Character-Theoretic Tools for Studying Linear Codes over Rings and Modules

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9. Using semigroup rings

- Report on joint work with Gnilke, Greferath, Honold, and Zumbrägel
- Semigroup rings
- Modules over semigroup rings
- Connections to EP
- The case of bi-invariant weights over Frobenius bimodules

Semigroups

- Consider a finite **semigroup**: one associative operation, written as multiplication.
- Main example for us: the multiplicative semigroup of a finite ring R with 1.
- This semigroup has a 1 (a **monoid**) and a 0.

Semigroup rings

- Analogous to group rings.
- We will use complex coefficients.
- One way: form $\mathbb{C}$-vector space with basis e_r, $r \in R$.
- Define multiplication of basis elements to be $e_r e_s = e_{rs}$, where rs is the product in R.
- Extend linearly.
- Note that $\mathbb{C}e_0$ is a two-sided ideal

Equivalent approach

- Define $\mathcal{R} = \{\alpha : R \to \mathbb{C}\}$ to be the $\mathbb{C}$-vector space of all $\mathbb{C}$-valued functions on R; $\dim \mathcal{R} = |R|$.
- Product on $\mathcal{R}$ ("multiplicative convolution"):

$$(\alpha * \beta)(r) = \sum_{st=r} \alpha(s)\beta(t), \quad r \in R,$$

where the sum is over pairs s, t in R with $st = r$.
- Then $\alpha \leftrightarrow \sum_{r \in R} \alpha(r) e_r$ of other approach.

R-modules induce $\mathcal{R}$-modules

- Let A be a finite left R-module.
- Set $\mathcal{A} = \{w : A \to \mathbb{C}\}$, a $\mathbb{C}$-vector space with $\dim \mathcal{A} = |A|$.
- Then $\mathcal{A}$ is a right $\mathcal{R}$-module via "right correlation":

$$(w \circledast \alpha)(a) = \sum_{r \in R} w(ra)\alpha(r), \quad a \in A.$$

- $w \circledast (\alpha * \beta) = (w \circledast \alpha) \circledast \beta$, $w \in \mathcal{A}$, $\alpha, \beta \in \mathcal{R}$.
- Similarly for right R-module; get left $\mathcal{R}$-module.

Some splittings

- Set $\mathcal{R}_0 = \{\alpha \in \mathcal{R} : \sum_{r \in R} \alpha(r) = 0\}$: augmentation ideal.
- $\mathcal{R}_0$ is a two-sided ideal of $\mathcal{R}$; $\mathcal{R} = \mathbb{C}e_0 \oplus \mathcal{R}_0$.
- Set $\mathcal{A}_0 = \{w \in \mathcal{A} : w(0) = 0\}$; $\mathcal{A}_0$ is a right $\mathcal{R}$-submodule, and $\mathcal{A} = \mathbb{C}1 \oplus \mathcal{A}_0$, where $1 \in \mathcal{A}$ is the constant function 1.

Recall the extension property EP

- Recall that a **weight** w on an alphabet A is any function $w : A \to \mathbb{C}$ with $w(0) = 0$; i.e., $w \in \mathcal{A}_0$.
- Recall that A has the extension property (EP) with respect to a weight w if every linear w-isometry $f : C \to A^n$ extends to a monomial transformation of A^n that is a w-isometry.

Isometries

Theorem (Greferath-Honold)

If f is a w-isometry, then f is a $(w \circledast \alpha)$-isometry for any $\alpha \in \mathcal{R}$.

$$
\begin{aligned}
(w \circledast \alpha)(xf) &= \sum_{r \in R} w(rxf)\alpha(r) \\
&= \sum_{r \in R} w(rx)\alpha(r) = (w \circledast \alpha)(x)
\end{aligned}
$$

Connections to EP

Corollary

*If A has EP with respect to $w \circledast \alpha$, then A has EP with respect to w.**

- If f is a w-isometry, then it is a $(w \circledast \alpha)$-isometry. By EP for $w \circledast \alpha$, f extends to a monomial transformation.
- *Fine print: need to worry about the right symmetry groups being different: $w \circledast \alpha$ may have more symmetry than w.

Case of bi-invariant weights over Frobenius bimodules

- For the rest of today, let A be a Frobenius bimodule over R. I.e., A is a bimodule over R with $A \cong \widehat{R}$ as left and as right R-modules. Ex.: bimodule $A = \widehat{R}$.
- A admits a left generating character χ, and χ is also a right generating character.
- $\alpha \in \mathcal{R}$, $w \in \mathcal{A}$ are **bi-invariant** if $\alpha(urv) = \alpha(r)$, $w(uav) = w(a)$ for all $r \in R$, $a \in A$, and units $u, v \in \mathcal{U}$.

Conditions on w

- Consider the poset $\{aR : a \in A\}$ of all cyclic right R-submodules of A, under set inclusion.
- Möbius function $\mu(0, aR)$.
- Suppose $w \in \mathcal{A}_0$ satisfies

$$\sum_{aR \subseteq B} w(a)\mu(0, aR) \neq 0, \tag{1}$$

for all nonzero right R-submodules $B \subseteq A$.

Main results

Theorem

Suppose A is a Frobenius bimodule over R, and suppose w is a bi-invariant weight in $\mathcal{A}_0$ satisfying (1), then A has EP with respect to w.

Corollary

If R is a finite Frobenius ring and w is a bi-invariant weight on R satisfying (1), then R has EP with respect to w.

Fourier transform

- The generating character χ of A is an element of $\mathcal{A}$.
- The map $\mathcal{R} \to \mathcal{A}$, $\alpha \mapsto \chi \circledast \alpha$, is a Fourier transform:

$$(\chi \circledast \alpha)(a) = \sum_{r \in R} \chi(ra)\alpha(r).$$

- Invert: $\chi \circledast \tilde{w} = w$, where

$$\tilde{w}(r) = \frac{1}{|A|} \sum_{a \in A} w(a)\chi(-ra).$$

Homogeneous weight

- Recall that the homogeneous weight w_{Hom} has EP on any Frobenius bimodule.
- Both symmetry groups of w_{Hom} are maximal: all of $\mathcal{U}$.
- Recall that

$$w_{\mathrm{Hom}}(a) = 1 - \frac{1}{|\mathcal{U}|} \sum_{u \in \mathcal{U}} \chi(ua), \quad a \in A.$$

Inverting w_{Hom}

- Define $\varepsilon \in \mathcal{R}$:

$$\varepsilon(r) = \begin{cases} -\frac{1}{|\mathcal{U}|}, & r \in \mathcal{U}, \\ 1, & r = 0, \\ 0, & \text{otherwise.} \end{cases}$$

- Then $\chi \circledast \varepsilon = w_{\mathrm{Hom}}$:

$$(\chi \circledast \varepsilon)(a) = \sum_{r \in R} \chi(ra)\varepsilon(r) = w_{\mathrm{Hom}}(a).$$

Outline of argument

- Suppose we can find $\gamma \in \mathcal{R}$ such that $\tilde{w} * \gamma = \varepsilon$.
- Then $w \circledast \gamma = w_{\mathrm{Hom}}$:

$$w \circledast \gamma = (\chi \circledast \tilde{w}) \circledast \gamma = \chi \circledast (\tilde{w} * \gamma) = \chi \circledast \varepsilon = w_{\mathrm{Hom}}.$$

- Apply earlier result, as w_{Hom} has EP.
- Condition (1) will allow us to solve $\tilde{w} * \gamma = \varepsilon$ for γ.

Condition (1)

Theorem

Condition (1) *is equivalent to*

$$\sum_{b \in B} w(b)\chi(b) \neq 0, \tag{2}$$

for all nonzero right R-submodules $B \subseteq A$.

Proof

- Break up into sum over right $\mathcal{U}$-orbits.
- Using results from lecture 7:

$$\sum_{b \in B} w(b)\chi(b) = \sum_{a\mathcal{U} \subseteq B} \sum_{b \in a\mathcal{U}} w(b)\chi(b)$$

$$= \sum_{aR \subseteq B} w(a)\mu(0, aR).$$

Solving $\tilde{w} * \gamma = \varepsilon$

- We want to solve $\tilde{w} * \gamma = \varepsilon$ for γ.
- Note that $\tilde{w}$ and ε are bi-invariant and in $\mathcal{R}_0$.
- We want γ to be bi-invariant and in $\mathcal{R}_0$, too.
- The equation, for any $r \in R$, is

$$\sum_{st=r} \tilde{w}(s)\gamma(t) = \varepsilon(r).$$

- Solve recursively, starting with $r \in \mathcal{U}$.

When $r \in \mathcal{U}$

- If $r \in \mathcal{U}$, then $st = r$ implies $s, t \in \mathcal{U}$.
- Using bi-invariance of $\tilde{w}$ and γ, equation becomes

$$-\frac{1}{|\mathcal{U}|} = \sum_{t \in \mathcal{U}} \tilde{w}(rt^{-1})\gamma(t) = |\mathcal{U}|\tilde{w}(1)\gamma(1).$$

- $\tilde{w}(1) \neq 0$ is the case $B = A$ of (2).
- So $\gamma(u) = -1/(|\mathcal{U}|^2 \tilde{w}(1))$ for $u \in \mathcal{U}$.

Recursive step

- Suppose γ has been defined to be bi-invariant and to satisfy $\tilde{w} * \gamma = \varepsilon$ for some values of $r \in R$.
- Let $r \in R$ be any element, neither zero nor a unit, such that Rr is maximal among principal left ideals of R where γ is not defined on $\mathcal{U}r$.
- Consider $(\tilde{w} * \gamma)(r) = \varepsilon(r) = 0$.

Recursive step, part 2

- If $st = r$, then $Rr \subseteq Rt$.
- If $Rr \subsetneq Rt$, then maximality of Rr implies that $\gamma(t)$ is already defined.
- Then $(\tilde{w} * \gamma)(r) = 0$ becomes

$$0 = \sum_{\substack{st=r \\ Rr \subsetneq Rt}} \tilde{w}(s)\gamma(t) + \sum_{\substack{st=r \\ Rr=Rt}} \tilde{w}(s)\gamma(t).$$

Recursive step, part 3

- Focus on sum with $Rr = Rt$.
- Then $\mathcal{U}r = \mathcal{U}t$, so $t = ur$ for some $u \in \mathcal{U}$.
- Thus $r = st = sur$, so that $(su - 1)r = 0$.
- Let $\mathrm{ann}_{\mathrm{lt}}(r) = \{q \in R : qr = 0\}$, a left ideal of R.
- Then $su - 1 \in \mathrm{ann}_{\mathrm{lt}}(r)$.
- Every factorization $st = r$ with $Rr = Rt$ has the form $s = (q + 1)u^{-1}$, $t = ur$, with $u \in \mathcal{U}$ and $q \in \mathrm{ann}_{\mathrm{lt}}(r)$.

Recursive step, part 4

- Using bi-invariance, the sum with $Rr = Rt$ becomes

$$\sum_{\substack{st=r \\ Rr=Rt}} \tilde{w}(s)\gamma(t) = \sum_{\substack{q\in\mathrm{ann}_{\mathrm{lt}}(r) \\ u\in\mathcal{U}}} \tilde{w}((q+1)u^{-1})\gamma(ur)$$

$$= |\mathcal{U}|\gamma(r) \sum_{q\in\mathrm{ann}_{\mathrm{lt}}(r)} \tilde{w}(q+1)$$

Recursive step, part 5

- But $\sum_{q\in \mathrm{ann}_{\mathrm{lt}}(r)} \tilde{w}(q+1)$ simplifies:

$$|A| \sum_{q\in \mathrm{ann}_{\mathrm{lt}}(r)} \tilde{w}(q+1) = \sum_{q\in \mathrm{ann}_{\mathrm{lt}}(r)} \sum_{a\in A} w(a)\chi(-(1+q)a)$$
$$= \sum_{a\in A} w(a)\chi(a) \sum_{q\in \mathrm{ann}_{\mathrm{lt}}(r)} \chi(qa).$$

- What about $\sum_{q\in \mathrm{ann}_{\mathrm{lt}}(r)} \chi(qa)$?

Recursive step, part 6

- $\sum_{q \in \mathrm{ann}_{\mathrm{lt}}(r)} \chi(qa)$ is a sum over the left submodule $\mathrm{ann}_{\mathrm{lt}}(r)a \subseteq A$.
- Since χ is a generating character, this sum vanishes unless $\mathrm{ann}_{\mathrm{lt}}(r)a = 0$. In that case, the sum equals $|\mathrm{ann}_{\mathrm{lt}}(r)|$.
- Set $B_r = \{a \in A : \mathrm{ann}_{\mathrm{lt}}(r)a = 0\}$, a right submodule of A. Then

$$\sum_{q \in \mathrm{ann}_{\mathrm{lt}}(r)} \chi(qa) = \begin{cases} |\mathrm{ann}_{\mathrm{lt}}(r)|, & a \in B_r, \\ 0, & a \notin B_r. \end{cases}$$

Recursive step, part 7

- Going back to $\sum_{q\in \mathrm{ann}_{\mathrm{lt}}(r)} \tilde{w}(q+1)$, we have

$$|A| \sum_{q\in \mathrm{ann}_{\mathrm{lt}}(r)} \tilde{w}(q+1) = |\mathrm{ann}_{\mathrm{lt}}(r)| \sum_{a\in B_r} w(a)\chi(a).$$

- This is nonzero: the $B = B_r$ case of (2). Thus,

$$\gamma(r) = -\left(\sum_{\substack{st=r \\ Rr\subsetneq Rt}} \tilde{w}(s)\gamma(t) \right) \Big/ \left(|\mathcal{U}| \sum_{q\in \mathrm{ann}_{\mathrm{lt}}(r)} \tilde{w}(1+q) \right)$$

Recursive step, part 8

- Check that γ is still bi-invariant.
- Continue recursively. Eventually get to case $r = 0$.
- Coefficient of $\gamma(0)$ term vanishes, so we are free to define $\gamma(0)$ so that $\gamma \in \mathcal{R}_0$.
- I'll spare you those details.