1992 are used in the first step, 11/1992-10/1997 in the second step, and 11/1997-10/2002 in the third step. Then the data from the period 11/1992-10/1997 are used in the first step, 11/1997-10/2002 in the second step, and 11/2002-12/2003 in the third step. Therefore, the time series average in the third step runs from 11/1992 to 12/2003, i.e., $\bar{\hat{\gamma}}_j$ is the time series average (11/1997-12/2003) of $\hat{\gamma}_{j,t}$ in the following cross-sectional regression (the third step):

$$R_{p,t} - R_{f,t} = \gamma_{1,t} \cdot \delta_t + \gamma_{2,t} \cdot (1-\delta_t) + \gamma_{3,t} \cdot \delta_t \cdot \hat{\beta}_p + \gamma_{4,t} \cdot (1-\delta_t) \cdot \hat{\beta}_p + \varepsilon_{p,t}\,, \qquad (5)$$

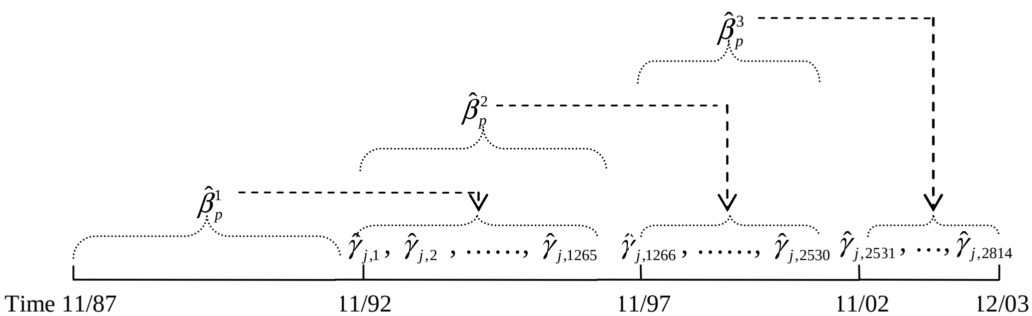
where $\delta = 1$ in the up market and $\delta = 0$ in the down market, and $\hat{\beta}_p$ is from the second step [see equation (4)].

Table 4

The Estimation results of the time-varying beta model and the constant beta model

	(A) Time-varying beta model	(B) Constant-beta model
$\bar{\hat{\gamma}}_1$	0.027 (0.841)	-0.239* (-4.807)
$\bar{\hat{\gamma}}_2$	-0.004 (-0.119)	0.215* (4.094)
$\bar{\hat{\gamma}}_3$	0.743* (20.558)	0.893* (18.785)
$\bar{\hat{\gamma}}_4$	-0.723* (-20.241)	-0.823* (-17.021)

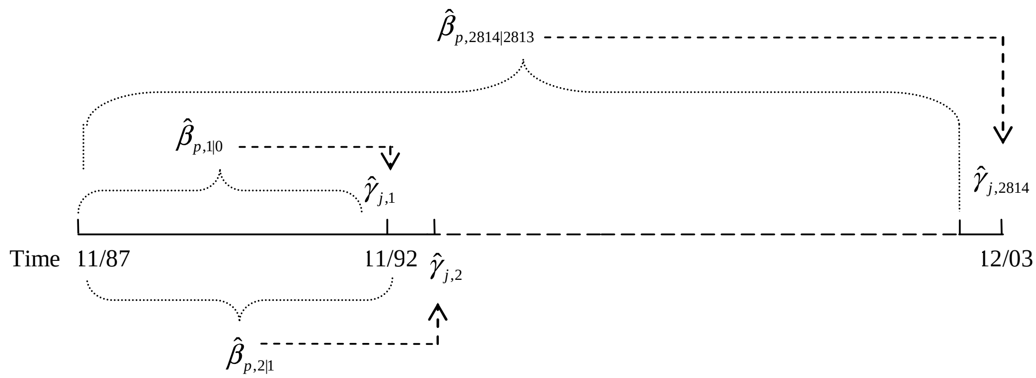
The numbers in parentheses are t-statistics. The asterisk indicates statistical significance at the 5% level. There are only two steps: the second step [Equation (4) or (4')] and the third step [Equation (5) or (5')] in the PSM model. The following graph shows how the estimated betas and risk-return relationships ($\hat{\gamma}_{j,t}$) are obtained in the constant-beta model.



The data from 11/1987 to 10/1992 are used in (4) to obtain the first constant beta ($\hat{\beta}_p^1$) and this beta and the data from 11/1992 to 10/1997 are used in (5) to estimate the first 1265 risk-return relationships. Then the data from 11/1992 to 10/1997 are used in (4) again to obtain the second constant beta ($\hat{\beta}_p^2$), which is used in (5) to obtain the next 1265 risk-return relationships. Finally, the data from 11/1997 to 10/2002 are used to obtain another constant beta ($\hat{\beta}_p^3$), which is used to obtain the risk-return relationships for the final period 11/2002-12/2003.

The following graph shows how the estimated betas and risk-return relationships are obtained in the time-varying-beta model.

<Table 4 footnote continues. . .>



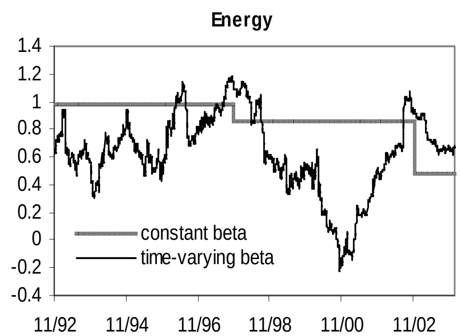
The data from 11/1987 to 10/1992 are used in (4') to obtain $\hat{\beta}_{p,1|0}$, which is used in (5') to obtain the risk-relationship on the first trading day of November, 1992 ($\hat{\gamma}_{j,1}$). The realized return on this day is then added to the sample and used in (4') to obtain $\hat{\beta}_{p,2|1}$, which is used in (5') to obtain the risk-relationship on the second trading day of November, 1992 ($\hat{\gamma}_{j,2}$), and so on.

The time series average in the third step runs from 11/1992 to 12/2003 (a total of 2814 days), i.e., $\bar{\hat{\gamma}}_j$ is the time series average (11/1997-12/2003) of $\hat{\gamma}_{j,t}$ from the cross-sectional regressions (5) and (5').

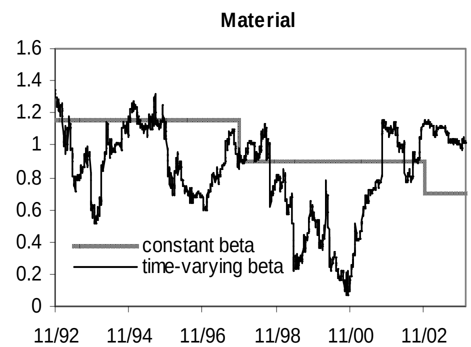
Figure 1

Constant beta and time-varying beta

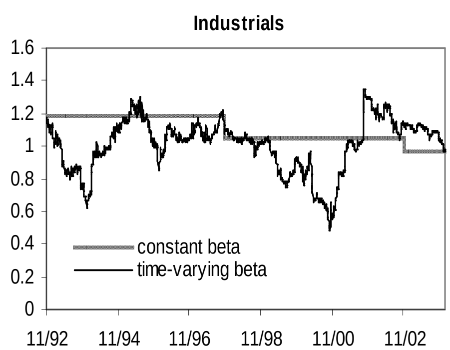
Each graph plots the constant betas estimated in the PSM model and the time-varying betas estimated by the ALSKF for one of the ten industries. These betas are the ones that are used to generate the results in Table 4. The time-series averages of the betas are listed under the graph.



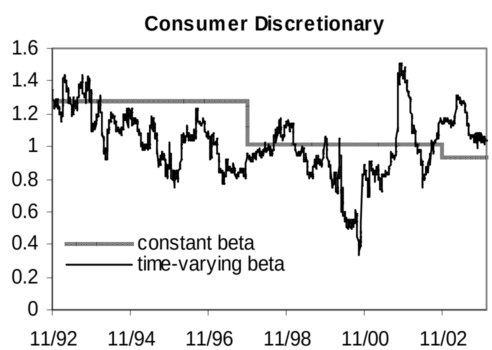
Average of the constant betas: 0.773
Average of the time-varying betas: 0.632



Average of the constant betas: 0.918
Average of the time-varying betas: 0.837

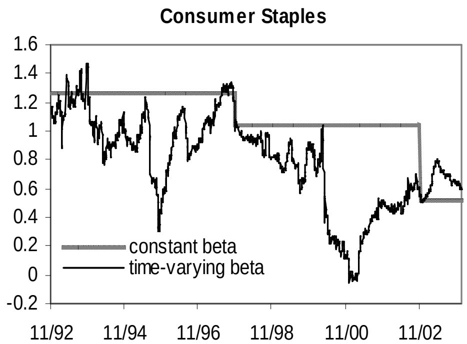


Average of the constant betas: 1.066
Average of the time-varying betas: 1.007

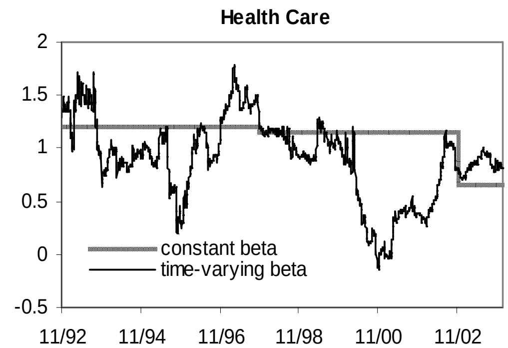


Average of the constant betas: 1.079
Average of the time-varying betas: 1.018

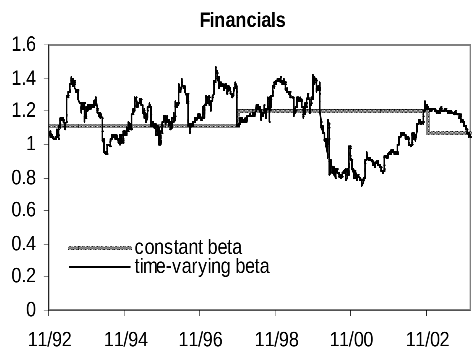
Figure 1 (continued)



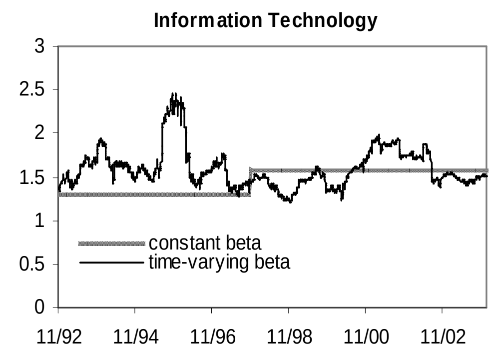
Average of the constant betas: 0.942
Average of the time-varying betas: 0.790



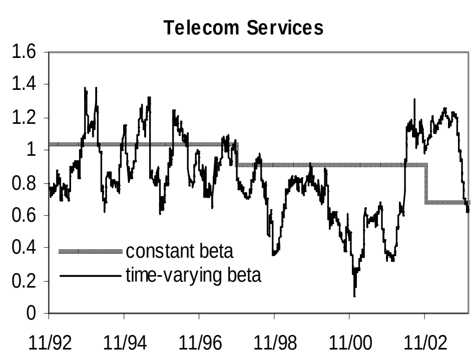
Average of the constant betas: 1.003
Average of the time-varying betas: 0.898



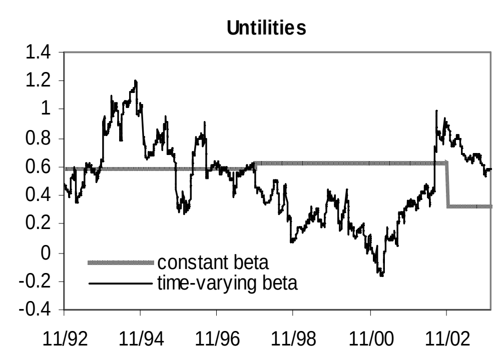
Average of the constant betas: 1.125
Average of the time-varying betas: 1.140



Average of the constant betas: 1.482
Average of the time-varying betas: 1.596



Average of the constant betas: 0.876
Average of the time-varying betas: 0.832



Average of the constant betas: 0.512
Average of the time-varying betas: 0.501