HNRS 2900-125
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The wave equation for a vibrating string is a second-order linear partial differential equation with constant coefficients. If $u = u(x,t)$ represents displacement from an equilibrium position, then the wave equation has the form:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}.$$

It is easier to write partial derivatives as subscripts, making the wave equation

$$u_{tt} = c^2 u_{xx}. \tag{1}$$

The constant c equals tension divided by linear density of the string. If the string has length L, then the wave equation must hold for all $0 < x < L$ and $t > 0$.

In addition to the wave equation itself, there are *boundary conditions* and *initial conditions*. The initial conditions describe the initial state of the string, i.e., the state at time $t = 0$. Both the initial position of the string ($u(x,0)$) and the initial velocity of the string ($u_t(x,0)$) are specified:

$$u(x,0) = f(x), \tag{2}$$

$$u_t(x,0) = g(x), \tag{3}$$

for $0 < x < L$, where f and g are given functions.

The boundary conditions will depend upon the physical situation. If both ends of the string are fixed, then the boundary conditions are:

$$u(0,t) = 0, \tag{4}$$

$$u(L,t) = 0, \tag{5}$$

for all $t > 0$. When describing waves in a tube, it matters if the ends are open or closed. The situation of two closed ends is the same as the fixed ends above. If one end is closed (say, at $x = 0$) and the other end is open (at $x = L$), then the boundary conditions are:

$$u(0,t) = 0, \tag{6}$$

$$u_x(L,t) = 0, \tag{7}$$

for all $t > 0$. If both ends are open, then the boundary conditions are:

$$u_x(0,t) = 0, \tag{8}$$

$$u_x(L,t) = 0, \tag{9}$$

for all $t > 0$.

In class, we discussed solving the wave equation (1), subject to the boundary conditions (4) and (5). To do this, we assumed that the displacement function u could be written as the product of two single-variable functions:

$$u(x,t) = X(x)T(t).$$

The wave equation (1) then becomes:

$$XT'' = c^2 X'' T,$$

where the primes indicate derivatives with respect to the appropriate single variable (x for X, t for T). The variables in this last equation can be separated:

$$\frac{T''}{c^2 T} = \frac{X''}{X}.$$

Since the left side is a function of t only, while the right side is a function of x only, both functions must be constant, say, equal to $-\lambda$. Thus the wave equation becomes a pair of equations:

$$X'' + \lambda X = 0, \tag{10}$$

$$T'' + c^2 \lambda T = 0. \tag{11}$$

In terms of the functions X and T, the boundary conditions for non-trivial solutions take on the following forms. Boundary conditions (4) and (5) become

$$X(0) = 0, \quad X(L) = 0. \tag{12}$$

You should convince yourself that the other boundary conditions are as follows. Conditions (6) and (7) become

$$X(0) = 0, \quad X'(L) = 0, \tag{13}$$

while conditions (8) and (9) become

$$X'(0) = 0, \quad X'(L) = 0. \tag{14}$$

In class, we then looked for solutions to (10), subject to the boundary conditions (12). We found that non-trivial solutions exist only for certain values of λ, namely for

$$\lambda_n = \frac{n^2 \pi^2}{L^2}, \quad n = 1, 2, \ldots.$$

For the value λ_n, the corresponding solution of (10) and (12) was

$$X_n(x) = \sin\left(\frac{n\pi}{L} x\right).$$

For the same value λ_n, the solutions of (11) were

$$T_n(t) = A_n \cos\left(\frac{cn\pi}{L} t\right) + B_n \sin\left(\frac{cn\pi}{L} t\right),$$

where A_n and B_n are arbitrary constants. Putting these together, we have a number of solutions of the wave equation (1), with boundary conditions (4) and (5):

$$u_n(x,t) = A_n \cos\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi}{L}x\right) + B_n \sin\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi}{L}x\right).$$

Since the wave equation and the boundary conditions are linear and homogeneous, the sum of an arbitrary number of solutions is again a solution. So, any expression of the form $u = \sum u_n$ is also a solution.

For $u = \sum u_n$, that is,

$$u = \sum A_n \cos\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi}{L}x\right) + \sum B_n \sin\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi}{L}x\right),$$

the initial conditions (2) and (3) now enter the picture. We showed in class that the initial conditions (2) and (3) imply the following equations:

$$f(x) = \sum A_n \sin\left(\frac{n\pi}{L}x\right), \tag{15}$$

$$g(x) = \sum \frac{cn\pi}{L} B_n \sin\left(\frac{n\pi}{L}x\right). \tag{16}$$

Exercise 1. Do all of the above when using different boundary conditions (either the pair (6) and (7) or the pair (8) and (9)).