Triacontagonal coordinates for the E_8 root system

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Abstract. This note gives an explicit formula for the elements of the E_8 root system. The formula is triacontagonally symmetric in that one may clearly see an action by the cyclic group with 30 elements. The existence of such a formula is due to the fact that the Coxeter number of E_8 is 30.

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1 Introduction

The Coxeter numbers of H_4 and E_8 are both equal to 30, [6]. Some artistically-inclined mathematicians have used this fact in order to depict these root systems. In so doing, each has necessarily produced a figure having triacontagonal symmetry, meaning the same symmetry as a regular 30-sided polygon. According to [3], where it is used as the frontispiece, van Oss first sketched such a projection of the regular polytope having Schläfli symbol $\{3, 3, 5\}$, also known as the "600-cell". The vertices of $\{3, 3, 5\}$ coincide with the elements of the (non-crystallographic) root system H_4. One may also find this sketch in the article [8] and a sketch of the dual polytope $\{5, 3, 3\}$ in [1].

The roots of E_8 coincide with the vertices of a highly symmetric convex polytope 4_{21} apparently discovered by Gosset, [3, 5], and also with the vertices of a regular complex polytope $3\{3\}3\{3\}3\{3\}3$ discovered by Witting, [4, 2]. The book [4] displays a triacontagonal projection of $3\{3\}3\{3\}3\{3\}3$ and [2] displays a triacontagonal projection of 4_{21}. According to [7], Peter McMullen sketched the triacontagonal projections of both of these polytopes by hand during a period lasting from August 18 until August 20 in 1964. Recently John Stembridge produced an 8-color computer sketch of 4_{21}, to supplement an announcement by a research group at the American Institute of Mathematics of having computed a particular class of representations of a real Lie group having the E_8 root system. A simplified sketch is offered here, showing only the relative locations of the 240 vectors, in 8 concentric cycles with 30 points per cycle.

Here is a streamlined version of the formula from [3] for the elements of the H_4 root system. Let $\omega = \exp\left(\frac{i\pi}{30}\right)$ and $a > b > c > d$ be the positive roots of the polynomial $45x^8 - 90x^6 + 60x^4 - 15x^2 + 1$ over $\mathbb{R}$. Then the 120 vectors

$$\begin{array}{ll} A_n = (a\omega^{2n}, d\omega^{22n}), & B_n = (b\omega^{2n+1}, c\omega^{22n+11}), \\ C_n = (c\omega^{2n+1}, -b\omega^{22n+11}), & D_n = (d\omega^{2n}, -a\omega^{22n}), \end{array}$$

where $n \in \{0, 1, 2, ..., 28, 29\}$, comprise the H_4 root system. This formula has the feature that projecting along the first coordinate, essentially forgetting the second coordinate, yields the triacontagonal projection of the H_4 roots. The purpose of this note is to give a similar formula for the E_8 root system.

The triacontagonal projections of the H_4 and E_8 root systems are closely related. The latter is the union of two scaled sizes of the former, where the ratio of the larger to the smaller is the golden ratio $\tau = \frac{1}{2}\left(1 + \sqrt{5}\right)$. Given this, one expects that a similar formula should exist for the E_8 roots. Indeed, the formula given here was obtained by "guess-and-check". Checking is tedious, but one may use Maple, for example, to compute as many inner products as are necessary to become convinced that the formula yields a root system isomorphic to E_8. (A more "ethical" way to obtain the formula would be to diagonalize a Coxeter element of the Coxeter group of E_8. However, this author believes that the ends could not possibly justify the enormity of the computations involved in such means.)

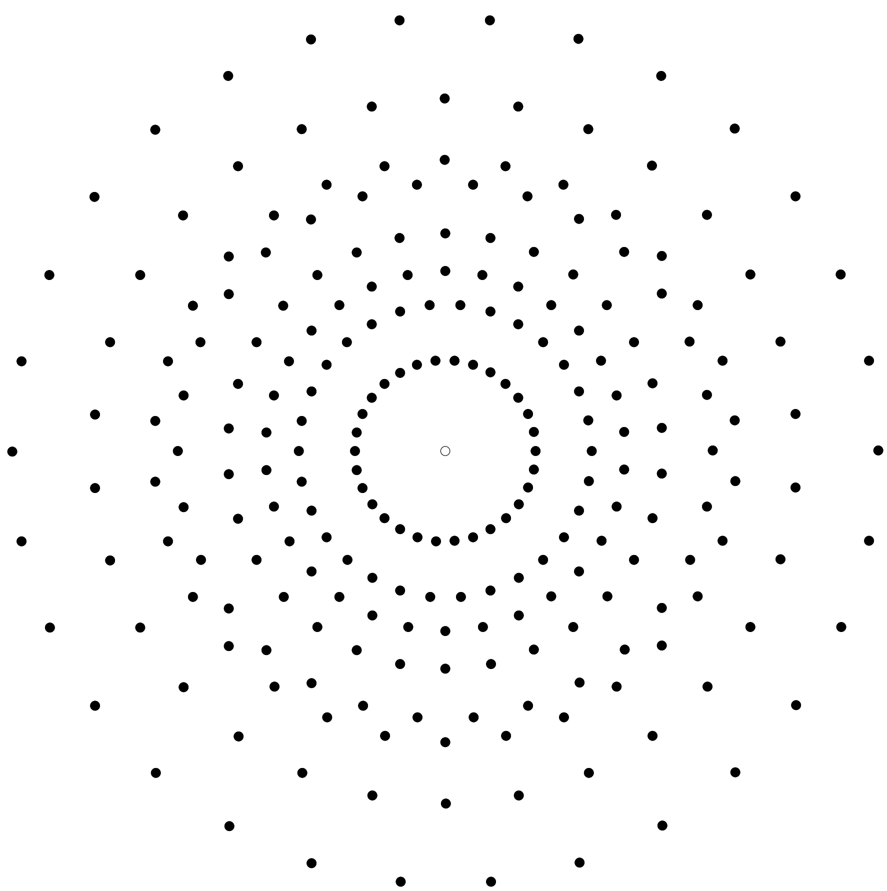

Figure 1. Triacontagonal projection of the E_8 root system.

2 The formula

Denote $\omega = \exp\left(\frac{i\pi}{30}\right)$ and define $\{a, b, c, d\}$ as above. More explicitly, a, b, c, and d are the positive numbers satisfying

$$
\begin{aligned}
2a^2 &= 1 + 3^{-1/2}5^{-1/4}\tau^{3/2}, \quad & 2b^2 &= 1 + 3^{-1/2}5^{-1/4}\tau^{-3/2}, \\
2c^2 &= 1 - 3^{-1/2}5^{-1/4}\tau^{-3/2}, \quad & 2d^2 &= 1 - 3^{-1/2}5^{-1/4}\tau^{3/2},
\end{aligned}
$$

where τ is the golden ratio. For any integer n, denote $c_n = \omega^n + \omega^{-n} = 2\cos\left(\frac{n\pi}{30}\right)$. Next, denote

$$
\begin{aligned}
r_1 &= a/c_9, \quad r_2 = b/c_9, \quad r_3 = c/c_9, \quad r_4 = d/c_9, \\
r_5 &= a/c_3, \quad r_6 = b/c_3, \quad r_7 = c/c_3, \quad r_8 = d/c_3.
\end{aligned}
$$

Then the 240 rows of the matrices

$$
\begin{bmatrix} A_n \\ B_n \\ C_n \\ D_n \\ E_n \\ F_n \\ G_n \\ H_n \end{bmatrix}
=
\begin{bmatrix}
r_1 & r_4 & r_6\omega & r_7\omega \\
r_2\omega^{29} & r_3\omega^{19} & -r_8\omega^{24} & -r_5\omega^{18} \\
r_3\omega^{29} & -r_2\omega^{19} & r_5\omega^{24} & -r_8\omega^{18} \\
r_4 & -r_1 & r_7\omega & -r_6\omega \\
r_5 & r_8 & -r_2\omega & -r_3\omega \\
r_6\omega^{29} & r_7\omega^{19} & r_4\omega^{24} & r_1\omega^{18} \\
r_7\omega^{29} & -r_6\omega^{19} & -r_1\omega^{24} & r_4\omega^{18} \\
r_8 & -r_5 & -r_3\omega & r_2\omega
\end{bmatrix}
\cdot
\begin{bmatrix}
\omega^{2n} & & & \\
& \omega^{22n} & & \\
& & \omega^{14n} & \\
& & & \omega^{26n}
\end{bmatrix},
$$

where $n \in \{0, 1, 2, ..., 28, 29\}$, as regarded as elements of $\mathbb{C}^4$, comprise a root system isomorphic to E_8. Moreover, each of these vectors has norm equal to 1.

3 Remarks

Projecting along the first coordinate, by forgetting the last three coordinates, yields the image depicted above as a subset of $\mathbb{C} \cong \mathbb{R}^2$.

Each cycle of 30 roots, as denoted by A_n, B_n, and so on, is expressed using the trigonometric function $n \mapsto (\omega^{2n}, \omega^{22n}, \omega^{14n}, \omega^{26n})$, composed with some amplitude and phase adjustments. Each phase shift was chosen so that $\{A_n, B_n, C_n, D_n, E_n, F_n, G_n, H_n\}$ is a system of simple roots for any fixed value of n.

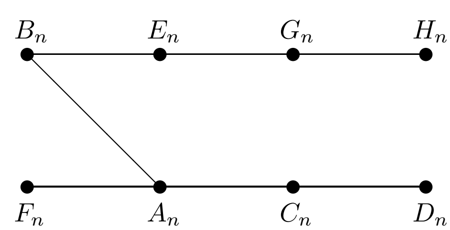

Figure 2. The Dynkin diagram of E_8.

The amplitudes r_k were chosen so that each vector has norm 1. Alternatively, one may use the amplitudes

$$\begin{array}{llll} r_1 = 1, & r_2 = c_{11}, & r_3 = c_6 c_{13}, & r_4 = c_6 c_{14}, \\ r_5 = c_{12}, & r_6 = c_{11} c_{12}, & r_7 = c_{13}, & r_8 = c_{14}. \end{array}$$

In so doing, one avoids the complicated definition/formulae for the values $\{a, b, c, d\}$ given above. For example, using this choice of amplitudes facilitates the verification of the formula, for then all the coordinates lie in the cyclotomic field $\mathbb{Q}(\omega)$. The tradeoff is that the norms of the vectors are more complicated to express.

References

[1] Chilton, B. L. On the projection of the regular polytope {5, 3, 3} into a regular triacontagon. *Canad. Math. Bull.* **7** (1964), 385-398.

[2] H. S. M. Coxeter and G. C. Shephard. *Leonardo.* **25**, No. 3/4, (1992), 239-244.

[3] H. S. M. Coxeter. *Regular Polytopes,* 3rd Ed. Dover Publications, New York, 1973.

[4] H. S. M. Coxeter. *Regular Complex Polytopes.* Cambridge University Press, 1974.

[5] Thorold Gosset. On the regular and semi-regular figures in space of n dimensions. *Messenger of Mathematics.* **29** (1900) 43-48.

[6] James Humphreys. *Reflection Groups and Coxeter Groups.* Cambridge University Press, 1990.

[7] Peter McMullen. Personal communication, April 24, 2007.

[8] John Stillwell. The Story of the 120-Cell. *Notices of the Amer. Math. Soc.* January 2001. 17-24.