Prescribability of ghost symmetries of planar configurations

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Abstract

A ghost symmetry of a point configuration is a symmetry which appears in some projection of the configuration. Generally, ghost symmetries are more interesting when they are abundant. This paper gives necessary and sufficient conditions for a list of three involutions to correspond to a configuration whose ghost symmetries realize those involutions. Associated with any such "triad" of hyperinvolutions is a graph, and the characterization is essentially a slight generalization of 3-connectedness. The main theorem appears to be a close analogue of Steinitz's theorem which characterizes edge graphs of convex 3-polytopes.

1 Introduction

The simplest non-trivial example of ghost symmetry occurs in the generic configuration of 4 points in $\mathbb{R}^2$. Choose $C = \{v_1, v_2, v_3, v_4\} \subset \mathbb{R}^2$ such that $v_1 + v_2 + v_3 + v_4 = 0$. Let λ_b, λ_g, and λ_r be the orthogonal complements of $v_1 + v_4$, $v_2 + v_4$, and $v_3 + v_4$, respectively, and let π_b, π_g, and π_r be the corresponding orthogonal projections onto these lines. Then, by construction, the configurations $\pi_b(C)$, $\pi_g(C)$, and $\pi_r(C)$ each have two-fold symmetry about the origin. However, the symmetries of these three images is different. The two-fold symmetries of $\pi_b(C)$, $\pi_g(C)$, and $\pi_r(C)$ respectively induce the permutations $(1,4)(2,3)$, $(1,3)(2,4)$, and $(1,2)(3,4)$.

A less trivial example, on five points, is as follows. Choose two points $v_1 = (x_1, y_1)$, $v_2 = (x_2, y_2)$ and a parameter μ, and let

$$v_3 = (1-\mu)v_1 + \frac{\mu - 1}{\mu} v_2,\ v_4 = -v_1 + \frac{-\mu + 1}{\mu} v_2,\ \text{and}\ v_5 = (-1+\mu)v_1 - v_2.$$

For generic v_1, v_2, and μ, the configuration $C = \{v_1, v_2, v_3, v_4, v_5\}$ has a property similar to that described above. Specifically, there are three 1-dimensional projections of C which each have 2-fold symmetry. These are given by projecting onto the orthogonal complements of v_1, v_2, and v_3. Respectively, these symmetries correspond to the permutations $(2,5)(3,4)$, $(1,4)(3,5)$ and $(1,5)(2,4)$. Thus, the configuration C has ghost symmetry because projecting along one of these three axes produces a figure having a symmetry which the configuration C may not have.

This raises the question of which configurations have non-trivial ghost symmetries. One may state this more precisely: Suppose $C \subset \mathbb{R}^n$ is a configuration, $V \subset \mathbb{R}^n$ is a subspace, and $\pi : \mathbb{R}^n \to V$ is the orthogonal projection. Call π "isonumerous" with respect to C if the restriction of π to C is a bijection. If π is isonumerous, then one may identify each linear symmetry of $\pi(C)$ with a permutation of C. The symmetries of these projections are then the "ghost symmetries" of C. Given a finite set S and list T of permutations on S, the prescribablity question asks whether or not there exists a configuration C such that $|C| = |S|$ and whose ghost symmetries realize the elements of T.

Even for configurations in the plane $\mathbb{R}^2$, this question is not easy to answer. However, one may say a few things. Letting $\mathbb{R}P^1$ denote the set of all 1-dimensional subspaces of $\mathbb{R}^2$, one has:

Proposition 1.1. *A pair of vectors $\{v, w\} \subset \mathbb{R}^2$ is linearly independent iff there is exactly one subspace $\lambda \in \mathbb{R}P^1$ such that the orthogonal projection $\pi : \mathbb{R}^2 \to \lambda$ satisfies $\pi(v + w) = 0$.*

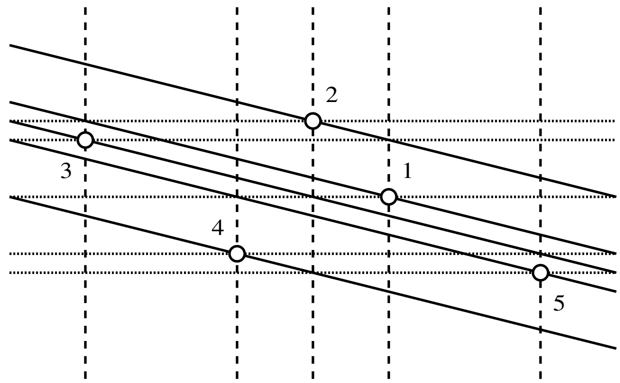

Figure 1: A Planar Configuration with Ghost Symmetries

Proof. Suppose $\{v, w\}$ is linearly independent and let λ be the orthogonal complement of $v+w \neq 0$. Then the orthogonal projection π onto λ satisfies $\pi(v+w) = 0$. On the other hand, suppose there are two distinct subspaces λ_1, λ_2 with this property. Then $v+w$ is orthogonal to both λ_1 and λ_2, implying that $v+w=0$. □

Thus, if $\{v, w\}$ is linearly independent, then there is exactly one subspace $\lambda \in \mathbb{R}P^1$ which realizes the transposition $v \leftrightarrow w$ as a ghost symmetry. More generally, if $C \subset \mathbb{R}^2$ is a configuration and λ is a 1-dimensional subspace, then every permutation induced by a non-trivial symmetry of $\pi_\lambda(C)$ must be a product of transpositions and it cannot have more than one fixed point. Call such a permutation a "hyperinvolution". If $|C|$ is even, then every hyperinvolution on C has no fixed points while if C is odd, then every hyperinvolution on C has exactly one fixed point.

The main result reported here gives a necessary and sufficient condition for a triple of hyperinvolutions on a given set to have a realization as a configuration. In both its statement and its proof, the theorem closely resembles Steinitz's theorem concerning realizability of graphs as edge-graphs of 3-dimensional convex polytopes, [2, 9]. As far as this author has been able to determine, the theorem proved here does not follow as a corollary to Steinitz's theorem. For example, planarity is not one of the conditions of the main theorem. Thus, the theorem is a close analogue of Steinitz's theorem. This author suspects that both Steinitz's theorem and the main result stated here follow as corollaries to some general statement. This generalization would necessarily deviate from the generalization of Steinitz's theorem which is most commonly sought, that being a combinatorial characterization of skeleta of convex polytopes in dimensions greater than 3.

The main theorem proved here uses some graph theory. For example, a triple of hyperinvolutions on a set with even cardinality is combinatorially indistinguishable from a Tait-colored graph. One of the conditions in the main theorem is essentially graph-theoretical. It is this author's impression that all of the graph theory used here is well-known but not widely well-known. The graph theory which is used here reduces down to some special cases of Tutte's results concerning 2-connected graphs and their decompositions, [6]. This author is not a graph theorist, and he is not thoroughly versed in Tutte's proofs of his results. Nevertheless, this author is confident in the validity of the graph theory which is used. Some details are therefore omitted for the sake of brevity; on the other hand, some graph theory results are proved here, but only if it sheds some light on the underlying geometry.

Since Tait-colored graphs are used to study compact connected surfaces, [3, 7], there may be some relation to Grünbaum's conjecture on their triangulations. This study uses combinatorial data, especially permutations, induced by isonumerous orthogonal projections to study properties of planar configurations. Curiously, so does [1]. However, our perspectives appear to differ because convexity is not an issue here.

The structure of this paper is as follows. First there is a discussion of the terminology and graph theory which is necessary to state the main result. Next appears a statement and proof of the result that says that it is possible to prescribe a triad of ghost symmetries iff the triad does not have what shall be called a “dibridge”. This is handled in two main cases, first where the underlying set has an even number of elements and the second where the set has an odd number of elements. The even case involves the most intricate work and the odd case follows quickly once the even case is settled. Unless stated otherwise, the reader should assume that the number of points is even. After this, there is some discussion of the realization space of a realizable triad and some open questions.

This paper has benefited from discussions with Ben Phillips and Jeff Strom.

2 Terminology

2.1 Graph Theory

The basic objects under study are triples of hyperinvolutions or, briefly, “triads”. Recall from above that a hyperinvolution on S is a permutation on S of order two and having at most one fixed point, and that a triad on S is a triple $T = \{b, g, r\}$ consisting of three hyperinvolutions on S. It is convenient to designate the hyperinvolutions of a triad using the colors blue, green, and red. In this article, all diagrams appear in black and white, and the colors are represented using solid lines, dashed lines with long dashes, and dashed lines with short dashes.

One should interpret a triad graph-theoretically. In particular, a triad of hyperinvolutions is the same as a Tait-colored multi-graph. Some authors call this a “3-graph”, [7]. The set S comprise the vertices. The colors are represented by T. One assigns the color $\sigma \in T$ to the edge (a, b) if the transposition (a, b) appears in the decomposition of σ into disjoint cycles. If $|S|$ is odd, then one allows that an edge may be a loop joining a vertex to itself.

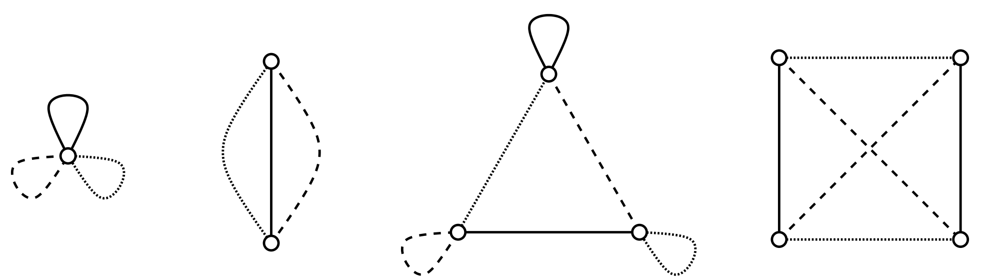

Figure 2: Some Small Triads

One may quickly verify that a transposition (a, b) commutes with a hyperinvolution σ if and only if (a, b) is one of the transpositions appearing in the decomposition into disjoint cycles. Thus, saying that (a, b) is an edge colored with σ is equivalent to saying that $(a, b) \circ \sigma = \sigma \circ (a, b)$ for some $\sigma \in T$. If $[x, y] = xyx^{-1}y^{-1}$ denotes the commutator of x and y, then (a, b) is an edge with color σ iff $[(a, b), \sigma] = \mathrm{id}$.

A “component” of a triad is a graph-theoretic component. Using a parity argument, one sees that if $|S|$ is even, then every connected component has an even number of vertices, while if $|S|$ is odd, then exactly one connected component has an odd number of vertices.

Suppose T is a triad of hyperinvolutions on S. A “monochrome pair” is a pair of edges $(a, b)(c, d)$ having the same color. In the notation of group theory, this means that a, b, c, and d

are distinct and $[(a,b),\sigma] = [(c,d),\sigma] = \mathrm{id}$ for some $\sigma \in T$. Suppose $(a,b)(c,d)$ is a monochrome pair appearing in the cyclic decomposition of $\sigma \in T$. Then σ also commutes with $(a,c)(b,d)$ and with $(a,d)(b,c)$, and one may obtain new triads by replacing $\sigma \in T$ by $(a,c)(b,d)$ or $(a,d)(b,c)$. Call such a transformation a "box switch". Given a monochrome pair in a triad, there are a total of three different box switches. These correspond to the non-trivial elements of the Klein-four group.

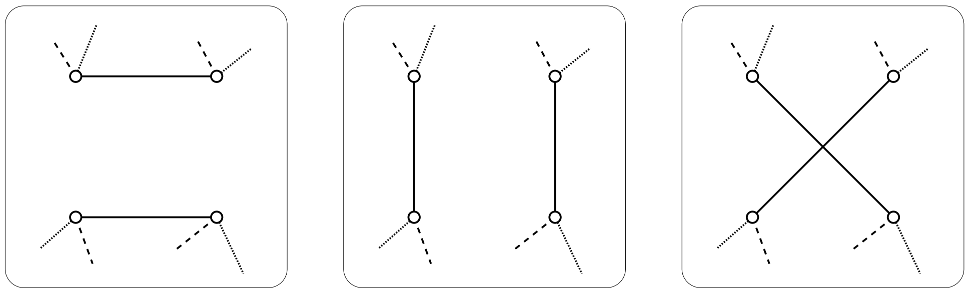

Figure 3: Monochrome Pairs and Box Switches

Call a monochrome pair $(a,b)(c,d)$ of a triad a "dibridge" if one of its box switches results in a graph with more components. Of the two possible box switches along a dibridge $\tau = (a,b)(c,d)$, exactly one results in a triad having more components than T. Denote this new triad by $\tau \cdot T$. In general, if T has k connected components and τ is a dibridge of T, then $\tau \cdot T$ has exactly $k+1$ components. By definition, every box switch in a triad which lacks a dibridge results in a connected graph.

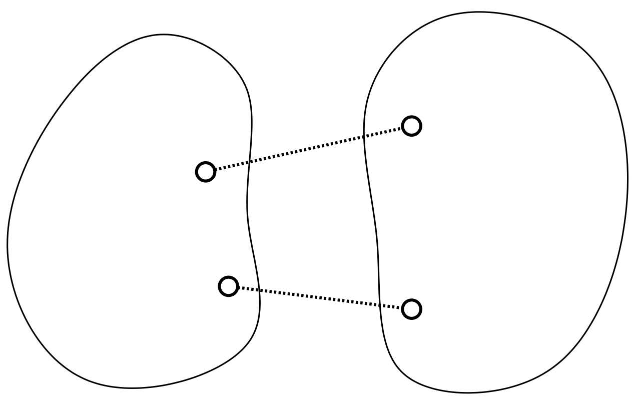

Figure 4: A Dibridge

The following elementary result relates dibridges to 3-connectedness.

Proposition 2.1. *Suppose T is a triad on S. Then T is 3-connected if and only if T is connected and has no dibridges.*

Call a triad "elementary" if it is connected and has no dibridges. It follows from Tutte's results on 2-connected graphs that every triad may be decomposed along dibridges into a connected sum of elementary triads and this decomposition is unique, [6]. In order to see this connection, it is necessary to be more precise about this decomposition. Suppose $S_1 \cap S_2 = \emptyset$ and $T_1 = \{b_1, g_1, r_1\}$ and $T_2 = \{b_2, g_2, r_2\}$ are triads on S_1 and S_2 respectively. Then the set

$T_1 \oplus T_2 = \{b_1 \circ b_2, g_1 \circ g_2, r_1 \circ r_2\}$ is a triad on the union $S_1 \cup S_2$. Call this the "direct sum" of T_1 and T_2. Using a box switch, one may join T_1 with T_2 to obtain a triad having a dibridge. Write $T \to T_1 \oplus T_2$ if one obtains $T_1 \oplus T_2$ by applying a box switch along a dibridge of T.

One may describe Tutte's decomposition as follows. Suppose T is a triad on S. One may inductively define a directed tree $\mathcal{T}$ associated with T. The vertices of the tree are represented by certain pairs (S_α, T_α) where $\{S_\alpha\}$ is a partition of S and T_α is a triad on S_α for all α. The vertex (S, T) appears at the root. Next, given any vertex (S_α, T_α), the triad T_α may or may not have a dibridge. If T_α has no dibridge, then (S_α, T_α) is a terminal branch, having no edges directed away from it. If T_α has a dibridge, then there is a non-trivial partition $S_\alpha = S_\beta \coprod S_\gamma$ and triads T_β on S_β and T_γ on S_γ such that $T_\alpha \to T_\beta \oplus T_\gamma$. This yields two vertices (S_β, T_β) and (S_γ, T_γ). One also draws directed edges from (S_α, T_α) to both (S_β, T_β) and (S_γ, T_γ). Since S is finite, this tree is also finite. The terminal vertices of this directed tree represent elementary triads. Tutte's results guarantee the uniqueness of these elementary triads. One also has:

Proposition 2.2. *Suppose T is a triad and T has a dibridge. Then there is a decomposition $T \to T_1 \oplus T_2$ such that T_1 is elementary.*

2.2 Realizability

Definition 2.3. *Suppose T is a triad of hyperinvolutions on S. A "realization" of T is a pair (ι, λ) where $\iota : S \to \mathbb{R}^2$ and $\lambda : T \to \mathbb{R}P^1$ are injections such that each orthogonal projection $\pi_\sigma : \mathbb{R}^2 \to \lambda_\sigma$ is isonumerous with respect to $\iota(S)$ and satisfies $\pi_\sigma(\iota(x) + \iota(\sigma \cdot x)) = 0$ for all $\sigma \in T$ and all $x \in S$.*

Call a triad "realizable" if it has at least one realization. The lines λ_σ are the "ghost symmetry axes" and one says that $\iota(S)$ has "ghost symmetry" due to the fact that each of the images $\pi_\sigma(S)$ has a linear symmetry whose action coincides with the permutation σ. The space of realizations of a fixed triad T carries an action by the group $GL(2, \mathbb{R})$, induced by the usual action on $\mathbb{R}^2$.

One may associate a particular set of lines in $\mathbb{R}^2$ with a realization (ι, λ) as follows. Suppose $\sigma \in T$ and $x \in S$. Then there is a unique line $L_{x,\sigma}$ containing $\iota(x)$ which is perpendicular to $\lambda(\sigma)$. The fact that π_σ is isonumerous with respect to $\iota(S)$ ensures that there are precisely $|S|$ lines for each $\sigma \in T$. Since λ is injective, there are precisely three distinct such sets of lines. If $\sigma \in T$, $x \in S$, and there is little danger of confusion, then it is convenient to refer to this line $L_{x,\sigma}$ as the "line colored σ passing through x".

The main result may be stated:

Theorem 2.4. *Suppose S is a set, $|S|$ is even, and T is a triad of hyperinvolutions on S. Then the following are equivalent.*
(a) T is realizable.
(b) T is a direct sum of elementary triads.
(c) T does not have a dibridge.

The equivalence of (b) and (c) follows from the graph theory results described above. The implication (a) $\implies$ (b) also follows quickly, although some geometry comes into play. The essential fact is that a triad which has a dibridge has a subgraph which realizability forbids. As one might expect, the proof of the implication (c) $\implies$ (a) is more involved. The proof is by is induction on the cardinality on $|S|$.

As in the inductive proofs of Steinitz's theorem, the work involves establishing some allowable reductions and transformations. There is one basic type of reduction, which can be described as "amputation along a tricolor bridge". One also needs a transformation which creates tricolor bridges without destroying realizablity. This is the "H-to-I transformation". The first of these preserve realizability and the number of dibridges, while the second preserves realizability and the number of dibridges provided the triad does not have a tribridge. The H-to-I transformation allows one to convert a triad with large girth to one with smaller girth. Thus, given an arbitrary triad, one applies as many H-to-I transformations as necessary until one has a triad which contains a non-trivial tricolor bridge. Then one amputates to obtain smaller triads which also lack dibridges.

2.3 Triads with dibridges

The proof that a triad which has a dibridge is not realizable depends on the fact that every elementary triad is realizable. First, one may say:

Proposition 2.5. *Suppose T_1 and T_2 are realizable triads on S_1 and S_2. Then the direct sum $T_1 \oplus T_2$ is realizable.*

Proof. Suppose (ι_1, λ_1) and (ι_2, λ_2) are realizations of T_1 and T_2 respectively. Due to the action of $GL(2, \mathbb{R})$, one may assume that $\lambda_1 = \lambda_2 = \lambda$. Next, one may dilate ι_2 by some factor k so that none of the colored lines of ι_1 and $k\iota_2$ coincide. Define $\iota : S_1 \cup S_2$ by $\iota(x) = \iota_1(x)$ if $x \in S_1$ and $\iota(x) = k\iota_2(x)$ if $x \in S_2$. Then (ι, λ) is a realization of $T_1 \oplus T_2$. $\square$

Call a finite subset $C \subset \mathbb{R}^2$ "balanced" if $\sum_{v \in C} v = 0$. If S is a set, $\iota : S \to \mathbb{R}^2$ is a function, $\lambda \in \mathbb{R}P^1$, and π_λ is the orthogonal projection onto λ, then say $\iota(S)$ is "balanced" with respect to λ if $\sum_{x \in S} \pi_\lambda(x) = 0$. By linear independence, $\iota(S)$ is balanced iff $\iota(S)$ is balanced with respect to two different subspaces $\lambda_1, \lambda_2 \in \mathbb{R}P^1$ iff $\iota(S)$ is balance with respect to every subspace of $\mathbb{R}^2$.

Proposition 2.6. *Suppose T_1 and T_2 are realizable triads on S_1 and S_2, $(1,3)(2,4)$ is a red pair of $T_1 \oplus T_2$, $(1,3)$ is an edge of T_1, $(2,4)$ is an edge of T_2, and T is obtained from $T_1 \oplus T_2$ by performing a box switch along $(1,3)(2,4)$. Then T is not realizable.*

Proof. Assume that the box switch is obtained by replacing $(1,3)(2,4)$ in $T_1 \oplus T_2$ by $(1,2)(3,4)$. Assume to the contrary that (ι, λ) is a realization of T. By linear independence, (ι, λ) restricts to realizations of T_1 and T_2. Let r_1, r_2, r_3, and r_4 respectively denote the red lines passing through 1, 2, 3, and 4. Then $r_3 = -r_1$ and $r_4 = -r_2$ because $(\iota|_{S_1}, \lambda|_{T_1})$ and $(\iota|_{S_2}, \lambda|_{T_2})$ are realizations. However, one also has $r_2 = -r_1$ and $r_4 = -r_3$ because (ι, λ) is a realization of T. This implies that $r_4 = -r_3 = -(-r_1) = r_1$. This contradicts the fact that π_r is isonumerous with respect to $\iota(S_1 \cup S_2)$. $\square$

Next, using Tutte's results, one has:

Proposition 2.7. *Suppose a triad T has a dibridge. Then there is a box switch and a decomposition $T \to T_1 \oplus T_2$ where T_1 is an elementary triad.*

These propositions establish the fact that if T_1 and T_2 are realizable triads, then any box switch which joins them along a dibridge is not realizable. A corollary to this proposition says that a subgraph of a triad obtained by deleting an edge of a realizable triad is forbidden as a subgraph of a realizable triad. Thus, the only thing remaining to prove is that every elementary triad is realizable.

2.4 Cycles

As in several proofs of Steinitz's theorem, the ability to perturb plays a role. Similarly, one of the basic strategies in the proof of the current theorem is to exploit the fact that for each cycle in the graph representing a realizable triad, there is a one-parameter family of perturbations whose generic member is also a realization. As one shall see, the property of 3-connectedness provides a sufficient abundance of cycles about which one may perturb.

A cycle of a triad T is represented by a sequence $(x_0, x_1, ..., x_l = x_0) \subset S$ of some length l such that each pair (x_{i-1}, x_i) is an edge of T (regarding subscripts modulo l). It is assumed that every cycle is simple, meaning that there are no repetitions among the vertices $\{x_1, ..., x_l\}$.

A perturbation along $\gamma = (x_0, x_1, x_2, ..., x_l = x_0)$ has the effect of moving all the points in $\{x_1, x_2, ..., x_l\}$ while leaving all other points fixed. The construction is as follows.

Assume $T = \{b, g, r\}$, (ι, λ) is a realization of T, and for each $\sigma \in T$, choose non-zero vectors $v_\sigma \in \lambda_\sigma^\perp = \ker(\pi_\sigma)$ such that $v_b + v_g + v_r = 0$. Such vectors exist because λ is injective. Choose a parameter $\epsilon \in \mathbb{R}$. Denote the color of the edge (x_{i-1}, x_i) by σ_i for $i = 1, 2, ..., l$. Since each vertex has exactly one edge of each color, one has $\sigma_i \neq \sigma_{i+1}$ for all i. Define τ_i as the unique third color at vertex x_i which is not equal to either σ_i or σ_{i+1}. Next, assign $s : \{1, 2, 3, ..., l\} \to \{-1, 1\}$ as follows. First let $s_1 = 1$. Then, assuming s_i has been defined, let

$$s_{i+1} = \begin{cases} -s_i & \text{if} \quad \tau_{i+1} = \tau_i, \\ s_i & \text{if} \quad \tau_{i+1} \neq \tau_i. \end{cases}$$

One may quickly check that s_i is well-defined. Next, let $\iota_\epsilon(x) = \iota(x)$ for all $x \notin \{x_1, x_2, ..., x_l\}$ and

$$\iota_\epsilon(x_i) = \iota(x_i) + s_i \epsilon v_{\tau_i}$$

for all $i \in \{1, 2, 3, ..., l\}$. Call ι_ϵ the "ϵ-perturbation" of ι along γ.

By construction, a perturbation of a realization has the property that $\pi_\sigma(\iota_\epsilon(x) + \iota_\epsilon(\sigma \cdot x)) = 0$ for all $x \in S$ and all $\sigma \in T$. However, the projections π_σ may fail to be isonumerous with respect to $\iota_\epsilon(S)$, so a perturbation is not always expected to yield a realization. Nevertheless, only finitely many perturbations of a given realization fail to be realizations, so the generic perturbation of a realization is still a realization. In particular, one has:

Lemma 2.8. *Suppose T is a triad of hyperinvolutions on S, (ι, λ) is a realization of T, and γ is a cycle with vertices $(x_0, x_1, ..., x_l = x_0)$. Then there exists $\delta > 0$ such that $(\iota_\epsilon, \lambda)$ is a realization of T for all $|\epsilon| < 0$.*

2.5 Tricolor Bridges

Definition 2.9. *Suppose $T = \{r, g, b\}$ is a triad on S. A "tricolor bridge" of T is a triple $\{x, y, z\}$ of transpositions such that (a) $[b, x] = [g, y] = [r, z] = \mathrm{id}$ and and (b) there is a partition $S = S_1 \coprod S_2$ such that $|S_1 \cap (b \cdot S_2)| = |S_1 \cap (g \cdot S_2)| = |S_1 \cap (r \cdot S_2)| = 1$.*

Note that condition (b) is equivalent to $|(b \cdot S_1) \cap S_2| = |(g \cdot S_1) \cap S_2| = |(r \cdot S_1) \cap S_2| = 1$ In graph-theoretic terms, $\{x, y, z\}$ is a tricolor bridge between S_1 and S_2 when every path from T_1 to T_2 must use one of the edges $\{x, y, z\}$. If T is a triad on S and $x \in S$, then $\{x\}$ is joined to $S \backslash \{x\}$ via a trivial tricolor bridge. Use the special term "tribridge" for a tricolor bridge for which both S_1 and S_2 contain at least 3 elements.

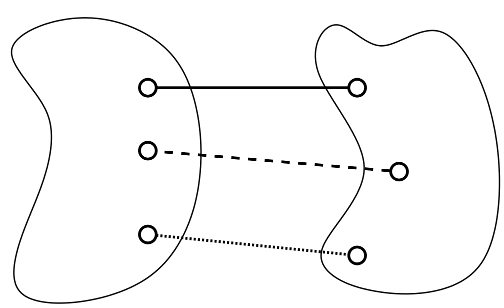

Figure 5: A Tricolor Bridge

One may use a tricolor bridge of a triad to manufacture a triad having more components. Suppose T is a triad on S and $\{x, y, z\}$ is a tricolor bridge of T. Let S_1 and S_2 be as in the definition, and choose points p, q not lying in S. Assume the edges are given as $x = (1, 4)$,

$y = (2, 5)$, and $z = (3, 6)$. Define a triads T_1 and T_2, respectively, on $S_1 \cup \{p\}$ and $S_2 \cup \{q\}$ by replacing x, y, and z by $(1, p)(4, q)$, $(2, p)(q, 5)$, and $(3, p)(6, q)$, respectively. This procedure thus creates a new triad T' on $S \cup \{p, q\}$. Moreover, the triad restricts to triads on $S_1 \cup \{p\}$ and $S_2 \cup \{q\}$, so T' has one more component than T. Call this "amputation" along the tricolor bridge $\{x, y, z\}$.

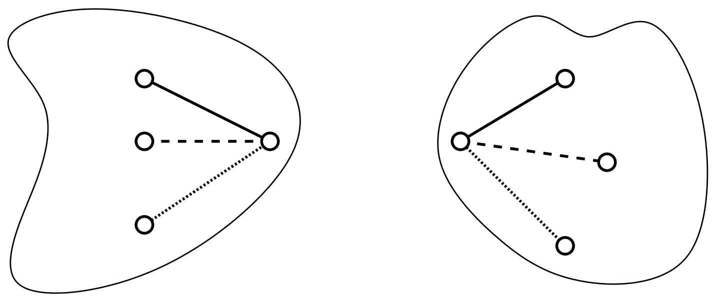

Figure 6: The Result of Amputation along a Tricolor Bridge

One may reverse this procedure. Suppose T_1 and T_2 are triads on S_1 and S_2 respectively and $p \in S_1$ and $q \in S_2$. Then there are elements, say $1, 2, 3 \in S_1$ and $4, 5, 6 \in S_2$ such that $(1, p)$ is an edge of r, $(2, p)$ is an edge of g, $(3, p)$ is an edge of b, etc. One may form a new triad T' containing a tricolor bridge by replacing $(1, p)(4, q)$ by $(1, 4)$, etc. As above, the result of this procedure instead reduces the number of components and the number of vertices. Following [3], call this composition a "star product".

It is fairly clear that these operations preserve the number of dibridges. It is equivalent to showing T is 3-connected iff T_1 and T_2 are both 3-connected. One also has:

Proposition 2.10. *(a) If T is a realizable triad and T_1 and T_2 are triads obtained by amputating T along a tricolor bridge, then T_1 and T_2 are realizable. (b) Suppose T_1 and T_2 are realizable triads on S_1 and S_2, $p \in S_1$, and $q \in S_2$. Then the star product of T_1 and T_2 along $\{p, q\}$ is realizable.*

Proof. (a) Suppose $S = \{1, 2, 3, ...\}$ has a partition $S = S_1 \cup S_2$ and $T = \{b, g, r\}$ is a triad on S, (ι, λ) is a realization of T, $S_1 \cap (b \cdot S_2) = \{1\}$, $S_1 \cap (g \cdot S_2) = \{2\}$, and $S_1 \cap (r \cdot S_2) = \{3\}$. Arguing as follows, one may show that the blue line through $\iota(1)$, the green line through $\iota(2)$, and the red line through $\iota(3)$ are concurrent. Let $v = b_1 \cap g_2$. One must show v lies on r_3. Since v is on b_1 and on g_2, one has $\pi_b(\iota(b \cdot 1)) + \pi_b(v) = 0$ and $\pi_g(\iota(g \cdot 2)) + \pi_b(v) = 0$. However, $S_1 \cap (b \cdot S_2) = \{1\}$ and $S_1 \cap (g \cdot S_2) = \{2\}$. Also, $\pi_b(\iota(x) + \iota(b \cdot x)) = 0$ and $\pi_g(\iota(x) + \iota(g \cdot x) = 0$ for all $x \in S_2 \subset S$. These imply

$$\pi_b(v) + \sum_{x \in S_2} \pi_b(\iota(x)) = \pi_g(v) + \sum_{x \in S_2} \pi_g(\iota(x)) = 0.$$

Thus, the configuration $\{v\} \cup \iota(S_2)$ is balanced with respect to both blue and green. Since λ is injective, the configuration is balanced with respect to red as well. In symbols, this means

$$\pi_r(v) + \sum_{x \in S_2} \pi_r(\iota(x)) = 0.$$

However, One also that $S_1 \cap (r \cdot S_2) = \{3\}$, so

$$\pi_r(\iota(3)) + \sum_{x \in S_2} \pi_r(\iota(x)) = 0.$$

These imply $\pi_r(v) = \pi(\iota(3))$, so v lies on the red line r_3.

(b) Choose realizations (ι_1, λ), (ι_2, λ) of T_1, T_2 respectively. Let $S = S_1 \cup S_2 \backslash \{p, q\}$. Using a dilation, one may assume without loss of generality that $\iota_1(p) + \iota_2(q) = 0$. Define a map $\iota : S \to \mathbb{R}^2$ by

$$\iota(x) = \begin{cases} \iota_1(x) & \text{if} \quad x \in S_1, \\ \iota_2(x) & \text{if} \quad x \in S_2. \end{cases}$$

Choose $x \in S$ and $\sigma \in T$. Then $\pi_\sigma(\iota(x) + \iota(\sigma \cdot x)) = 0$ if $(x, \sigma \cdot x)$ is an edge of either T_1 or T_2. One must also check that this condition holds for the three new edges in T which are constructed when joining T_1 and T_2. Each of these edges has the form $(\sigma \cdot p, \sigma \cdot q)$ for some $\sigma \in T$. Since (ι_1, λ) and (ι_2, λ) are realizations, one has

$$\pi_\sigma(\iota_1(p) + \iota_1(\sigma \cdot p)) = \pi_\sigma(\iota_2(q) + \iota_2(\sigma \cdot q)) = 0.$$

Adding these and using the above assumption, one obtains

$$\pi_\sigma(\iota(\sigma \cdot p) + \iota(\sigma \cdot q)) = \pi_\sigma(\iota_1(\sigma \cdot p) + \iota_2(\sigma \cdot q)) = 0.$$

One still must verify that ι can perturbed in such a way to guarantee that all three projection π_σ be isonumerous with respect to $\iota(S)$. Suppose there is a degeneracy. This means that there are $x, y \in S$ and $\sigma \in T$ such that $\pi_\sigma(x) = \pi_\sigma(y)$. Since (ι_1, λ) and (ι_2, λ) are realizations, the points x and y cannot lie in the same component S_1 and S_2. Thus, one may assume $x \in S_1 \backslash \{p\}$ and $y \in T_2 \backslash \{q\}$. Also, without loss of generality, one may assume $\sigma = r$. Recall that T_2 is 3-connected. Hence there is a cycle in T_2 of the form $\gamma = (r \cdot y, y, b \cdot y, ..., r \cdot y)$ which does not pass through q. This induces an identical cycle in the star product T which never enters T_1. Hence the cycle γ does not pass through $x \in T_1$. Any non-trivial ϵ-perturbation along γ eliminates this degeneracy. Since S is finite, one may eliminate every such degeneracy. $\square$

If a triad T contains a triangle, then it has a tribridge. In this case, amputation along the tricolor bridge corresponds to a Δ-to-Y transformation as described in [9].

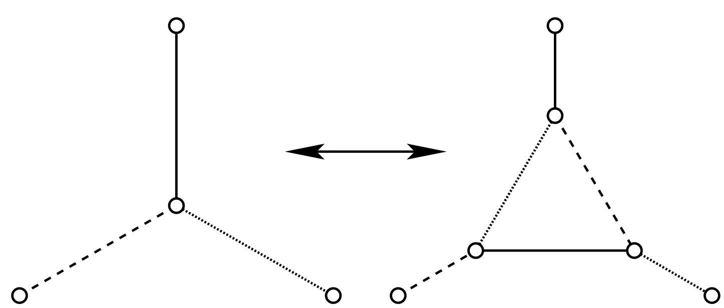

Figure 7: The Y-to-Δ transformation

2.6 The H-to-I Transformation

Suppose T is a triad on S, $\{1, 2, 3, 4, 5, 6\}$ is a subset of S, $(5, 6)$ is green, $(1, 5)(4, 6)$ is red, and $(2, 5)(3, 6)$ is blue. Form a new triad as follows. First let $5'$ and $6'$ denote points not lying in S. Let $S' = \{5', 6'\} \cup (S \backslash \{5, 6\})$. The triad T' on S' is obtained by replacing

$$\begin{array}{rcl} (5,6) & \longrightarrow & (5',6'), \\ (1,5)(4,6) & \longrightarrow & (1,5')(4,6'), \\ (2,5)(3,6) & \longrightarrow & (2,6')(3,5') \end{array}$$

for each of the respective elements of T. The ability to perform this transformation relies on S having at least 6 points. In the graph corresponding to T, this transformation appears to replace a figure resembling a capital letter H with a figure resembling a capital letter I (with serifs) or vice-versa. Thus, call this an "H-to-I" transformation along the edge $(5, 6)$.

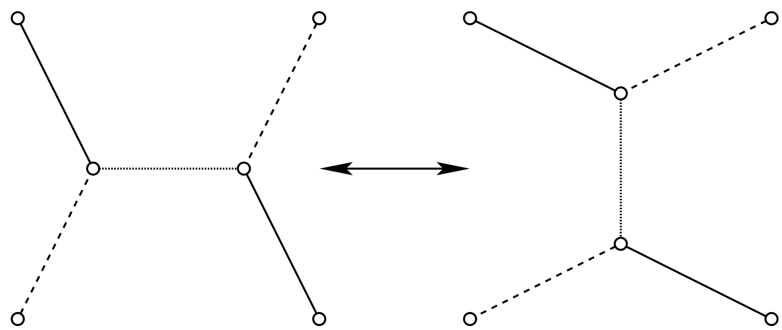

Figure 8: The H-to-I transformation

One may regard an H-to-I transformation as a "local" box switch. Indeed, an H-to-I transformation is a special type of box switch, applied to a monochrome pair of edges, two of whose vertices are joined by an edge of another color. Thus, if one regards $5' = 5$ and $6' = 6$ in the construction above, then this transformation amounts to replacing the blue pair $(2, 5)(3, 6)$ with $(2, 6)(3, 5)$. Also, this box switch is indistinguishable from that which replaces the red pair $(1, 5)(4, 6)$ with $(1, 6)(4, 5)$. In general, write $T \sim T'$ if T and T' are triads and T' is obtained from T by performing an H-to-I transformation.

Proposition 2.11. *If T is a triad on S, $T \sim T'$, T has no dibridges, and T' has a dibridge, then T has a tribridge.*

Proof. Assume as above that $\{1, 2, 3, 4, 5, 6\} \subset S$, $(5, 6)$ is green, $(1, 5)(4, 6)$ is red, and $(2, 5)(3, 6)$ is blue. Assume T' is obtained by replacing

$$\begin{array}{rcl} (5,6) & \longrightarrow & (5',6'), \\ (1,5)(4,6) & \longrightarrow & (1,5')(4,6'), \\ (2,5)(3,6) & \longrightarrow & (2,6')(3,5'). \end{array}$$

It is clear that the dibridge of T' is not blue or red. Hence there is green dibridge of the form $(5', 6')(7, 8)$. Assume that the box switch that results in more components is obtained by replacing $(5', 6')(7, 8)$ with $(5', 8)(6', 7)$. Thus, $\{1, 3, 5', 7\}$ lie in one of the components and $\{2, 4, 6', 8\}$ lie in the other component after this box switch. Let T_1 be the former of these components, with vertex set $S_1 \subset S$ and let T_2 be the latter, with vertices S_2. Next, consider the triad T. The edge $(2, 5)$ is blue, the edge $(4, 6)$ is red, and the edge $(7, 8)$ is green. Moreover, $S_1 \cap (b \cdot S_2) = \{5\}$, $S_1 \cap (r \cdot S_2) = \{6\}$, and $S_1 \cap (g \cdot S_2) = \{7\}$. This is therefore a tricolor bridge. It remains to show that this bridge is non-trivial. In order to see this, one must consider two cases. First, it is possible that 2 and 4 denote the same element of S. In this case, the subset $\{2 = 4, 5, 6\}$ is connected to the rest of S via a non-tricolor bridge. In the latter case, with $2 \neq 4$, the result follows because both $|S_1| \geq 3$ and $|S_2| \geq 3$. □

If T is a triad on at least 6 points and T has a dibridge, then T has a tribridge. Thus, the preceding proposition has the corollary that if T has no tribridges and $T \sim T'$, then T' cannot have a dibridge.

Proposition 2.12. *If T is a realizable triad, $T \sim T'$, and T' has no dibridges, then T' is realizable.*

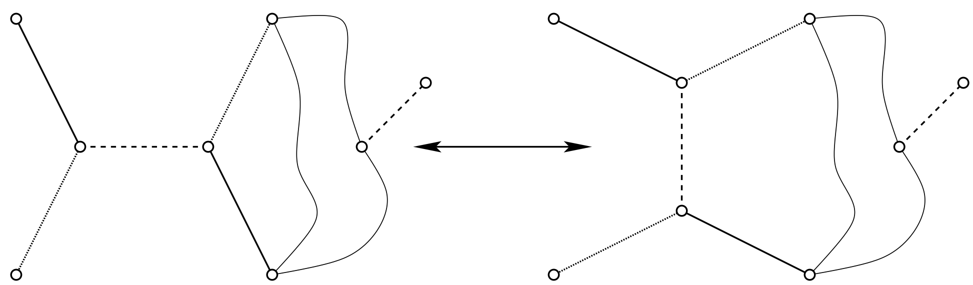

Figure 9: A Guide to the Preceding Proof

Proof. Assume as above that $\{1,2,3,4,5,6\} \subset S$, $(5,6)$ is green, $(1,5)(4,6)$ is red, and $(2,5)(3,6)$ is blue. Let $S' = \{5', 6'\} \cup S \backslash \{5,6\}$ and suppose the triad T' on S' is obtained by replacing

$$\begin{array}{rcl} (5,6) & \longrightarrow & (5',6'), \\ (1,5)(4,6) & \longrightarrow & (1,5')(4,6'), \\ (2,5)(3,6) & \longrightarrow & (2,6')(3,5'). \end{array}$$

Suppose (ι, λ) is a realization of T. For each color σ and each $x \in S$, let σ_x be the line colored σ which passes through $\iota(x)$. Define a map $\iota' : S' \to \mathbb{R}^2$ as follows. First, let $\iota'(5')$ be the intersection of lines r_5 and b_6 and let $\iota'(6')$ be the intersection of lines r_6 and b_5. Next, if $x \in S \backslash \{5,6\}$, define $\iota'(x) = \iota(x)$. Let $g_{5'}$ and $g_{6'}$ be the green lines through $\iota'(5')$ and $\iota'(6')$.

The pair (ι', λ) is a likely candidate for a realization of T'. Indeed, since (ι, λ) is a realization of T, one also has that $\pi_\sigma(\iota'(x) + \iota'(\sigma x)) = 0$ for each $\sigma \in T$. Also, r_5 contains only one point of $\iota(S)$, namely $\iota(5)$, and similarly for r_6. This shows that $\iota' : S' \to \mathbb{R}^2$ is injective and the projections π_r and π_b are isonumerous with respect to $\iota'(S')$. However, a degeneracy might arise as one might have, for example, $g_{5'} = g_x$ for some $x \in S'$. This would be a degeneracy because in this case π_g would not not isonumerous. The rest of this proof is devoted showing that one may avoid this degeneracy by using a perturbation along some cycle.

First, suppose $g_{5'} = g_{6'}$. This is equivalent to having $\pi_g(\iota'(5) - \iota'(6)) = 0$. It is desired to perturb the initial realization so that $\pi_g(\iota'(5) - \iota'(6)) \neq 0$. Consider perturbing ι along a cycle $\gamma = (1, 5, 2, ..., 1)$ which does not pass through the vertex 6. Such a cycle exists because T is 3-connected. The cycle γ contains the path $(1,5,2)$, and the edges $(1,5)$ and $(5,2)$ are red and blue. Thus, the image of 5 under any perturbation along γ also lies on g_5. However, such a perturbation does not alter the location of the image of 6. Thus, one may choose the initial realization so as to avoid the first type of degeneracy.

Next, consider the latter case. Without loss of generality, one may assume that $g_{5'} = g_x$ for some $x \in S$. Since T' is 3-connected, there is a cycle in T' of the form $\gamma = (3, 5', 6', ..., 3)$ which does not include the vertex x. Thus, one may purturb ι' along the cycle γ without perturbing $\iota(x)$. This therefore avoids the second type of degeneracy. □

2.7 Putting the Pieces Together

At this point, all of the requisite pieces of the induction are in place. Here is a summary.

Suppose k is a positive integer with $k \geq 3$ and assume that a triad on a set with cardinality less than $2k$ is realizable whenever it has no dibridges. Let S be a set with $|S| = 2k$ and suppose T is a triad on S which has no dibridges. If T has a tribridge, then one may use amputation and the induction hypothesis to conclude that T is realizable.

On the other hand, suppose T has no tribridges. Choose a cycle $\gamma = (x_0, x_1, x_2, ..., x_l = x_0)$ in T which has edges of all three colors. Since T has no tribridges, one has $l \geq 4$. Thus, γ has an edge, say (x_i, x_{i+1}) such that the path $(x_{i-1}, x_i, x_{i+1}, x_{i+2})$ uses all three colors and at least one other edge of γ has the same color as (x_i, x_{i+1}). Perform an H-to-I transformation along (x_i, x_{i+1}) to obtain a triad $T' \sim T$.

One is faced with two possibilities for T'. First, if T' has a tribridge, then T' is realizable and, from above, so is T. On the other hand, T' may not have a tribridge. Since $(x_{i-1}, x_i, x_{i+1}, x_{i+2})$ employs all three colors, the triad T' contains the cycle $\gamma' = (x_0, x_1, ..., x_i, x_{i+2}, ..., x_l = x_0)$. Moreover, γ' employs all three colors, so one may apply another H-to-I transformation along an edge of γ' which has the same properties as x_i, x_{i+1} above. For triads which have no tricolor bridges, this does not destroy realizablity. Notice, in particular, that γ' is shorter than γ. Thus, one may apply a sequence of H-to-I transformations, each time reducing by 1 the lengths of the focused cycles γ', γ'', γ''', etc. Eventually, one will have a triad which has a triangle. If the triad has a triangle, then it has a tribridge and induction again takes over.

3 The Odd Case

For brevity, call a triad "odd" if its underlying set has odd cardinality and "even" if its underlying set has even cardinality. The purpose of this section is to formulate an analogous criterion for prescribability of odd triads. As it turns out, a realization of an odd triad arises as a degenerate realization for a corresponding uniquely-determined even triad.

The statement of the main theorem requires slightly more general notions of box-switch and dibridge. Suppose T is an odd triad on S and $\sigma \in T$. Then σ has a unique fixed point, say c. Suppose that (a, b) is a factor in the cyclic decomposition of σ. Then one can construct two new triads defined by

$$T_1 = (T \backslash \{\sigma\}) \cup \{(a, b, c) \circ \sigma\} \text{ and } T_2 = (T \backslash \{\sigma\}) \cup \{(a, c, b) \circ \sigma\},$$

where (a, b, c) denotes the 3-cycle which maps $a \mapsto b \mapsto c$. Call these "generalized box switches". If one obtains a disconnected graph by applying a generalized box switch, then say that T has a "generalized dibridge". Regard the usual box switches and dibridges defined earlier also as generalized box switches and generalized dibridges.

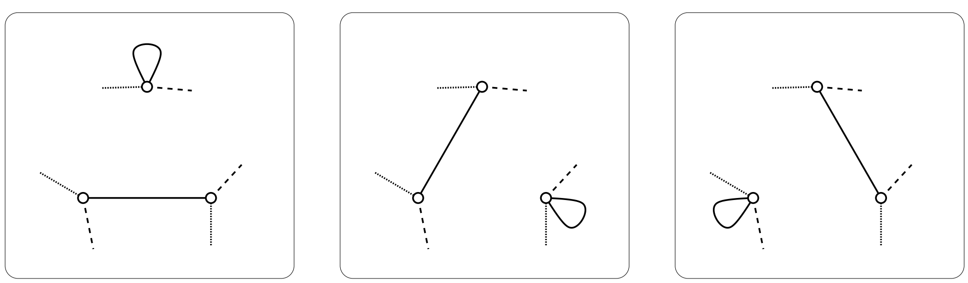

Figure 10: Generalized Box-Switches on a Triple

Theorem 3.1. *The following are equivalent for an odd triad T.*
(a) T is realizable.
(b) T is a direct sum of elementary even triads and exactly one elementary odd triad.
(c) T does not have a generalized dibridge.

As before, that statements (b) and (c) are equivalent is clear. Hence, the aim is to show the equivalence of (a) and (c). Most of the work has already been done, thanks to the proof of the main theorem given in the preceding section.

There are a couple transformations which allow one to pass between from an odd triad to an even triad and vice-versa. Suppose $T = \{b, g, r\}$ is an odd triad on S and $w \notin S$. Suppose that

x is the fixed point of b, y is the fixed point of g, and z is the fixed point of r. Let $b' = b \circ (w, x)$, $g' = g \circ (w, y)$, and $r' = r \circ (w, z)$. Then the set $T' = \{b', g', r'\}$ is a uniquely-determined triad of hyperinvolutions on the set $S' = S \cup \{w\}$. Call T' the "completion" of T. "Cauterization" is the reverse of this transformation. Thus, to cauterize an even triad, one selects a vertex w, deletes w, and replaces the edges connecting to it by loops at the vertices $b \cdot w, g \cdot w, r \cdot w$ lying at the other ends of those edges.

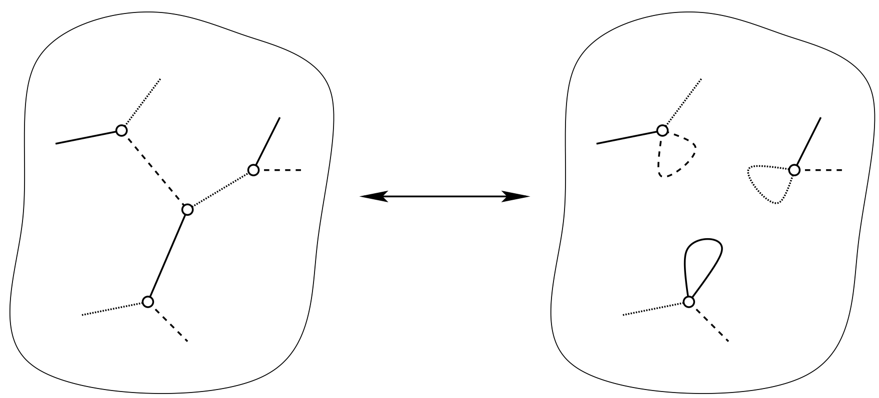

Figure 11: Cauterization and Completion

It is notable that cauterization does not preserve 3-connectedness. However, when the number of vertices is odd, 3-connectedness is not equivalent to a lack of generalized dibridges. Nevertheless, one may quickly show:

Proposition 3.2. *(a) If T' is the completion of T and T has no generalized dibridges, then T' has no generalized dibridges. (b) If T' is a cauterization of T and T has no generalized dibridges, then T' has no generalized dibridges.*

Using this, one has:

Proposition 3.3. *If T is an odd triad which has no generalized dibridges, then T is realizable.*

Proof. Suppose T' is the completion of T. Then T' has no dibridges and therefore T' is realizable. Let (ι', λ) be a realization of T' and assume T' is obtained by adjoining the vertex w. The idea is to perturb ι' to a map ι in such a way that $\iota(w) = 0$ and (ι, λ) restricts to a realization of T. That this is possible may be argued thus:

Since T' is realizable, it is 3-connected. Hence, there are paths $\alpha = (b \cdot w, ..., g \cdot w)$, $\beta = (r \cdot w, ..., b \cdot w)$, and $\gamma = (g \cdot w, ..., r \cdot w)$ no two of which share vertices in common and none of which passes through w. Let (w, α, w) denote the cycle which starts at w follows α and terminates at w, and similarly for (w, β, w) and (w, γ, w). First, there is a unique perturbation $\iota' \to \iota''$ along the cycle (w, α, w) in such a way that $\pi_g(\iota''(w)) = 0$. Next, there is a unique perturbation $\iota'' \to \iota'''$ along the cycle (w, β, w) in such a way that $\pi_r(\iota'''(w)) = 0$. By construction, one has that $\iota'''(w) = 0$.

The pair (ι''', λ) is a reasonable candidate for the realization of T. If each projection π_σ is isonumerous with respect to $\iota'''(S)$, then one may let $\iota = \iota'''$ and the proof is complete. On the other hand, suppose there exist $\sigma \in T$ and vertices $x, y \in S$ such that $\pi_\sigma(x) = \pi_\sigma(y)$. One may then show there is a perturbation $\iota''' \to \iota$ which eliminates this degeneracy. One performs this perturbation along the cycle (α, β, γ) so as to retain $\iota'''(w) = \iota(w) = 0$.

One of the points, say x, must lie on the path (β, α). Assume without loss of generality that the colors of the edges along the cycle (w, β, α, w) having x as a vertex are blue and green, so the color at x which is not part of the cycle is red. Let r_x be the red line passing through $\iota'''(x)$.

Then r_x coincides with the red line passing through $\iota(x)$. The other point y then either lies on this path or it doesn't, and one may handle these cases separately.

Suppose first that y does not lie on the path (β, α). In this case, σ cannot be red, for otherwise ι' would not have been a realization to begin with. Thus, σ is either blue or green. In either case, every non-trivial perturbation $\iota''' \to \iota$ along (α, β, γ) satisfies $\pi_\sigma(\iota(x)) \neq \pi_\sigma(\iota(y))$. Suppose instead that y lies on the path (β, α). Then, as above, any non-trivial perturbation $\iota''' \to \iota$ along (α, β, γ) satisfies $\pi_\sigma(\iota(x)) \neq \pi_\sigma(\iota(y))$.

Finally, since S is finite, one may eliminate all the degneracies in turn and obtain a realization of T. $\square$

Proposition 3.4. *If T is an odd triad which has a generalized dibridge, then T is not realizable.*

Proof. This follows from Tutte's decomposition theorems. If T has a generalized dibridge, then there is a generalized box-switch decomposition $T \to T_1 \oplus T_2$ such that T_1 is elementary. Thus, T has a subgraph which agrees with T_1 except for one loop or edge. This is a forbidden subgraph, so T is not realizable. $\square$

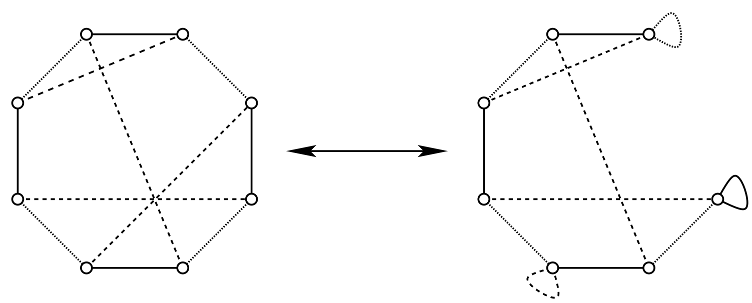

Figure 12: Cauterization Destroys 3-Connectedness

4 Conclusion

The fact that a realization may be perturbed along a cycle was instrumental in the proof of the main theorem. By induction on the number of points in a configuration, one may also say:

Proposition 4.1. *Suppose $k \geq 2$. (a) The realization space of a connected realizable triad on $2k$ points has dimension k. (b) The realization space of a connected realizable triad on $2k-1$ points has dimension $k-2$.*

This is closely related to the fundamental group of the graph underlying the triad. It is also apparent that the realization space of a realizable triad is an elementary semialgebraic set. (See [5] for definitions.) However, few other properties of the realization spaces are known. For example, it has not been shown that they are contractible or even connected.

In the proof of the main theorem, one might try to rephrase the reduction argument in terms of blow-ups/blow-downs of general cycles instead of triangles. Thus, in the argument used, if the triad T has no tricolor bridges, then one uses a sequence of H-to-I transformations in order to arrive at a triad which contains a tricolor bridge. One may instead think of this as a single step, where a cycle of length l is replaced by a tree having $l-2$ vertices. The proof that an odd triad is realizable iff it has no generalized dibridges relied on the even case. It is reasonable to expect that there is a proof which is analogous to the proof of the even case. It would be interesting to see a non-constructive proof of the main theorem.

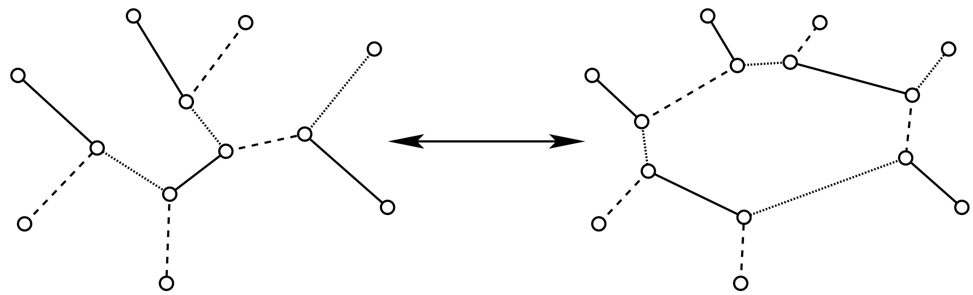

Figure 13: Blowing Up and Down along a Tree

It is re-emphasized here that the purpose of this study is to lay a foundation for configurations which have more than three ghost symmetries. Clearly the definition of “realization” allows that one may generalize realizability to sets having more than three hyperinvolutions.

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