

The Use of Ratios in the Player Piano Studies of Conlon Nancarrow

Julie Scrivener
1721 Sunnyside Drive
Kalamazoo, MI 49048, U.S.A.
E-mail: scrivener_julie@hotmail.com

Abstract

The forty-nine Studies for player piano by Conlon Nancarrow are pervaded by the use of mathematical ratios, particularly ratios based on intervals found in the justly-tuned scale. Nancarrow found a remarkable variety of ways in which to deploy ratios in his Studies. The most obvious and pervasive use is in the establishment of different simultaneous tempos related by ratios as simple as 3:4 to as complex as e/π . Ratios were also used to establish relationships between pitch materials and other details, and in certain structural features. This paper is an introductory survey of the use of ratios in these Studies.

Conlon Nancarrow (1912–1997) was a remarkable musical pioneer of the twentieth century who, while working in virtual isolation in Mexico, turned to the player piano as a means to realize complex rhythmic and metric structures that were unplayable by human performers. His often transparent compositional processes are heavily influenced by the use of ratios, and he deployed ratios in his compositions in remarkably inventive ways to control many of his musical materials.

Over a span of almost 50 years, Nancarrow wrote 49 numbered “Studies” for the player piano. He used ratios in several different ways in these pieces, and many of his ideas—particularly the earlier ones—are obviously influenced by the ideas of Henry Cowell as set forth in *New Musical Resources* of 1930 [3]. It was Cowell who first presented the idea of relating simultaneous tempos to acoustical pitch ratios found in musical intervals and chords. For instance, the interval ratio for the purely-tuned perfect fifth, 3:2, could be represented rhythmically in a passage such as shown in Example 1.



Example 1: Conjectural passage showing a 3:2 rhythmic relationship.

Rhythmic relationships of 3-divisions in one voice against 2-divisions in another are very common in music going all the way back to medieval times. More complex rhythmic ratios, such as 4:3 (corresponding to the musical fourth), 5:4 (the major third), 6:5 (the minor third), and 5:3 (the major sixth) occur in much Romantic piano music such as that of Chopin (see Example 2); however, these uses are primarily decorative and not representative of large-scale polytempo as Cowell envisioned it.

Cowell went much further in proposing a system in which correspondences are established between rhythm and interval ratios. The illustration in Example 3 shows one such relationship based on the second-inversion major triad, in which the ratio between the three chord members is 5:4:3. This experimental use of three different tempo values at one time, none related by powers of 2, was unprecedented in Western music.

Nancarrow was especially inclined to use ratios found in the Fibonacci series 1, 1, 2, 3, 5, 8, 13, ..., with the ratios 3:5, 3:8, and 5:8 being particularly prevalent in his compositions. This is at least partially because it happens that each of these ratios is also representative of a purely-tuned musical interval, with the ratios 3:5 and 5:8 representing the pitch intervals major sixth and minor sixth, respectively, while the

Example 2: Rhythmic ratios in music of Chopin [2]: (a) 4-against-3, “Prelude No. 23 in F major”; (b) 5-against-6, “Prelude No. 13 in F-sharp major.”

Example 3: Diagram from Henry Cowell’s *New Musical Resources* (cited in [4], p. 6).

ratio 3:8 represents the interval of an octave plus a fourth. Gann notes that while Nancarrow denied ever intentionally using the Fibonacci series ([4], p. 260), he seemed attracted to using pairs of numbers that resulted in proportionately similar ratios. (He was also partial to series of prime numbers.) And, of course, in basing his rhythmic ideas so clearly on Cowell’s, which are generated by pitch ratios that often appear naturally in the series, his work inevitably gravitates toward these numbers.

1. Ratios in Nancarrow’s Music

Ratios are used in Nancarrow’s player piano Studies to control parameters such as the following:

1. **Tempo relationships.** Nancarrow wrote almost two dozen “tempo canons” that borrow directly from Cowell’s ideas in exploring various tempo relationships (what Cowell describes as “a harmony of several different rhythms played together”). These works have subtitles such as “Canon 12/15/20,” “Canon 60/61,” and the ambitious “Canon 150/160 $\frac{5}{7}$ /168 $\frac{3}{4}$ /180/187 $\frac{1}{2}$ /200/210/225/240/250/262 $\frac{1}{2}$ /281 $\frac{1}{4}$ ”² in which twelve voices represent the pitches of the justly-tuned chromatic scale. Nancarrow also explored tempo relationships involving irrational numbers, such as “Canon $\sqrt{2}/2$ ” and “Canon e/π .”

2. **Melodic and harmonic materials.** In some cases Nancarrow used ratios to establish relationships between pitch materials and other details.

3. **Structure.** This category would include the use of ratios to determine levels of imitation between voices, positioning of entrances, and other structural concerns on both small and large levels.

¹[4], p. 1.

²It is interesting to note that Nancarrow borrowed this series of proportions directly from Cowell’s *New Musical Resources*.

1.1. Tempo Relationships. Nancarrow began exploring relationships between pitch intervals and different simultaneous tempos in his very first player piano piece, “Rhythm Study No.1,” in which the two tempos—120 and 210—are related by a 4:7 ratio. The ratio 4:7 defines a purely-tuned minor seventh ([4], p. 6).

The ratios given in the subtitles of Nancarrow’s tempo canons relate directly to the tempo markings for the individual voices; the number of components in the ratio indicates the number of canonic voices in the study (it should be noted that a canonic “voice” is not necessarily a monophonic [unharmonized] musical line, but is more likely a polyphonic musical stratum, sometimes quite dense, that is repeated at different pitch levels in successive voices). Table 1 describes most of Nancarrow’s tempo canons and the tempo markings used in the scores.

Table 1: Relationship of Canon Ratios and Tempo Markings

Study	Tempo Markings
No. 14, “Canon 4/5”	$\text{♩} = 88, \text{♩} = 110$
No. 15, “Canon 3/4”	$\text{♩} = 165, \text{♩} = 220$
No. 17, “Canon 12/15/20”	$\text{♩} = 138, \text{♩} = 172.5, \text{♩} = 230$
No. 18, “Canon 3/4”	$\text{♩} = 168, \text{♩} = 224$
No. 19, “Canon 12/15/20”	$\text{♩} = 144, \text{♩} = 180, \text{♩} = 240$
No. 24, “Canon 14/15/16”	$\text{♩} = 149\frac{1}{3}, \text{♩} = 160, \text{♩} = 170\frac{2}{3};$ $\text{♩} = 224, \text{♩} = 240, \text{♩} = 256$
No. 31, “Canon 21/24/25”	$\text{♩} = 105, \text{♩} = 120, \text{♩} = 125$
No. 32, “Canon 5/6/7/8”	$\text{♩} = 85, \text{♩} = 102, \text{♩} = 119, \text{♩} = 136$
No. 33, “Canon $\sqrt{2}/2$ ”	$\text{♩} = 140\sqrt{2}, \text{♩} = 280$
No. 34, “Canon $\frac{9}{4/5/6} / \frac{10}{4/5/6} / \frac{11}{4/5/6}$ ”	the three voices generally exhibit a 9:10:11 tempo ratio—the opening markings are $\text{♩} = 72, \text{♩} = 80,$ and $\text{♩} = 88$; the application of the 4:5:6 ratio results in a lengthy succession of progressively faster tempos in each voice, peaking at $\text{♩} = 264$ in the fastest voice
No. 36, “Canon 17/18/19/20”	$\text{♩} = 85, \text{♩} = 90, \text{♩} = 95, \text{♩} = 100$
No. 37, “Canon $150/160\frac{5}{7}/168\frac{3}{4}/180/187\frac{1}{2}/200/210/225/240/250/262\frac{1}{2}/281\frac{1}{4}$ ”	$\text{♩} = 150, \text{♩} = 160\frac{5}{7}, \text{♩} = 168\frac{3}{4}, \text{♩} = 180,$ $\text{♩} = 187\frac{1}{2}, \text{♩} = 200, \text{♩} = 210, \text{♩} = 225,$ $\text{♩} = 240, \text{♩} = 250, \text{♩} = 262\frac{1}{2}, \text{♩} = 281\frac{1}{4}$
No. 40, “Canon e/π ”	tempo markings not used—Nancarrow notates a simple diagram which gives timings that establish a relationship of <i>roughly</i> e to π
No. 41A, “Canon $\frac{1}{\sqrt{\pi}}/\sqrt{2}/3$ ” No. 41B, “Canon $\frac{1}{3\sqrt{\pi}}/3\sqrt{13}/16$ ”	Nancarrow does not give tempo markings; the canons are intended to be played together, and he establishes the following relationships using total playing times: $\frac{1}{\sqrt{\pi}} = 5'00$ ”; $\sqrt{2}/3 = 7'40$ ”; $\frac{1}{3\sqrt{\pi}} = 4'35$ ”; $3\sqrt{13}/16 = 6'10$ ”
No. 43, “Canon 24/25”	$\text{♩} = 120, \text{♩} = 125$

Nancarrow's stated purpose in writing these canons

was [to explore] my interest in temporally dissonant relationships. Temporal dissonance is as hard to define as tonal dissonance. I certainly would not define a temporal relation of 1 to 2 as dissonant, but I would call a 2 to 3 relation mildly dissonant, and more and more so up to the extreme of the irrational ones.³

It is easy to see in Table 1 where some of these tempo canons reach extremes of temporal dissonance. Study No. 33, with its ratio of $\sqrt{2}/2$, is one such case. The irrational number $\sqrt{2}$ (1.4142136...) approximates the equal-tempered tritone (among the most dissonant and unstable intervals in tonal music), which is commonly given the ratio 7:5. Nancarrow casts attention in this piece on this irrational number by using a variety of ratios that are rational approximations of $\sqrt{2}$. Gann identifies the ratios 7:5 and 10:7 as being used in the final section, where in one voice only note values of five and seven sixteenth notes are used while the other voice consists only of values of seven and ten sixteenth notes.

Study No. 37, with the relative speed of its twelve voices defined by the justly-tuned scale, is one of a few pieces in which Nancarrow uses ratios that clearly refer to pitch ratios. The canons listed in Table 1 contain the most obvious references to use of ratios in establishing tempo relationships, but some intriguing examples can be found in Nancarrow's non-canonic pieces. Study No. 28, a non-canonic piece based on large-scale acceleration, finds the composer again using the ratios of the justly-tuned chromatic scale to establish tempo relationships, this time to establish a "scale" of acceleration. Gann ([4], p. 164) describes how an incremental acceleration in this study from $\text{♩} = 252$ to $\text{♩} = 864$ adheres to a "scalar" progression as shown in Table 2.

Table 2: Scalar Acceleration in Study No. 28

Tempo (♩ =)	Ratio*	Corresponding note	Tempo (♩ =)	Ratio*	Corresponding note
252	7:10	F#	480	4:3	F
270	3:4	G	504	7:5	F#
288	4:5	A \flat	540	3:2	G
300	5:6	A	576	8:5	A \flat
315	7:8	B \flat	600	5:3	A
337.5	15:16	B	630	7:4	B \flat
360*	1:1*	C*	675	15:8	B
384	16:15	C#	720	2:1	C
405	9:8	D	765	17:8	C#
432	6:5	E \flat	810	9:4	D
450	5:4	E	864	12:5	E \flat

*Ratios are in relationship to C, representing 1:1 and the "tonic tempo" at $\text{♩} = 360$

As in the justly-tuned scale, the increments between intervals in this "scale" of tempos are of varying sizes. Gann notes ([4], p. 197) that there are four different interval ratios for adjacent pitches in this scale: 15:14, 16:15, 21:20, and 25:24. They are represented in Study No. 28 as follows: the largest increment, 15:14 (increment of 7.1%), occurs between F#–G and B \flat –B; the ratio 16:15 (6.7%) occurs between G–A \flat , B–C, C–C#, D–E \flat , and E–F; the ratio 21:20 (5%) occurs between A–B \flat and F–F#; and the smallest increment, 25:24 (4.2%) occurs between A \flat –A and E \flat –E.⁴ The resulting overall acceleration ($\text{♩} = 252$ to

³[11], p. 5.

⁴Inexplicably, Nancarrow uses two different ratios for the two occurrences of C#, and neither of them is the 15:14 ratio used by Cowell in *New Musical Resources*. The 16:15 ratio used for the first occurrence of C# is actually what Cowell specifies for D \flat ([3], p. 99). That does not explain, however, why Nancarrow's second occurrence of C#

(♩ = 864) is equivalent to the pitch interval of an octave plus a diminished seventh, or the pitch ratio 24:7 (10:7 [interval from F# to C] + 12:5 [interval from C to E♭ the octave above] = 24:7).

In another application of a pitch ratio to a rhythmic feature, Nancarrow uses the pitch ratio 9:4 (major ninth) in Study No. 23 to determine the relative length of 71 consecutive pitches from B₀ to A₆. For each major ninth higher the pitch progresses in this series, the note duration decreases by half ([1], p. 49). In Example 4 from this Study, pitches are arranged in five-note groupings, with the duration of all notes in each group determined by its first note. Note the progressively longer notes as the first pitch of the group becomes lower.



Example 4: Excerpt from Study No. 23, p. 4. (In this piece, as in many others, Nancarrow uses proportional notation to indicate duration—that is, note duration is directly proportional to the space between notes on the staff.)

1.2. Melodic and Harmonic Materials. There are also instances in Nancarrow’s Studies of apparent uses of pitch ratios to define melodic and harmonic materials. In Study No. 28, which was cited above for its “scale of tempos,” there are eight clusters of three pitches a half-step apart that are repeatedly articulated but in an irregular fashion. Gann claims that these clusters of pitches, which represent a pitch ratio of 15:16:17 (two adjacent half-steps), also exhibit a 15:16:17 speed ratio, with the highest note in each cluster being rearticulated 17:16 as fast as the middle note and the lower note rearticulated 15:16 the speed of the middle note. This is difficult to verify due to the nature of measuring duration in Nancarrow’s score. What Gann has done is to measure in millimeters the distance between rearticulated notes and determine an average. In the cluster shown in Example 5, my own measurements show that the average distance between successive articulations of the note A is about 25.8 mm in the score (distances in Example 5 are reduced). To effect a 15:16:17 ratio with this measurement as the middle figure, the measurements for the faster (upper) voice would need to be about 24.3 mm and the slower (lower) voice about 27.5 mm. The actual measurements are 24.0 mm for the upper voice and 27.5 mm for the lower voice, which is remarkably close to the ratio in question. One must keep in mind that Nancarrow’s handwritten scores represent only a rough approximation of the piano rolls, which are more precise.



Example 5: Three-note cluster from Study No. 28 (p. 3). Some rearticulations of the upper and lower notes are omitted.

Also in Study No. 28, Gann points out that the pitch ratios for the middle note of each of these eight clusters correlate to the “scale of tempos” shown in Table 2, with the highest pitch (D₆) corresponding to the “tonic tempo” of C (360); four of the cluster tempos are lower than 252, the slowest tempo in the piece, with the lowest pitch corresponding to the note G below the scale (tempo=135); see Example 6.

Another application of the 7:5 (tritone) ratio occurs in Study No. 5, where two rhythmic ostinatos are established which are related by that ratio and the opening interval between these two voices is the tritone C–F# (see Example 7). The two lines share a $\frac{35}{16}$ meter signature, but the distance between notes in the top line is multiples of seven sixteenth notes and in the bottom line multiples of five sixteenth notes.

(17:8) is not an octave of the first (and nowhere does Cowell use the lower octave of this ratio, 17:16, for either C# or D♭); the second ratio would have to be 32:15 rather than 17:8 to be an octave duplication of 16:15. As a result, the two intervals C#–D differ: the first C#–D interval is an increment of 5.5%, or the rather unwieldy ratio of 135:128, while the second C#–D interval is a 5.9% increment, ratio 18:17.

Pitch:

Tempo:	360	315	288	252	216	192	160	135
Ratio to "C" Tempo:	1/1	7/8	4/5	7/10	3/5	8/15	4/9	3/8
Tempo-scale pitch:	C	B _♭	A _♭	F#	E _♭	D _♭	B _♭	G

Example 6: Correspondence of middle pitches in eight 3-note clusters and scale of tempos in Study No. 28 ([4], p. 165).

Although the tonal implications of these two lines are actually C major and B pentatonic, the use of the tritone at the very beginning of this rhythmic context is a striking parallel.

brackets indicate ♪ values

35 14 7 14 10 15 5 25

staccato sempre

Example 7: Opening of Study No. 5, showing 7:5 rhythmic ratio and opening tritone interval between voices.

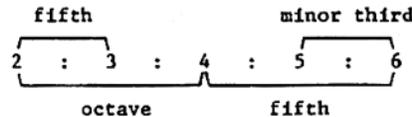
1.3. Structural Use of Ratios. Some of Nancarrow's most remarkable uses of ratios involve musical structure. The use of ratios in structural details occurs on both small and large levels. On the small side, Nancarrow sometimes used ratios in setting up the length of isorhythmic (repeating rhythmic) patterns, as he did in Study No. 7. Here Nancarrow creates his major themes out of three different isorhythms in a 3:4:5 ratio: an 18-beat isorhythm (5+4+2+3+4), a 24-beat isorhythm (5+5+2+4+ 3+2+3), and a palindromic 30-beat isorhythm (3+2+2+3+2+3+3+2+3+2+2+3). The ratio 3:4:5 (representing the second-inversion major triad as shown in Example 3) seems to be a favorite of Nancarrow's. He uses it in numerous other Studies, including No. 9 where the three main themes are related rhythmically by this ratio.

The ratio 4:5:6 has structural significance in Study No. 34, which is subtitled "Canon $\frac{9}{4/5/6} / \frac{10}{4/5/6} / \frac{11}{4/5/6}$." Here the tempos of the three canonic voices are related by 9:10:11 and the ratio 4:5:6 is applied twice to each of these voices to create progressively faster tempos. The ratio 4:5:6 also describes the major triad, and it is significant in No. 34 that the three entrances of the canon spell out a major triad, beginning on E, A_♭ (G#), and B, respectively.

Carlsen uncovers an extraordinary case of structural integrity in Study No. 8. The final section is a 3-voice canon in which the lowest voice first states the canon heterophonically in octaves, the middle voice in fifths, and the upper voice in minor thirds, creating a texture of root-position triads. The positioning of

the entrances in this canon is carefully worked out so that:

When the second voice enters, it does so at a point exactly halfway through the first voice's isorhythmic pattern; the third voice enters exactly three-quarters of the way through the pattern The positioning of these entrances is specifically related to the divisions of successive intervals in the overtone series. Thus, the entrance of the second voice exactly halfway through the pattern parallels, in the overtone series, the division of the octave (2:4) into fifth (2:3) and fourth (3:4); by the same token, the further subdivision of that half into quarters parallels the subdivision of the fifth (4:6) into major and minor thirds (4:5 and 5:6). Obviously, it is no coincidence that the bass voice is characterized by octaves, the next by fifths, and the highest by minor thirds:



([1], p. 43)

Example 8: Concluding texture of Study No. 3a. Ostinato lines 2, 4, and 7 express a 2:3:5 rhythmic ratio, while the entire texture expresses a 3:5 ratio (the three ostinato lines against five lines of irregularly-spaced chords [lines 1, 3, 5a/b, 6, and 8]).

In another particularly fine example of a ratio being used structurally in different ways, Study No. #3a (from the “Boogie-Woogie Suite”) uses a 3:5 ratio on two structural levels. On a smaller level Nancarrow

creates a 2:3:5 rhythmic ratio between three ostinatos (repeating rhythmic patterns); see Example 8, where the ostinatos are in lines 2, 4, and 7 (the note symbol $\tilde{\sim}$ in line 2 is Nancarrow's own symbol for a 5-division note, in this case in the meter $\frac{20}{16}$). The piece climaxes with an 8-voice texture (also Example 8) in which five voices are expressing irregularly-spaced staccato chords against the three ostinato lines: a textural expression of the 3:5 ratio.

2. Conclusion

Conlon Nancarrow's interest in the rhythmic ideas expressed by Henry Cowell led him to the player piano as his primary performing medium, allowing him to clearly and exactly express ratios in his music in ways that had not been possible with human performers. Nancarrow extended Cowell's ideas to go beyond the pitch ratios of just intonation to ratios as close as 60:61 and even ratios involving irrational numbers, and he further extended Cowell's work by using ratios not only to control rhythmic parameters but also to establish relationships with other features such as pitch and structure.

Except for Nancarrow, the use of ratios in twentieth-century music was mostly limited to simultaneous expressions of different tempos related by simple ratios. Elliott Carter, for instance, another composer influenced by the ideas of Cowell, used techniques similar to Nancarrow's in establishing tempo relationships, but did so within the limitations of human performance. It appears that no other composer ventured as far as Nancarrow did in deploying ratios to affect so many different musical parameters.

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