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Lie Superalgebras Associated with Constant Sectional Curvature

DAVID A. RICHTER
Department of Mathematics, Western Michigan University, Kalamazoo MI, 49008-5244, U.S.A. e-mail: david.richter@wmich.edu

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Abstract. This note describes an observation connecting Riemannian manifolds of constant sectional curvature with a particular class of Lie superalgebras. Specifically, it is shown that the structural equations of a space M with constant sectional curvature, of one variety or another, nearly coincide with some identities satisfied by tensors which can be used to construct some specific families of Lie superalgebras. In particular, one obtains either $\mathrm{osp}(n, 2)$, $\mathrm{spl}(n, 2)$, or $\mathrm{osp}(4, 2n)$ if the Riemannian manifold has constant curvature, constant holomorphic curvature or constant quaternion-holomorphic curvature, respectively.

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1. Introduction

Lie algebras lie at the foundation of differential geometry and topology, and it should not come as a surprise that their close descendents, Lie superalgebras, might also serve some similar purpose. A notable example has already been established in the Hodge theory on compact Kähler manifolds, [7] and [10]. Also, as was pointed out in Kac' classification [3] of Lie superalgebras, the axioms defining a Lie superalgebra coincide with some properties of Whitehead's homotopy operations. In this paper we focus on some identities in classical differential geometry, namely in the study of spaces of constant curvature, found in some well-known textbooks [1,2,4,11]. The observation made in this paper is that if the sectional curvature tensor of a Riemannian manifold is required to generate a Lie superalgebra of a type to be described below, then the manifold must have constant sectional curvature, constant holomorphic sectional curvature, or constant quaternion-holomophic sectional curvature. These particular Lie superalgebras will be constructed explicitly. Necessarily, the manner in which the sectional curvature tensor 'generates' a Lie superalgebra will be described as well.

Here is some background on the observation. Sudbery [9] found realizations of the Lie superalgebras $G(3)$ and $F(4)$ using Cayley numbers or 'octonions'. These two Lie superalgebras have the following property: Suppose

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

is the decomposition into the even and odd subspaces of $\mathfrak{g}$. Then the even subspace decomposes as

$$\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{sl}_2\mathbb{C},$$

where $\mathfrak{k}$ is reductive, and the odd subspace factors as a tensor-product representation of $\mathfrak{g}_0$,

$$\mathfrak{g}_1 = V \otimes \mathbb{C}^2.$$

For $\mathfrak{g} = G(3)$, V is the standard representation of the exceptional simple Lie algebra $\mathfrak{g}_2$, while for $\mathfrak{g} = F(4)$, V is the spin representation of $\mathfrak{b}_3 \cong \mathfrak{so}_7\mathbb{C}$. In both cases, $\mathbb{C}^2$ denotes the standard representation of $\mathfrak{sl}_2\mathbb{C}$. The observation which prompted this study was that a few of the identities Sudbery needed in his construction resemble some of the structural equations describing a Riemannian manifold with constant sectional curvature.

It is natural to try to determine which Lie superalgebras have this type of decomposition. Using Kac's classification [3], one may quickly obtain Table I:

(Here α is a complex parameter determining an orbit in $\mathbb{C}$ under the action of the 6-element group of linear substitutions,

$$\langle \alpha \mapsto (1-\alpha), \alpha \mapsto \alpha^{-1} \rangle.$$

The orbits are in one-to-one correspondence with isomorphism-equivalence classes of simple classical Lie superalgebras of dimension 17.) One may rightfully refer to this class of Lie superalgebras as being of 'Sudbery's type'. All of these Lie superalgebras admit an elegant explicit description, and more significance will be borne out by the relationship to classical differential geometry described later.

The coefficient field throughout is $\mathbb{R}$, and the Lie superalgebras, which are obtained are real forms of some of the Lie superalgebras found in the table. The real forms of Lie superalgebras have been classified, and can be found in [6]. The appendices are to provide the notational conventions for Lie superalgebras and representations of $\mathfrak{sl}_2\mathbb{R}$ which are needed throughout the paper.

Table I. Lie superalgebras of Sudbery's type

$\mathfrak{g}$	$\mathfrak{k}$	V
$\mathrm{osp}(n,2)$	$\mathfrak{so}(n)$	$\mathbb{C}^n$
$\mathrm{spl}(n,2)$	$\mathbb{C} \oplus \mathfrak{sl}_n\mathbb{C}$	$\mathbb{C}^n \oplus (\mathbb{C}^n)^*$
$\mathrm{osp}(4,2n)$	$\mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sp}(n)$	$\mathbb{C}^2 \otimes \mathbb{C}^{2n}$
$D(2,1;\alpha)$	$\mathfrak{sl}_2\mathbb{C} \oplus \mathfrak{sl}_2\mathbb{C}$	$\mathbb{C}^4$
$G(3)$	$\mathfrak{g}_2$	$\mathbb{C}^7$
$F(4)$	$\mathfrak{b}_3$	$\mathbb{C}^8$ (spin)

2. Structure of Lie Superalgebras of Sudbery's Type

Suppose that a Lie superalgebra $\mathfrak{g}$ is real and of Sudbery's type. Thus, the even and odd subspaces of $\mathfrak{g}$ are given by

$$\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{sl}_2\mathbb{R} \quad \text{and} \quad \mathfrak{g}_1 = V \otimes \mathbb{R}^2,$$

where $\mathfrak{k}$ is a real reductive Lie algebra, V is a representation of $\mathfrak{k}$, and $\mathbb{R}^2$ is the standard representation of $\mathfrak{sl}_2\mathbb{R}$ by traceless 2×2 real matrices. The purpose of this and the subsequent section is to study Lie superalgebras which have this structure, with little reference to differential geometry. In this section we study them in general, and in Sections we focus on particular families. The construction of Lie superalgebras of this type depends on the existence of a tensor $F: V \otimes V \to \mathfrak{k}$ with some particular properties to be described in Theorem 2.1. Later we will see how these algebras are related to some Riemannian manifolds.

It is a general fact that a Lie superalgebra can be generated from its odd subspace $\mathfrak{g}_1$, the even subspace being spanned by anticommutators of elements of the odd subspace. This is how we shall proceed. Start with a vector space V. Using notation developed in Appendix A, let $\mathfrak{g}_1$ be spanned by vectors of the form

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = u_1 e_1 + u_2 e_2, \quad u_1, u_2 \in V.$$

Now that we have our odd subspace, the anti-commutator between odd elements is built from some accessory maps defined here. Endow V with an inner product g, and choose a Lie subalgebra $\mathfrak{k} \subset \mathfrak{so}(V, g)$, which preserves g, i.e., assume that

$$g(Au, v) + g(u, Av) = 0$$

for all $A \in \mathfrak{k}$ and $u, v \in V$. Suppose that F is a bilinear map from $V \times V$ into $\mathfrak{k}$, and choose a couple of odd vectors $\mathbf{u} = u_1 e_1 + u_2 e_2$ and $\mathbf{v} = v_1 e_1 + v_2 e_2$ from $\mathfrak{g}_1$. For the four-tuple (u_1, u_2, v_1, v_2), define

$$D(u_1, u_2, v_1, v_2) = F(u_1, v_2) + F(v_1, u_2),$$
$$G(u_1, u_2, v_1, v_2) = \begin{bmatrix} g(u_1, v_1) & g(u_2, v_1) \\ g(u_1, v_2) & g(u_2, v_2) \end{bmatrix}$$

and

$$M = (G + G^T)J,$$

where J is the matrix

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Explicitly, this is

$$M(u_1, u_2, v_1, v_2) = \begin{bmatrix} -g(u_1, v_2) - g(u_2, v_1) & 2g(u_1, v_1) \\ -2g(u_2, v_2) & g(u_1, v_2) + g(u_2, v_1) \end{bmatrix}.$$

Finally, define the bracket between the two odd vectors $\mathbf{u} = u_1 e_1 + u_2 e_2$ and $\mathbf{v} = v_1 e_1 + v_2 e_2$ by

$$\{u_1 e_1 + u_2 e_2, v_1 e_1 + v_2 e_2\} = (D(u_1, u_2, v_1, v_2), \lambda M(u_1, u_2, v_1, v_2)), \tag{1}$$

where $\lambda \in \mathbb{R}$ is a parameter. The sign of λ determines some of the structure of $\mathfrak{g}$. Indeed, notice that if $\lambda = 0$, then this product 'degenerates'. This then gives the anti-commutator between odd vectors. Notice that $\{\mathbf{u}, \mathbf{v}\}$ does indeed lie in the direct sum $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{sl}_2\mathbb{R}$.

The definitions of the even/even and even/odd commutators are now apparent. First, the commutator between even elements is defined in an obvious way because $\mathfrak{g}_0$ is defined as an external direct sum of Lie algebras. Second, notice that the even/odd commutator is the manifestation of a representation ρ of $\mathfrak{g}_0$ on $\mathfrak{g}_1$. However, in this case, $\mathfrak{g}_0$ is a direct sum, so there is a tensor-product action on $\mathfrak{g}_1$: if $A \in \mathfrak{k}$ and $L \in \mathfrak{sl}_2\mathbb{R}$, define the commutator on decomposable tensors by

$$[(A, L), u \otimes e] = \rho(A, L)(u \otimes e) = Au \otimes e + u \otimes L(e), \tag{2}$$

where $u \in V$ and $e \in \mathbb{R}^2$. More explicitly, if

$$L = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathfrak{gl}_2\mathbb{R}$$

with $a + d = 0$, and $u_1, u_2 \in V$, then

$$[(A, L), u_1 e_1 + u_2 e_2] = (Au_1 + au_1 + bu_2)e_1 + (Au_2 + cu_1 + du_2)e_2. \tag{3}$$

The odd/even commutator is obtained by skew-symmetrizing

$$[u \otimes e, (A, L)] = -[(A, L), u \otimes e]. \tag{4}$$

The following theorem describes the conditions required of the map F in order that $\mathfrak{g}$ be a Lie superalgebra.

THEOREM 2.1. *The space* $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *defined above is a Lie superalgebra iff the bilinear function* $F\colon V \times V \to \mathfrak{k}$ *satisfies* (i) $F(u, v) + F(v, u) = 0$, (ii) $F(Au, v) + F(u, Av) = [A, F(u, v)]$, *and* (iii) $F(u, v)u = \lambda[g(u, v)u - g(u, u)v]$ *for all* $u, v \in V$ *and all* $A \in \mathfrak{k}$.

This result is implicit in [9], and it is not difficult to establish. One merely verifies super skew symmetry and the super Jacobi identity through tedious case-by-case analysis. With that, finding Lie superalgebras which have this apparently arbitrary structure is equivalent to finding tensors F satisfying the criteria in this theorem.

3. Solutions

This section gives some solutions F to the criteria outlined in Theorem 2.1. These solutions yield particular realizations of certain real forms of the Lie superalgebras osp$(n,2)$, spl$(n,2)$, osp$(4,2n)$, (notation from [3,6]), and the members of the exceptional continuous family $D(2,1;\alpha)$. Since the other two exceptional cases $G(3)$ and $F(4)$ are described in [9], this section therefore completes the descriptions of the types of Lie superalgebras discussed in Section 1.

Before doing this, it is informative to have explicit descriptions of three sequences of real Lie algebras. Namely, it is useful to define $\mathfrak{so}(n)$, $\mathfrak{u}(n)$ and $\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)$ as specific sets of real matrices, with the Lie bracket given by the Lie bracket between matrices

$$[X,Y]=XY-YX.$$

First of all, define $\mathfrak{so}(n)$ as the set of real skew-symmetric matrices

$$\mathfrak{so}(n)=\{X\in\mathbb{R}^{n\times n}:X^T+X=0\}.$$

Before defining the other two sequences, complex and quaternionic structure matrices are required. Let I denote the $4n\times 4n$ matrix

$$I=\begin{bmatrix} 0 & I_n & 0 & 0 \\ -I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_n \\ 0 & 0 & I_n & 0 \end{bmatrix},$$

where I_n is the $n\times n$ identity matrix, and let J denote the $2n\times 2n$ matrix

$$J=\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}.$$

It is convenient, albeit annoying, to let J also denote the $4n\times 4n$ matrix, where each I_n appearing here is replaced by I_{2n}. In that case, there is also the $4n\times 4n$ matrix $K=IJ$. Define $\mathfrak{u}(n)$ as the subspace of $\mathfrak{so}(2n)$, which commute with the $2n\times 2n$ matrix J. Explicitly,

$$\mathfrak{u}(n)=\left\{\begin{bmatrix} X & Y \\ -Y & X \end{bmatrix}:X^T=-X,\,Y^T=Y\right\}.$$

Finally, $\mathfrak{sp}(1)\oplus\mathfrak{sp}(n)$ needs to be defined in a couple stages. First, define $\mathfrak{sp}(1)$ as the real span of the $4n\times 4n$ matrices I, J, and K, and let $\mathfrak{sp}(n)$ be the subspace of $\mathfrak{so}(4n)$, which commute with both I and J. Explicitly,

$$\mathfrak{sp}(n)=\left\{\begin{bmatrix} W & X & Y & Z \\ -X & W & Z & -Y \\ -Y & -Z & W & X \\ -Z & Y & -X & W \end{bmatrix}:\begin{matrix} W^T=-W,\,X^T=X, \\ Y^T=Y,\,Z^T=Z \end{matrix}\right\}.$$

Then $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ is the internal direct sum of $\mathfrak{sp}(1)$ and $\mathfrak{sp}(n)$.

As one may perhaps be aware, it is possible to describe these Lie algebras $\mathfrak{u}(n)$ and $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ using complex numbers $\mathbb{C}$ and Hamilton's quaternions $\mathbb{H}$, respectively. That is, both $\mathfrak{u}(n)$ and $\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$ are maximal compact Lie algebras acting on $\mathbb{C}^n$ and $\mathbb{H}^n$, respectively. Thus, they are certain generalizations of $\mathfrak{so}(n)$ acting on $\mathbb{R}^n$. However, it is more convenient to use real matrices because one may use an inner product that is defined uniformly for all three sequences: for $u, v \in \mathbb{R}^n$, let $g(u, v) = u^T v$. That is, let g be the standard inner product on $\mathbb{R}^n$.

Using these real Lie algebras, one now proceeds to describe three sequences of real forms of the Lie superalgebras $\mathrm{osp}(n, 2)$, $\mathrm{spl}(n, 2)$, and $\mathrm{osp}(4, 2n)$. Recall that, for each $\mathfrak{k}$ described above, there should be a representation V and a function F: $V \times V \to \mathfrak{k}$ having the properties (i), (ii), and (iii) of Theorem 2.1. For each of the three cases, define Π and Ω according to Table II:

PROPOSITION 3.1. *Set* $F(u, v) = \lambda[\Pi(uv^T - vu^T) - \Omega(u, v)]$, *where* Π *and* Ω *are given in Table II. Then* F *has the properties* (i), (ii), *and* (iii) *described in Theorem* 2.1.

Again, as with Theorem 2.1, this is established through a routine computation. This therefore yields the three sequences $\mathrm{osp}(n, 2)$, $\mathrm{spl}(n, 2)$, and $\mathrm{osp}(4, 2n)$ of Lie superalgebras. Note that the Lie superalgebras are 'degenerate' if $\lambda = 0$. The differential-geometric significance of this will be seen in Section 4.

3.1. THE LIE SUPERALGEBRAS $D(2, 1; \alpha)$

The continuous family $D(2, 1; \alpha)$ of exceptional 17-dimensional Lie superalgebras deserves some separate attention. One considers these upon specialization of the $\mathrm{osp}(4, 2n)$ superalgebras to the case, when $n = 1$. If $n = 1$, then

$$\mathfrak{k} \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \cong \mathfrak{so}(4).$$

Every element in this family has a copy of $\mathfrak{so}(4)$ as a subalgebra. Both copies of $\mathfrak{sp}(1)$ act on $\mathbb{R}^4$, but they are obtained differently; specifically, one is a right-action and the other is a left-action. Denote $I_+ = I$, $J_+ = J$, and $K_+ = K$, where I, J, and K are the 4×4 quaternion structure matrices defined earlier, but now introduce three more matrices defined by

Table II. Tensors for Lie superalgebras

$\mathfrak{g}^{\mathbb{C}}$	$\mathrm{osp}(n, 2)$	$\mathrm{spl}(n, 2)$	$\mathrm{osp}(4, 2n)$
$\mathfrak{k}$	$\mathfrak{so}(n)$	$\mathfrak{u}(n)$	$\mathfrak{sp}(1) \oplus \mathfrak{sp}(n)$
V	$\mathbb{R}^n$	$\mathbb{R}^{2n} \cong \mathbb{C}^n$	$\mathbb{R}^{4n} \cong \mathbb{H}^n$
$\Pi(X)$	X	$X - JXJ$	$X - IXI - JXJ - KXK$
$\Omega(u, v)$	0	$(u^T Jv)J$	$(u^T Iv)I + (u^T Jv)J + (u^T Kv)K$

$$I_- = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \quad \text{and} \quad J_- = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

and $K_- = J_- I_-$. Notice that $\{I_\pm, J_\pm, K_\pm\}$ is a basis for $\mathfrak{so}(4)$, and that the matrices with '−' for a subscript commute with the matrices with '+' for a subscript. Let $\mathfrak{so}(3)_+$ and $\mathfrak{so}(3)_-$ denote the Lie subalgebras generated by $\{I_+, J_+, K_+\}$ and $\{I_-, J_-, K_-\}$, respectively. The functions Π and Ω using I_+, J_+, and K_+ have already been defined, and the same could be done for the new 'left-handed' quaternion matrices $\{I_-, J_-, K_-\}$, but an identity given below will show that this is not necessary.

It is useful to consider these six matrices as constituting a basis for the six-dimensional vector space $\bigwedge^2(\mathbb{R}^4)$, under the identification

$$\begin{aligned} \textstyle\bigwedge^2(\mathbb{R}^4) &\longleftrightarrow \mathfrak{so}(4), \\ e_i \wedge e_j &\longleftrightarrow E_{ij} - E_{ji}. \end{aligned}$$

This identification induces the Hodge star automorphism $* = \mathrm{st}$ of $\mathfrak{so}(4)$. This map is determined uniquely by specifying its eigenspaces

$$\mathrm{st}(I_\pm) = \pm I_\pm, \qquad \mathrm{st}(J_\pm) = \pm J_\pm, \qquad \mathrm{st}(K_\pm) = \pm K_\pm.$$

Thus $\mathfrak{so}(3)_\pm$ is the ± 1 eigenspace for st. Related to this identification is an interesting and useful identity.

PROPOSITION 3.2. $\Pi(uv^T - vu^T) + 2\Omega(u, v) = 4(uv^T - vu^T)$.

Proof. First, since Π and Ω are alternating and there is an identification between $\bigwedge^2(\mathbb{R}^4)$ and $\mathfrak{so}(4)$, one may regard F as a linear function from $\mathfrak{so}(4)$ to itself. Thus, choose $W \in \mathfrak{so}(4)$. One may then write W as the linear combination

$$W = x_+ I_+ + y_+ J_+ + z_+ K_+ + x_- I_- + y_- J_- + z_- K_-$$

with real coefficients. Denote the projections of W onto $\mathfrak{so}(3)_\pm$ by $W_\pm$, so that

$$W_+ = x_+ I_+ + y_+ J_+ + z_+ K_+ \quad \text{and} \quad W_- = x_- I_- + y_- J_- + z_- K_-.$$

One checks that

$$\Pi(W) = 4W_- \quad \text{and} \quad \Omega(W) = 2W_+,$$

yielding

$$\Pi(W) + 2\Omega(W) = 4(W_- + W_+) = 4W.$$

This is exactly what is required, again using the identification between $\bigwedge^2(\mathbb{R}^4)$ and $\mathfrak{so}(4)$. □

Given a real number $\alpha \in \mathbb{R}$, let

$$F_\alpha(u,v) = \tfrac{1}{2}\alpha\Pi(uv^T - vu^T) + (1-\alpha)\Omega(u,v).$$

Then the following is easily established.

PROPOSITION 3.3. *F_α has the properties described in Theorem* 2.1 *for all α.*

Thus, for each real α there is a Lie superalgebra $\mathfrak{g}_\alpha$, constructed using F_α. However, similar to the complex case, one has from the following therom.

THEOREM 3.4. *The isomorphism equivalence classes of the $\mathfrak{g}_\alpha$ correspond to orbits in $\mathbb{R}$ under the 2-element group $\langle \alpha \mapsto (1-\alpha) \rangle$.*

Proof. Replacing α with $(1-\alpha)$ is equivalent to interchanging the left and right actions of $\mathfrak{sp}(1)$ on $\mathbb{R}^4$. Therefore, one constructs an isomorphism between the Lie superalgebras $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{(1-\alpha)}$ by using the corresponding outer automorphism. □

One should contrast this with the complex case. The even subspace of $D(2,1;\alpha)$ is a tensor product of three copies of the same Lie algebra $\mathfrak{sl}_2\mathbb{C}$, whereas for this real form the even subspace is a tensor product of one copy of $\mathfrak{sl}_2\mathbb{R}$ with two copies of $\mathfrak{so}(3)$.

4. Riemannian Geometry

In this section, we finally come to the observation promised in Section 1. In this setting, the solutions are derived from the Riemannian curvature operator for some specialized Riemannian manifolds. It is shown here that Riemannian manifolds with constant curvature, constant holomorphic curvature, and constant quaternion-holomorphic curvature yield the Lie superalgebra families $\mathrm{osp}(n,2)$, $\mathrm{spl}(n,2)$, and $\mathrm{osp}(4,2n)$ given above. Moreover, for the special case of constant holomorphic curvature, one also obtains the family $\mathrm{spl}(n,1)$ in a natural way.

Suppose (M,g) is a Riemannian manifold with Levi-Civita connection ∇. Recall that one may use this to construct the curvature tensor given by

$$R(u,v) = [\nabla_u, \nabla_v] - \nabla_{[u,v]}.$$

For each point of M, this tensor R represents a map from the tangent space to itself, and this map is an orthogonal derivation with respect to the metric g. The basic idea is to identify V with the tangent space, and use the curvature endomorphism R and the metric g to obtain elements of $\mathfrak{g}_0$. Under this hypothesis, $\mathfrak{k}$ is identified with a Lie subalgebra of the holonomy algebra of R. A Lie superalgebra arises using a slight modification of R, provided the sectional curvature satisfies some strong conditions.

To connect these ideas, introduce the notion of an abstract curvature operator.

DEFINITION 4.1. Suppose $\mathfrak{k}$ is a Lie algebra equipped with a representation V and a $\mathfrak{k}$-invariant inner product g on V. A bilinear function $R: V \times V \to \mathfrak{k}$ is an 'algebraic curvature operator' for $(\mathfrak{k}, V, g)$ if (a) $R(v, u) + R(u, v) = 0$, (b) $R(v, w)u + R(w, u)v + R(u, v)w = 0$, and (c) $g(R(u, v)w, t) = g(R(w, t)u, v)$ for all $t, u, v, w \in V$.

The following shows the connection to the Lie superalgebras described in the preceding Section:

PROPOSITION 4.2. *Let* $R(u, v) = F(u, v) + 3\lambda\Omega(u, v)$ *where* F *and* Ω *are defined in the preceding section. Then* R *is an algebraic curvature operator for* $(\mathfrak{k}, V, g)$.

This is trivial to establish from the definitions. The idea in this section is that this proposition can be turned around so that one obtains Lie superalgebras from the curvature tensors of some particular Riemannian manifolds. There are some subtle differences between the three types, so they are discussed separately.

4.1. CONSTANT CURVATURE

Suppose (M, g) is a space of constant curvature. Recall that this means that there is a constant κ such that the curvature tensor satisfies

$$R(u, v)w = \kappa[g(w, v)u - g(w, u)v]$$

for all tangent vector fields u, v, w. If $\Omega = 0$, as when $\mathfrak{k} = \mathfrak{so}(n)$, F is already an abstract curvature operator, i.e. it requires no modification. Indeed, if M is a space of constant curvature, then the Riemannian curvature operator R for (M, g) has the algebraic properties given in Theorem 2.1, yielding the Lie superalgebra osp$(n, 2)$.

Notice that if (M, g) has zero sectional curvature, then the the holonomy is trivial and the Lie superalgebra degenerates as the tensor product $\mathfrak{g} = V \otimes \mathbb{R}^2$.

4.2. CONSTANT HOLOMORPHIC CURVATURE

Suppose that (M, g, J) is a Kähler manifold whose curvature tensor satisfies

$$\begin{aligned} R(u, v)w = {} & -\tfrac{1}{4}\kappa[g(v, w)u - g(u, w)v - g(v, Jw)Ju + \\ & + g(u, Jw)Jv + 2g(u, Jv)Jw] \end{aligned}$$

for all tangent vector fields u, v, w, where κ is a constant. Then M is a space of constant holomorphic curvature, and one uses R to construct a Lie superalgebra as follows. Define the tensor F by

$$F(u, v) = R(u, v) + \tfrac{3}{4}\kappa g(u, Jv)J = R(u, v) + \tfrac{3}{4}\kappa\Omega(u, v).$$

Then F has the properties required by Theorem 2.1, and the resulting Lie superalgebra is spl$(n, 2)$. In this case (and the one that follows), however, a modification of R by adding a multiple of the almost complex structure J is necessary.

There is a little more one can say about this case. In fact, the associated Lie superalgebra has a subalgebra still bearing some differential-geometric significance. Let $\mathfrak{h}_1$ be the subspace of elements of $\mathfrak{g}_1$ having the form

$$\begin{bmatrix} u \\ Ju \end{bmatrix},$$

where J is the almost complex structure tensor. Evidently we have

$$\mathfrak{h}_1 \cong \mathbb{R}^{2n} \otimes \mathbb{R} \cong \mathbb{R}^{2n}.$$

Let $\mathfrak{h}$ be the Lie superalgebra generated by $\mathfrak{h}_1$ under the super Lie bracket defined above. One can quickly determine the structure of $\mathfrak{h}_0$ as follows. Suppose u and v are tangent vectors. Then the anticommutator is

$$\{ue_1 + Jue_2, ve_1 + Jve_2\} = (D, \lambda M),$$

as before. However, in this case we have

$$D = F(u, Jv) + F(v, Ju) = 2F(u, Jv)$$

and

$$\begin{aligned} M &= \begin{bmatrix} -g(u, Jv) - g(v, Ju) & 2g(u, v) \\ -2g(Ju, Jv) & g(u, Jv) + g(v, Ju) \end{bmatrix} \\ &= 2\begin{bmatrix} 0 & g(u, v) \\ -g(u, v) & 0 \end{bmatrix} \end{aligned}$$

using the properties of the metric and curvature tensors of a Kähler manifold, [4]. Evidently the even subalgebra $\mathfrak{h}_0$ has the structure

$$\mathfrak{h}_0 = \mathfrak{u}(n) \oplus \mathfrak{so}(2).$$

One can then also verify that this acts on $\mathfrak{h}_1$. Furthermore, the actions of the Lie subalgebras $\mathbb{R}J$ and $\mathfrak{so}(2)$ of $\mathfrak{h}_0$ on $\mathfrak{h}_1$ are identical, and one may quotient to obtain a real form of $\mathrm{spl}(n, 1)$.

4.3. CONSTANT $\mathbb{H}$-HOLOMORPHIC CURVATURE

Suppose now that $(M, g, (I, J))$ is a quaternion-Kähler manifold. Recall that although none of the quaternion structures I, J, and $K = IJ$ may be global almost complex structures on M, every point has a coordinate chart in which they are; moreover, the space spanned by $\{I, J, K\}$ is parallel over M. If M has constant $\mathbb{H}$-holomorphic curvature, then there is a constant κ such that

$$\begin{aligned} R(u, v)w = -\tfrac{1}{4}\kappa[&g(v, w)u - g(u, w)v - g(v, Iw)Iu - g(v, Jw)Ju - \\ &-g(v, Kw)Ku + g(u, Iw)Iv + g(u, Jw)Jv + g(u, Kw)Kv + \\ &+2g(u, Iv)Iw + 2g(u, Jv)Jw + 2g(u, Kv)Kw)] \end{aligned}$$

for all tangent vectors fields u, v, w. (see [5,8]). Then $F = R + \frac{3}{4}\kappa\Omega$ has all the properties required by Theorem 2.1, so there is an associated Lie superalgebra isomorphic to a particular real form of osp$(4, 2n)$.

5. Conclusions and Speculations

There is a geometric construction related to the forms Ω, when the complexification of $\mathfrak{g}$ is spl$(n, 2)$ or osp$(4, 2n)$. For the former case one has $\Omega(u, v) = (u^T J v)J$. If u and v are unit vectors, therefore, $\Omega(u, v)$ gives the angle α between the planes $u \wedge Ju$ and $v \wedge Jv$

$$\cos^2 \alpha = (u^T J v)^2.$$

Similarly, in the latter case, one has the 'angle function' α given by

$$\cos^2 \alpha = |\Omega \wedge \Omega| = (u^T I v)^2 + (u^T J v)^2 + (u^T K v)^2.$$

This can be found in [5]. Thus, when M has a metric of constant holomorphic or constant $\mathbb{H}$-holomorphic curvature, the superalgebra is obtained by adding to R a multiple of this angle function $\cos\alpha$. The tensor F has the following property.

PROPOSITION 5.1. *Suppose $F = R + \frac{3}{4}\kappa\Omega$, where R is one of the curvature tensors described above. Then if F is one of the functions given above, then*

$$g(F(u, v)u, v) = -\tfrac{1}{4}\kappa[g(u, v)^2 - g(u, u)g(v, v)]$$

for all tangent vector fields u, v.

Thus, in seeking the constant of holomorphic or $\mathbb{H}$-holomorphic curvature, it makes more sense to use F than R. Contrast this for the case when (M, g) has constant curvature, where κ has the opposite sign.

Appendix A. Lie Superalgebras

By definition, a Lie superalgebra decomposes as $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where $\mathfrak{g}_0$ and $\mathfrak{g}_1$ denote the even and odd subspaces respectively. In order that $\mathfrak{g}$ have a Lie superalgebra structure, there must be three types of brackets in $\mathfrak{g}$, (a) a commutator for pairs of even elements, $[\ ,\ ]\colon \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathfrak{g}_0$, (b) a commutator for pairs with one even and one odd element, $[\ ,\ ]\colon \mathfrak{g}_0 \times \mathfrak{g}_1 \to \mathfrak{g}_1$, and (c) an anticommutator for pairs of odd elements, $\{\ ,\ \}\colon \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$. Denote the super Lie bracket on $\mathfrak{g}$, whether it is a commutator or an anticommutator, by

$$\langle\ ,\ \rangle\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}.$$

This graded bracket must be super skew-symmetric,

$$(-1)^{ab}\langle A, B\rangle + \langle B, A\rangle = 0, \qquad (A \in \mathfrak{g}_a,\ B \in \mathfrak{g}_b), \quad a, b \in \{0, 1\} \tag{A.1}$$

and it must satisfy the super Jacobi identity,

$$(-1)^{ac}\langle A, \langle B, C\rangle\rangle + \text{cyclic} = 0, \quad (A \in \mathfrak{g}_a,\ B \in \mathfrak{g}_b,\ C \in \mathfrak{g}_c), \quad a, b, c \in \{0, 1\}. \tag{A.2}$$

If $\mathfrak{g}$ is a vector space endowed with a super skew symmetric bracket $\langle\ ,\ \rangle$ that satisfies the super Jacobi identity, then it is a Lie superalgebra.

Appendix B. The Standard Representation of $\mathfrak{sl}_2\mathbb{R}$

There is some useful notation for the action of $\mathfrak{sl}_2\mathbb{R}$ on $\mathbb{R}^2$. In particular, use the standard basis $\{e_1, e_2\}$ for $\mathbb{R}^2$, where

$$e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Finally, denote the 2×2 matrices by $E_{ij} \in \mathfrak{gl}_2\mathbb{R}$ which satisfy

$$E_{ij}(e_k) = \delta_{jk} e_i,$$

where δ_{jk} is the Kronecker delta symbol.

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