Failure of the MacWilliams Identities for the Lee Weight Enumerator over $\mathbb{Z}_m$, $m \geq 5$

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Abstract

In this paper we show the nonexistence of any version of the MacWilliams identities for Lee weight enumerators over $\mathbb{Z}_m$, $m \geq 5$.

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1. Introduction

The MacWilliams identities give a relation between the Hamming weight enumerator of a linear code and the Hamming weight enumerator of its dual. For a linear code $C$ over a finite field $\mathbb{F}_q$, the MacWilliams identities are given by

$$\text{lwe}_C(X,Y) = \frac{1}{|C|} \text{lwe}_C(X + (q-1)Y, X - Y),$$

where lwe refers to the Lee weight enumerator [5]. The same statement of the MacWilliams identities is valid for linear codes over a finite Frobenius ring of size $q$ with respect to the Hamming weight [8]. We are interested in the question of whether there is some version of the MacWilliams identities for other alphabets and other weight functions.

In this paper we consider the Lee weight over $\mathbb{Z}_m$, the integers modulo $m$. The Lee weight and the Hamming weight are equal when $m = 2$ or $m = 3$; thus the MacWilliams identities are valid in those cases. For codes over $\mathbb{Z}_4$, it is known from [3] that the Lee weight enumerator of a linear code $C$ over $\mathbb{Z}_4$ and its dual are related:

$$\text{lwe}_{C^\perp}(X,Y) = \frac{1}{|C|} \text{lwe}_C(X + Y, X - Y).$$

For $m \geq 5$, it was shown in [7] that the change of variables $X \rightarrow X + (q-1)Y$ and $Y \rightarrow X - Y$ does not give a version of the MacWilliams identities for any prime power $q|m$. This leaves open the possibility of other changes of variables that might give a relation between the Lee weight enumerators of a code and its dual. The main result of this paper is that there is no well-defined relation between the Lee weight enumerators of a code and its dual for $m \geq 5$. Specifically, this paper proves the following

**Theorem 1.1.** Suppose $m \geq 5$. There exist linear codes $C_1, C_2$ over $\mathbb{Z}_m$ satisfying $\text{lwe}_{C_1} = \text{lwe}_{C_2}$ and $\text{lwe}_{C_1^\perp} \neq \text{lwe}_{C_2^\perp}$.

Let $\mathcal{LC}(\mathbb{Z}_m)$ denote the collection of all linear codes over $\mathbb{Z}_m$, and let $\dashv: \mathcal{LC}(\mathbb{Z}_m) \rightarrow \mathcal{LC}(\mathbb{Z}_m)$ denote the map sending a linear code $C$ to its dual code $C^\perp$. Then Theorem 1.1 says that it is impossible to find a well-defined map making the following diagram commute.

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Corollary 1.2. There are no MacWilliams identities relating the Lee weight enumerators of linear codes and their dual codes over $\mathbb{Z}_m$, for $m \geq 5$.

Here is an outline of the proof of Theorem 1.1, which will also serve as a guide to the rest of the paper. Explicit examples of linear codes $C_1, C_2$ over $\mathbb{Z}_m$ satisfying $lwe_{C_1} = lwe_{C_2}$ and $lwe_{C_1^⊥} \neq lwe_{C_2^⊥}$ are constructed: in an ad hoc fashion for $m = 5, 6, 8, 9$ (in Section 2), and in a systematic fashion for all primes $p \geq 7$ (in Section 3).

To handle other values of $m \geq 5$, which necessarily are integer multiples of the preceding cases, in Section 4 we analyze the relationship between a linear code $C \subseteq \mathbb{Z}_m^n$ and the linear code $aC \subseteq \mathbb{Z}_{am}^n$, defined by scalar multiplying each codeword of $C$ by $a$. In particular, Lemma 4.1 shows that $lwe$ determines $lwe_{ac}$, so that $lwe_{C_1} = lwe_{C_2}$ will imply $lwe_{ac_{C_1}} = lwe_{ac_{C_2}}$, which is Corollary 4.2. On the other hand, Lemma 4.3 allows us to compare the number of codewords of sufficiently small weight in $C^⊥$ and $(aC)^⊥$. Consequently, if $lwe_{C_1^⊥} \neq lwe_{C_2^⊥}$ because the number of codewords of a sufficiently small weight differ, then the same will be true for $(aC_1)^⊥$ and $(aC_2)^⊥$, so that $lwe_{(ac_{C_1})^⊥} \neq lwe_{(ac_{C_2})^⊥}$, which is Corollary 4.4.

2. Preliminaries and examples for small values of $m$

In this paper we are interested in linear codes defined over the ring $\mathbb{Z}_m$ of integers modulo $m$. A linear code $C$ of length $n$ over $\mathbb{Z}_m$ is a submodule of $\mathbb{Z}_m^n$. Vectors $v \in \mathbb{Z}_m^n$ have the form $v = (v_1, \ldots, v_n)$. The dual code $C^⊥$ in $\mathbb{Z}_m^n$ is defined by $C^⊥ = \{v \in \mathbb{Z}_m^n : \sum_{i=1}^n v_i c_i = 0, \text{ for all } c \in C\}$.

The Lee weight is defined on $\mathbb{Z}_m$ by $l_m(i) = \left|\frac{i}{m}\right|$, the ordinary absolute value on $\mathbb{Z}$, where $\mathbb{Z}_m$ is thought of as $\mathbb{Z}_m = \{i \in \mathbb{Z} : -m/2 < i \leq m/2\}$. We will write just $l(i)$ when $m$ is obvious from context. For vectors $v \in \mathbb{Z}_m^n$, define $l(v) = \sum_{i=1}^n l(v_i)$. For a linear code $C$ of length $n$ over $\mathbb{Z}_m$, the maximum Lee weight in $C$ is $n[m/2]$. The Lee weight enumerator of $C$ is an element in the polynomial ring $\mathbb{C}[X,Y]$ given by

$$lwe_C(X,Y) = \sum_{c \in C} X^{l(c)} Y^{l(c)}.$$

Let $A_i(C)$ denote the number of codewords of Lee weight $i$ in $C$, that is, $A_i(C) = |\{c \in C : l(c) = i\}|$, for $0 \leq i \leq n[m/2]$. The Lee weight enumerator of $C$ can then be rewritten as

$$lwe_C(X,Y) = \sum_{i=0}^{n[m/2]} A_i X^{l(i)} Y^{l(i)}.$$

In the following we will present examples of pairs of codes $C_1, C_2$ with $lwe_{C_1} = lwe_{C_2}$ and $lwe_{C_1^⊥} \neq lwe_{C_2^⊥}$ over $\mathbb{Z}_m$, for $m = 5, 6, 8, 9$.

Example 2.1. Let $m = 5$. Let $G_1$ and $G_2$ be generator matrices for $C_1$ and $C_2$ given by the following table. The table shows how many times each column type appears in $G_1$ and $G_2$.

<table>
<thead>
<tr>
<th>Column Type</th>
<th>0 0 1 1 1 1</th>
<th>1 2 2 2 2 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Multiplicity $\eta$ in $G_1$</td>
<td>5 5 5 5 5 5</td>
<td>1 2 2 2 2 2</td>
</tr>
<tr>
<td>Multiplicity $\eta$ in $G_2$</td>
<td>9 6 8 5 9 5</td>
<td>2 6 5 5 2</td>
</tr>
</tbody>
</table>

Then $lwe_{C_1}(X,Y) = lwe_{C_2}(X,Y) = X^{140} + 4X^{65}Y^{75} + 20X^{50}Y^{90}$ but $lwe_{C_1^⊥} \neq lwe_{C_2^⊥}$. In fact, $A_2(C_1^⊥) = 380$ and $A_2(C_2^⊥) = 400$. The counts for $A_2$ are obtained as follows. A dual codeword of Lee weight 2 has either
one nonzero entry of the form $\pm 2$ or two nonzero entries of the form $\pm 1$, $\pm 1$. There are no dual codewords of the first type (i.e., a single one nonzero entry of the form $\pm 2$) because there are no zero columns in $G_1$ or $G_2$. Dual codewords of the second type (i.e., two $\pm 1$s) must occur with opposite signs, because no column type is annihilated by 2 and no two column types sum to zero. The only dual codewords of the second type arise by subtracting, in either order, columns of the same type. Thus

$$A_2(C_i^\perp) = 2\sum_{\eta} \binom{n}{2} = \begin{cases} 380, & i = 1, \\ 400, & i = 2, \end{cases}$$

where the sum is over the multiplicities $\eta$ given in the table.

Example 2.2. Let $m = 6$. Consider the codes $C_1$ and $C_2$ generated by

$$G_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 & 1 & 3 & 3 \end{bmatrix}. $$

Then $C_1$ and $C_2$ are given by

$$C_1 = \{(0,0,0,0),(1,1,1,1),(2,2,2,2),(3,3,3,3),(4,4,4,4),(5,5,5,5)\},$$

$$C_2 = \{(0,0,0,0),(1,1,3,3),(2,2,0,0),(3,3,3,3),(4,4,0,0),(5,5,3,3)\}.$$

Then $\text{lwe}_{C_1}(X,Y) = \text{lwe}_{C_2}(X,Y) = X^{12} + 2X^8Y^4 + 2X^4Y^8 + Y^{12}$.

We show that $\text{lwe}_{C_1} \neq \text{lwe}_{C_2}$ by showing that $A_2(C_1^\perp) \neq A_2(C_2^\perp)$. As we saw in Example 2.1, a dual codeword of weight 2 either contains a single nonzero entry of $\pm 2$ or two nonzero entries of $\pm 1$. For $C_1$, since none of the columns of $G_1$ is annihilated by $\pm 2$, then all the codewords of weight 2 in $C_1^\perp$ contain two nonzero entries of $\pm 1$. It is easy to see that if the two nonzero entries of such a codeword are equal, both equal to 1 or both equal to $-1$, then that codeword does not annihilate $G_1$. But if one entry is 1 and the other is $-1$ then that codeword does indeed annihilate $G_1$. There are $4 \cdot 3 = 12$ such vectors of length 4. Thus $A_2(C_1^\perp) = 12$.

For $C_2^\perp$, the third and the fourth column of $G_2$ are annihilated by $\pm 2$, therefore the four codewords $(0,0,\pm 2,0)$, $(0,0,0,\pm 2)$ are elements of $C_2^\perp$. Also, adding or subtracting the last two columns of $G_2$ gives the zero column, therefore the four codewords $(0,0,\pm 1,\pm 1)$ are all elements of $C_2^\perp$. Finally, since the first two columns of $G_2$ are identical, then we get that $(1,-1,0,0),(-1,1,0,0)$ are codewords in $C_2^\perp$. Thus $A_2(C_2^\perp) = 10$, and $\text{lwe}_{C_1} \neq \text{lwe}_{C_2}$.

The following example is due the referee of Example 5.5 and also appeared in Example 2.3.

Example 2.3. Let $m = 8$. Consider the codes $C_1$ and $C_2$ generated by

$$G_1 = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 & 3 & 4 \end{bmatrix}. $$

We get $\text{lwe}_{C_1}(X,Y) = \text{lwe}_{C_2}(X,Y) = X^{12} + 2X^8Y^4 + 5X^4Y^8$.

We show that $\text{lwe}_{C_1} \neq \text{lwe}_{C_2}$ by showing that $C_1^\perp$ and $C_2^\perp$ have different numbers of codewords of weight 3. Suppose that $c = (c_1,c_2,c_3)$ has weight 3. Then $c$ is an element of $C_1^\perp$ if and only if $G_1c^T = 0$, i.e., $c_1 + c_2 + 2c_3 = 0$. Since $1(c) = 3$, the entries $c_1$, $c_2$, and $c_3$ can have values from $\{0,\pm 1,\pm 2,\pm 3\}$ only. It is straightforward to find that the solutions of $c_1 + c_2 + 2c_3 = 0$ with those restrictions are

$$\{-1, -1, 1, 1, 1, 1, 2, 0, -1, 0, 2, -1, -2, 0, 1, 0, -2, 1\}. $$

Thus $A_3(C_1^\perp) = 6$. Similarly, we find that the codewords of weight 3 in $C_2^\perp$ are

$$\{(1,1,1),(-1,-1,-1),(1,1,-1),(1,-1,1),(1,1,1)\}. $$

Thus $A_3(C_2^\perp) = 4$. This shows that $A_3(C_1^\perp) \neq A_3(C_2^\perp)$ and hence $\text{lwe}_{C_1} \neq \text{lwe}_{C_2}$. 

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Example 2.4. Let \( m = 9 \). Define \( G_1 \) to be a \( 1 \times 133 \) matrix consisting of 33 entries equal to 1, 24 entries equal to 2, 70 entries equal to 3, and 6 entries equal to 4. Let \( G_2 \) be a \( 1 \times 133 \) matrix consisting of 9 entries equal to 1, 81 entries equal to 2, 36 entries equal to 4, and 7 entries equal to 0. Then \( \text{lwe}_{C_1} = \text{lwe}_{C_2} \), but \( \text{lwe}_{C_1} \neq \text{lwe}_{C_2} \). In fact,
\[
\text{lwe}_{C_1}(X, Y) = \text{lwe}_{C_2}(X, Y) = X^{532} + 2X^{343}Y^{189} + 2X^{217}Y^{315} + 4X^{154}Y^{378}.
\]
Because of the 0 entries in \( G_2 \), \( C^{\perp}_2 \) contains codewords of weight 1, while \( C^{\perp}_1 \) does not. I.e., \( A_1(C^{\perp}_1) \neq A_1(C^{\perp}_2) \).

3. Prime Modulus \( p, p \geq 7 \)

In this section we construct examples \( C_1 \) and \( C_2 \) with \( \text{lwe}_{C_1} = \text{lwe}_{C_2} \) and \( \text{lwe}_{C_1} \neq \text{lwe}_{C_2} \) over the integers modulo a prime \( p, p \geq 7 \).

Fix a prime \( p \), with \( p \geq 7 \). Let \( t = \frac{(p - 1)}{2} \). Let \( \mathbb{Z}_p^\times \) denote the set of nonzero elements of \( \mathbb{Z}_p \); \( \mathbb{Z}_p^\times \) is a group under multiplication, and \( \{ \pm 1 \} \) forms a subgroup. Let \( H \) denote the quotient group \( \mathbb{Z}_p^\times /\{ \pm 1 \} \), and let \( \pi \) be the canonical quotient map \( \pi : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times /\{ \pm 1 \} \). We choose the positive representative for each element in \( H \), so we identify the elements of \( H \) with the set \( \{ 1, 2, \ldots, t \} \).

Remark 3.1. Under the above identification, the quotient map \( \pi \) equals the Lee weight map \( L : \mathbb{Z}_p^\times \to \{ 1, 2, \ldots, t \} \).

We fix the following generator matrices \( G_1 \) and \( G_2 \), with corresponding codes \( C_1 \) and \( C_2 \). The matrix \( G_1 \) has size \( 1 \times (t(t - 1)) \), consisting of \( t \) entries of each \( i \) in \( \{ 2, \ldots, t \} \). That is
\[
G_1 = \begin{bmatrix} 2 & \ldots & 2 & 3 & \ldots & 3 & \ldots & t & \ldots & t \end{bmatrix},
\]
where every number is repeated \( t \) times. And \( G_2 \) is the matrix of size \( 1 \times (t(t - 1)) \), consisting of \( 2(t - 1) \) entries of 1 and \( t - 2 \) entries of each \( i \) in \( \{ 2, \ldots, t \} \).

Lemma 3.2. For the linear codes given above, \( \text{lwe}_{C_1} = \text{lwe}_{C_2} \), with a common weight distribution given by
\[
A_0 = 1 \quad \text{and} \quad A_i\left(\frac{t(t+1)}{2} - i\right) = 2, \quad \text{for } 1 \leq i \leq t.
\]

Proof. Let \( i \) be in \( \{ 1, \ldots, t \} \). As an element of the group \( H \), we have that \( iH = H = \{ 1, 2, \ldots, t \} \). Let \( * \) denote multiplication in \( H \). In the following we reindex summations by using the fact that multiplying by a group element is a permutation of \( H \). For \( G_1 \),
\[
L(iG_1) = \sum_{j=2}^{t} L(ij)t = t \left( \sum_{j \in H} i* j - i* 1 \right) = t \left( \sum_{k=1}^{t} k - i \right) = t \left( \frac{t(t+1)}{2} - i \right).
\]
Since \( L(iG) = L(-iG) \) over \( \mathbb{Z}_m \), then we get that \( A_i\left(\frac{t(t+1)}{2} - i\right)(C_1) = 2 \) for \( i \) in \( \{ 1, \ldots, t \} \). For \( G_2 \),
\[
L(iG_2) = 2(t-1) L(i) + \sum_{j=2}^{t} (t-2) L(ij) = 2(t-1)i + (t-2) \left( \sum_{k=1}^{t} k - i \right)
\]
\[
= 2(t-1)i + (t-2) \left( \frac{t(t+1)}{2} - i \right) = t \left( \frac{(t-2)(t+1)}{2} + i \right),
\]
for all \( 1 \leq i \leq t \). Therefore, \( A_0 = 1 \) and \( A_i\left(\frac{t(t+1)}{2} - i\right)(C_2) = 2 \), for \( 1 \leq i \leq t \). Consider the substitution \( i = t + 1 - j \), then we have \( 1 \leq j \leq t \), and
\[
t \left( \frac{(t-2)(t+1)}{2} + i \right) = t \left( \frac{(t-2)(t+1)}{2} + t + 1 - j \right) = t \left( \frac{t(t+1)}{2} - j \right)
\]
This shows that we indeed have \( A_0(C_2) = 1 \) and \( A_i\left(\frac{t(t+1)}{2} - i\right)(C_2) = 2 \), for \( 1 \leq i \leq t \), as desired. \( \Box \)

Now we show that \( \text{lwe}_{C_1} \neq \text{lwe}_{C_2} \) by showing that the number of codewords of weight three in \( C^{\perp}_1 \) does not equal the number of codewords of weight three in \( C^{\perp}_2 \). We will study the cases when \( p = 1 \) \text{ mod } 4 and when \( p = 3 \) \text{ mod } 4 separately.
3.1. Primes congruent to 1 modulo 4

In this subsection \( p = 1 \mod 4 \). That is, \( t = (p - 1)/2 \) is an even number, and so \( t/2 \) is an integer. Suppose that \( C \) is a code generated by \( G \) of size \( 1 \times n \) with no zero columns. The types of columns are \( \{1, 2, \ldots, t\} \). Let \( a_i \) denote the number of columns in \( G \) whose entry is \( i \), for \( i \in \{1, 2, \ldots, t\} \). There are, up to a sign, five possible types of codewords in \( C^\perp \) of weight 3. Since \( C \) does not contain zero columns, a codeword with a single 3 or -3 will not appear in \( C^\perp \). So only four types are considered here, each in a separate sub-subsection. We will count the number of codewords in \( C^\perp \) of each of the four types and then apply the count to \( C_1^\perp \) and \( C_2^\perp \).

3.1.1. A 1 and a -2

Let \( c \) be a codeword in \( C^\perp \) with a 1 and a -2. Let \([x]\) denote a column in \( G \) whose entry is \( x \). Suppose that 1 corresponds to column \([x]\) and -2 corresponds to column \([y]\) in the generator matrix, so that \( Gc^T = x - 2y \).

Since \( C \) has no zero columns, then \( 1 \leq x, y \leq t \) and so \( -(2t - 1) \leq x - 2y \leq t - 2 \). This implies that the only way to have \( x - 2y \equiv 0 \mod p \) is to have \( x - 2y = 0 \). This forces \( x \) to be even. So if \( x = 2i \), then \( y = i \) and the number of ways that this happens is \( a_2a_i \). Therefore, the number of such codewords in \( C^\perp \) is

\[
\sum_{i=1}^{t/2} a_2a_i.
\]

Therefore, the count for \( C_1^\perp \), given that \( a_1 = 0 \) and \( a_i = t \) for all \( 2 \leq i \leq t \), is:

\[
\sum_{i=1}^{t/2} a_2a_i = \sum_{i=2}^{t/2} i^2 = \frac{t^2(t - 2)}{2}.
\]

And, the count for \( C_2^\perp \), given that \( a_1 = 2(t - 1) \) and \( a_i = t - 2 \) for all \( 2 \leq i \leq t \), is:

\[
\sum_{i=1}^{t/2} a_2a_i = 2(t - 1)(t - 2) + \sum_{i=2}^{t/2} (t - 2)^2 = 2(t - 1)(t - 2) + \left( \frac{t}{2} - 1 \right) (t - 2)^2
\]

\[
= \frac{(t - 2)}{2} \left( 4(t - 1) + (t - 2)^2 \right) = \frac{t^2(t - 2)}{2}.
\]

3.1.2. A 1 and a 2

Let \( c \) be a codeword in \( C^\perp \) with a 1 and a 2. Suppose that 1 corresponds to column \([x]\) and 2 corresponds to column \([y]\) in the generator matrix, so that \( Gc^T = x + 2y \).

Since \( C \) has no zero columns, then \( 1 \leq x, y \leq t \) and so \( -2t \leq x + 2y \leq 2t + 1 \). This implies that the only way to have \( x + 2y \equiv 0 \mod p \) is to have \( x + 2y = 2t + 1 \). This implies that \( x \) is an odd number. So if \( x = 2i + 1 \), then \( y = t - i \) and the number of ways that this happens is \( a_{2i+1}a_{t-i} \). Therefore, the number of such codewords in \( C^\perp \) is

\[
\sum_{i=0}^{(t-2)/2} a_{2i+1}a_{t-i}.
\]

Thus, the count for \( C_1^\perp \) is:

\[
\sum_{i=0}^{(t-2)/2} a_{2i+1}a_{t-i} = \sum_{i=1}^{(t-2)/2} t^2 = \frac{t^2(t - 2)}{2}.
\]

And, the count for \( C_2^\perp \) is:

\[
\sum_{i=0}^{(t-2)/2} a_{2i+1}a_{t-i} = 2(t - 1)(t - 2) + \sum_{i=1}^{(t-2)/2} (t - 2)^2 = \frac{t^2(t - 2)}{2}.
\]
3.1.3. A 1 and two \(-1\)'s

Let \(c\) be a codeword in \(C^\perp\) with a 1 and two \(-1\)'s. Suppose that 1 corresponds to column \([x]\) and the \(-1\)'s correspond to columns \([y]\) and \([z]\) in the generator matrix, so that \(Gc^T = x - y - z\). Since \(1 - 2t \leq x - y - z \leq t - 1\), then the only way to have \(x - y - z \equiv 0 \mod p\) is to have \(x - y - z = 0\) and so \(x = y + z\). Therefore, the number of such codewords in \(C^\perp\) is
\[
\sum_{i=1}^{t/2} a_{2i} \left( \sum_{j=1}^{i-1} a_j a_{2i-j} + \frac{a_i}{2} \right) + \sum_{i=1}^{(t-2)/2} a_{2i+1} \left( \sum_{j=1}^{i} a_j a_{2i+1-j} \right).
\]

Therefore, the count for \(C_1^\perp\) is:
\[
\frac{t^2}{2} \sum_{i=2}^{t/2} t \left( \sum_{j=2}^{i-1} t^2 + \frac{t(t-1)}{2} \right) + \frac{(t-2)/2}{i=1} t \left( \sum_{j=2}^{i} t^2 \right) = \frac{t^2}{2} \sum_{i=2}^{t/2} t \left( \frac{(i-2)t^2 + \frac{t(t-1)}{2}}{2} \right) + \frac{(t-2)/2}{i=1} t \left( \frac{t^2(i-1)}{4} \right)
\]

And, the count for \(C_2^\perp\) is:

First part:
\[
\frac{t^2}{2} \sum_{i=2}^{t/2} t \left( \sum_{j=2}^{i-1} t^2 + \frac{t(t-1)}{2} \right) = a_2 \left( \frac{a_1}{2} \right) + \sum_{i=2}^{t/2} a_{2i} \left( a_{1}a_{2i-1} + \sum_{j=2}^{i-1} a_j a_{2i-j} + \frac{a_i}{2} \right)
\]
\[
= (t-2) \left( \frac{2(t-2)(2t-3)}{2} \right) + \sum_{i=2}^{t/2} \left( 2(t-1)(t-2)^2 + \sum_{j=2}^{i-1} (t-2)^2 + \frac{(t-2)^2(t-3)}{2} \right)
\]
\[
= \frac{t^2(t-2)(t^2-2)}{8}.
\]

Second part:
\[
\sum_{i=1}^{(t-2)/2} a_{2i+1} \left( a_1 a_{2i+1-1} + \sum_{j=2}^{i} a_j a_{2i+1-j} \right)
\]
\[
= \sum_{i=1}^{(t-2)/2} (t-2) \left( 2(t-1)(t-2) + (t-2)^2(i-1) \right) = \frac{(t-2)^3(t+2)}{8}.
\]

Adding the two parts together, the total count for \(C_2^\perp\) for this type is:
\[
\frac{t^2(t-2)(t^2-2)}{8} + \frac{t(t-2)^3(t+2)}{8} = \frac{t(t-2)(t^3-2t^2-3t+4)}{4}.
\]

3.1.4. Three \(-1\)'s

Let \(c\) be a codeword in \(C^\perp\) with three \(-1\)'s. Suppose that the ones correspond to columns \([x]\), \([y]\) and \([z]\). Now since \(3 \leq x + y + z \leq 3t\), the only way for \(Gc^T\) to be 0 \mod \(p\) is when \(x + y + z = p\). Since \(3 \nmid p\), then \(x, y, z\) cannot all be equal. Therefore, we have two cases.

First we consider the case when two of \(x, y, z\) are the same. Assume that \(y = z\). Then \(x\) is odd, and when \(x = 2i + 1\), \(y = z = t - i\). Therefore, the number of such codewords in \(C^\perp\) is
\[
\sum_{i=0}^{(t-2)/2} a_{2i+1} \left( \frac{a_{t-i}}{2} \right).
\]
Thus, the count for $C_1^+$ is:

$$\sum_{i=1}^{(t-2)/2} \frac{t^2(t-1)}{2} = \frac{1}{4}t^2(t-1)(t-2).$$

And, the count for $C_2^+$ is:

$$2(t-1)\frac{(t-2)(t-3)}{2} + \sum_{i=1}^{(t-2)/2} \frac{(t-2)^2(t-3)}{2} = \frac{1}{4}t^2(t-2)(t-3).$$

Let $M$ be the number of ways $p$ can be written as a sum of three distinct integers between 1 and $t$. Notice that 1 will never appear in such a partition. Indeed, if a partition contains 1, then the sum of the other two parts is 2$t$, this means that the other two parts are each equal to $t$, and so the parts are not distinct. Therefore, this situation accounts for $Mt^3$ codewords in $C_1^+$ and $M(t-2)^3$ codewords in $C_2^+$. To find $M$, let $Q(n,k)$ be the number of ways to write $n$ as a sum of $k$ distinct positive integers. Then by [1, p. 116] and [4, p. 45], $Q(p,3) = \lfloor (p-3)^2/12 \rfloor$, where $\lfloor \rfloor$ is the nearest integer function. Since our range is from 1 to $t$, we need to subtract the partitions when $t+1, t+2, \ldots, p-2$ appear in the partition. But $p-i$ appears $Q(i,2)$ times, for $i = 2, \ldots, t$. By [1, p. 116], $Q(i,2) = \lfloor (i-1)/2 \rfloor$.

$$\sum_{i=2}^{t} Q(i,2) = \sum_{i=2}^{t} \left\lfloor \frac{i-1}{2} \right\rfloor.$$

Notice that $\lfloor k/2 \rfloor = (k/2) - (1/2)$ for positive odd $k$, and $\lfloor k/2 \rfloor = k/2$ for even $k$. Since the interval $1 \leq i-1 \leq t-1$ contains $t/2$ positive odd integers, then

$$\sum_{i=2}^{t} \left\lfloor \frac{i-1}{2} \right\rfloor = \sum_{i=2}^{t} \frac{i-1}{2} - \frac{t}{4} = \frac{t(t-1)}{4} - \frac{t}{4} = \frac{1}{4}t(t-2),$$

and

$$M = Q(p,3) - \sum_{i=2}^{t} Q(i,2) = \lfloor (p-3)^2/12 \rfloor - \frac{1}{4}t(t-2) = \lfloor (2t-2)^2/12 \rfloor - \frac{1}{4}t(t-2) = \lfloor (t-1)^2/3 \rfloor - \frac{1}{4}t(t-2).$$

Recall $\lfloor (t-1)^2/3 \rfloor$ is the nearest integer function. Since $(t-1)^2$ is an integer, then the possible values for $\lfloor (t-1)^2/3 \rfloor$ are $(t-1)^2/3$, $(t-1)^2 + 1/3$ or $(t-1)^2 - 1)/3$. This implies that the possible values for $M$ are:

$$\frac{1}{12}(t^2 - 2t + 4), \quad \frac{1}{12}(t^2 - 2t + 8), \quad \text{or} \quad \frac{1}{12}t(t-2). \quad (3.1)$$

Recall that $A_3(C_i^+)$ is the number of codewords of weight 3 in $C_i^+$ for $i = 1, 2$. Remember, we need to double our count in each type to account for the negatives of our types. Therefore,

$$A_3(C_1^+) = 2 \left( 2 \cdot \frac{(t-2)^2}{2} + \frac{t^2(t-2)(t^2 - 3t - 1)}{4} + \frac{t^2(t-1)(t-2)}{4} + Mt^3 \right)$$

$$= \frac{1}{2}t^2(t-2)(t^2 - 2t + 2) + 2Mt^3.$$
For $A_3(C^0_2)$, the count is
\[
A_3(C^0_2) = 2 \left( 2 \cdot \frac{(t-2)t^2}{2} + \frac{t(t-2)(t^3 - t^2 - 3t + 4)}{4} + \frac{t^2(t-3)(t-2)}{4} + M(t-2)^3 \right) \\
= \frac{1}{2}t(t-2)(t+2)(t^2-2t+2) + 2M(t-2)^3.
\]

Therefore, $A_3(C^1_2) = A_3(C^2_2)$ if and only if
\[
0 = \frac{1}{2}t^2(t-2)(t^2-2t+2) - \frac{1}{2}t(t-2)(t+2)(t^2-2t+2) + 2Mt^3 - 2M(t-2)^3 \\
= -t(t-2)(t^2-2t+2) + M(12t^2 - 24t + 16).
\]

It follows that $A_3(C^1_2) = A_3(C^2_2)$ if and only if
\[
M = \frac{t(t-2)(t^2-2t+2)}{12t^2 - 24t + 16}.
\]

This expression clearly does not match any of the formulas for $M$ from equation 3.1. Nonetheless, the above expression and the earlier formulas may yield the same value for $M$ for certain values of $t$. The values of $t$ on which the expression $((t^2 - 2t + 2)(t-2)/t)/(12t^2 - 24t + 16)$ agrees with the first two formulas in equation 3.1 are not integers. But, the expression $((t^2 - 2t + 2)(t-2)/t)/(12t^2 - 24t + 16)$ agrees with $t(t-2)/12$, the third formula from equation 3.1, when $t = 0$ and $t = 2$. Notice that $t = 2$ when $p = 5$. This means that $A_3(C^1_2)$ and $A_3(C^2_2)$ are in fact equal when $p = 5$, so that this construction does not provide a counterexample in the case $p = 5$. Otherwise, for $p \geq 13$, we indeed have $A_3(C^0_2) \neq A_3(C^1_2)$, and therefore $\text{lwe}_{C^0_2} \neq \text{lwe}_{C^1_2}$ for all $p \equiv 1 \mod 4, p \geq 13$.

### 3.2. Primes congruent to 3 modulo 4

In this subsection $p \equiv 3 \mod 4$. Then $t = (p-1)/2$ is an odd number. We will use the same setup from the previous subsection. In most of the cases, the only differences are the upper limits of the summations.

#### 3.2.1. $A_1$ and $a = -2$

To have $x - 2y = 0, x$ must be even. So if $x = 2i$, then $y = i$, and the number of ways that this happens is $a_{2i}a_i$. Therefore, the number of such codewords in $C^1_2$ is
\[
\sum_{i=1}^{(t-1)/2} a_{2i}a_i.
\]

Therefore, the count for $C^1_2$, given that $a_1 = 0$ and $a_i = t$ for all $2 \leq i \leq t$, is:
\[
\sum_{i=1}^{(t-1)/2} a_{2i}a_i = \sum_{i=2}^{(t-1)/2} t^2 = t^2 \left( \frac{t-1}{2} - 1 \right) = \frac{t^2(t-3)}{2}.
\]

And, the count for $C^2_2$, given that $a_1 = 2(t-1)$ and $a_i = (t-2)$ for all $2 \leq i \leq t$, is:
\[
\sum_{i=1}^{(t-1)/2} a_{2i}a_i = 2(t-1)(t-2) + \sum_{i=2}^{(t-1)/2} (t-2)^2 = \frac{(t-2)(t^2 - t + 2)}{2}.
\]
3.2.2. A 1 and a 2

To have \( x + 2y = p = 2t + 1 \), \( x \) must be odd. So if \( x = 2i + 1 \), then \( y = t - i \), and the number of ways that this happens is \( a_{2i+1}a_{t-i} \). Therefore, the number of such codewords in \( C^\perp_1 \) is

\[
\sum_{i=0}^{(t-1)/2} a_{2i+1}a_{t-i}.
\]

Thus, the count for \( C^\perp_1 \) is:

\[
\sum_{i=0}^{(t-1)/2} a_{2i+1}a_{t-i} = \sum_{i=1}^{(t-1)/2} t^2 = \frac{t^2(t-1)}{2}.
\]

And, the count for \( C^\perp_2 \) is:

\[
\sum_{i=0}^{(t-1)/2} a_{2i+1}a_{t-i} = 2(t-1)(t-2) + \sum_{i=1}^{(t-1)/2} (t-2)^2 = \frac{(t-1)(t-2)(t+2)}{2}.
\]

3.2.3. A 1 and two \(-1\)'s

We need to solve \( x = y + z \). If \( x = 2i \) is even, then the possibilities for \( y \) and \( z \) are \( y = j \) and \( z = 2i - j \) for \( 1 \leq j \leq i \). Similarly, if \( x = 2i + 1 \), then \( y = j \) and \( z = 2i + 1 - j \) for \( 1 \leq j \leq i \). Therefore, the number of such codewords in \( C^\perp_1 \) is

\[
\sum_{i=1}^{(t-1)/2} \left( \begin{array}{c} (t-1)/2 \\ i-1 \\ \end{array} \right) a_{2i} \left( \sum_{j=1}^{i-1} a_ja_{2i-j} + \left( \begin{array}{c} a_i \\ 2 \\ \end{array} \right) \right) + \sum_{i=1}^{(t-1)/2} \left( \begin{array}{c} i \\ t \\ \end{array} \right) a_{2i+1} \left( \sum_{j=1}^{i} a_ja_{2i+1-j} \right).
\]

Therefore, the count for \( C^\perp_1 \) is:

\[
\sum_{i=1}^{(t-1)/2} t \left( \sum_{j=2}^{i-1} j^2 + \frac{t(t-1)}{2} \right) + \sum_{i=1}^{(t-1)/2} t \left( \sum_{j=2}^{i} j^2 \right) = \sum_{i=1}^{(t-1)/2} t (i-2)^2 + \frac{t(t-1)}{2} + \sum_{i=1}^{(t-1)/2} t (t^2(i-1)) = \frac{t^2(t-3)(t^2-2t-1)}{4}.
\]

And, the count for \( C^\perp_2 \) is:

First part:

\[
= a_2 \left( \begin{array}{c} (t-1)/2 \\ 2 \\ \end{array} \right) + \sum_{i=2}^{(t-1)/2} a_{2i} \left( a_1a_{2i-1} + \sum_{j=2}^{i-1} a_ja_{2i-j} + \left( \begin{array}{c} a_i \\ 2 \\ \end{array} \right) \right) = (t-2) \frac{(2t-2)(2t-3)}{2} + \sum_{i=2}^{(t-1)/2} \left( 2(t-1)(t-2)^2 + \sum_{j=2}^{i-1} (t-2)^3 + \frac{(t-2)^2(t-3)}{2} \right) = \frac{(t-1)(t-2)(t^2 - t + 2)}{8}.
\]

Second part:

\[
\sum_{i=1}^{(t-1)/2} \left( \begin{array}{c} (t-1)/2 \\ i \\ \end{array} \right) (2t-1)(t-2) + (t-2)^2(i-1) = \frac{(t-1)(t-2)^2(t^2 + 3t - 2)}{8}.
\]
Adding the two parts together, the total count for $C^1_z$ for this type is:

$$
\frac{(t-2)(t+2)(t-1)^3}{4}.
$$

3.2.4. Three 1’s

We consider the same two cases as in paragraph 3.1.4. First we consider the case when two of $x, y$ and $z$ are the same. Assume that $y = z$. Then $x$ must be odd, and when $x = 2i + 1, y = z = t - i$. Therefore, the number of such codewords in $C^1_z$ is

$$
\sum_{i=0}^{(t-1)/2} a_{2i+1}\binom{a_{t-1}}{2}.
$$

Thus, the count for $C^1_z$ is:

$$
\frac{(t-1)/2}{2} t^2(t-1) = \frac{1}{4} t^2(t-1)^2.
$$

And, the count for $C^2_z$ is:

$$
2(t-1)\frac{(t-2)(t-3)}{2} + \sum_{i=1}^{(t-1)/2} \frac{(t-2)^2(t-3)}{2} = \frac{1}{4} (t-1)(t-2)(t-3)(t+2).
$$

Recall $M$ is the number of ways $p$ can be written as a sum of three distinct integers between 1 and $t$. Notice that 1 will never appear in such a partition. This case accounts for $Mt^3$ codewords in $C^1_z$ and $M(t-2)^3$ codewords in $C^2_z$.

We know that $M = Q(p, 3) - \sum_{i=2}^{t} Q(i, 2) = Q(p, 3) - \sum_{i=2}^{t}(i-1)/2$. Since $1 \leq i - 1 \leq t - 1$ and there are $(t-1)/2$ odd numbers in the interval $[1, t-1]$, then

$$
\sum_{i=2}^{t} \left| \frac{i-1}{2} \right| = \sum_{i=2}^{t} \frac{i-1}{2} - \frac{t-1}{4} = \frac{t(t-1)}{4} - \frac{t-1}{4} = \frac{(t-1)^2}{4}.
$$

Hence,

$$
M = \left[ \frac{(t-1)^2}{3} \right] - \frac{1}{4} (t-1)^2.
$$

The possible values for $[(t-1)^2/3]$ are $(t-1)^2/3$, $(t-1)^2/3 + 1/3$ or $(t-1)^2 - 1)/3$. This implies that the possible values for $M$ are:

$$
\frac{1}{12}(t-1)^2, \frac{1}{12}(t^2 - 2t + 5), \text{ or } \frac{1}{12}(t^2 - 2t - 3). \tag{3.2}
$$

Here are the total counts of codewords of weight 3 in $C^1_z$ and $C^2_z$.

$$
A_3(C^1_z) = \frac{(t-3)^2}{2} + \frac{(t-1)^2}{2} + \frac{t^2(t-3)(t^2-2t-1)}{4} + \frac{t^2(t-1)^2}{4} + Mt^3.
$$

$$
A_3(C^2_z) = 2 \left( \frac{(t-2)(t^2-t+2)}{2} + \frac{(t-1)(t-2)(t+2)}{4} + \frac{(t-2)(t+2)(t-1)^3}{4} \right)
\frac{(t-1)(t-2)(t-3)(t+2)}{4} + M(t-2)^3.
$$

$$
= \frac{(t-2)(t^4-t^2+4)}{2} + 2M(t-2)^3.
$$
Therefore, $A_3(C_1^+) = A_3(C_2^+)$ if and only if

$$0 = \frac{t^2(t - 1)(t^2 - 3t + 4)}{2} - \frac{(t - 2)(t^4 - t^2 + 4)}{2} + 2Mt^3 - 2M(t - 2)^3$$

$$= -t^4 + 4t^3 - 3t^2 - 2t + 4 + M(12t^2 - 24t + 16).$$

It follows that $A_3(C_1^+) = A_3(C_2^+)$ if and only if

$$M = \frac{t^4 - 4t^3 + 3t^2 + 2t - 4}{12t^2 - 24t + 16}.$$

This expression clearly does not match any of the formulas for $M$ from equation 3.2. Moreover, the only integers at which the expression $(t^4 - 4t^3 + 3t^2 + 2t - 4)/(12t^2 - 24t + 16)$ agrees with any of the formulas in equation 3.2 are $t = 0$ and $t = 2$. These are not possible values for $t$ when $p \equiv 3 \mod 4$. Hence, for $p \geq 7$, we indeed have $A_3(C_1^+) \neq A_3(C_2^+)$, and therefore $\text{lwe}_{C_1^+} \neq \text{lwe}_{C_2^+}$ for all $p \equiv 3 \mod 4$, $p \geq 7$.

We summarize the results for primes $p \geq 7$.

**Proposition 3.3.** For each prime $p \geq 7$, there exist linear codes $C_1$ and $C_2$ over $\mathbb{Z}_p$ with $\text{lwe}_{C_1} = \text{lwe}_{C_2}$ and $A_3(C_1^+) \neq A_3(C_2^+)$.  

4. Propagation of examples and proof of the main theorem

We would like to say that if $C_1$ and $C_2$ give an example with $\text{lwe}_{C_1} = \text{lwe}_{C_2}$ and $\text{lwe}_{C_1^+} \neq \text{lwe}_{C_2^+}$ over $\mathbb{Z}_m$, then $aC_1$ and $aC_2$ provide a corresponding example over $\mathbb{Z}_{am}$, for any positive integer $a$. Although the previous statement is not true in that generality, a weaker version is true and is sufficient to cover all integers $m \geq 5$. We need the following construction.

Let $C$ be a linear code over $\mathbb{Z}_m$. For any positive integer $a$, define $aC$ to be the code over $\mathbb{Z}_{am}$ given by

$$aC = \{(ac_1, \ldots, ac_n) \in \mathbb{Z}_{am}^n : (c_1, \ldots, c_n) \in C\}.$$  

**Lemma 4.1.** Let $C$ be a linear code of length $n$ over $\mathbb{Z}_m$ and $a$ be a positive integer. Then the weight distribution of the code $aC$ is given in terms of the weight distribution of $C$ as follows.

$$A_q(aC) = \begin{cases} 0, & a \nmid q, \\ A_q(a,C), & a \mid q, \end{cases}$$

for all $0 \leq q \leq \lfloor am/2 \rfloor n$.

**Proof.** Let $c \in C$. Assume all entries of $c$ satisfy $-m/2 < c_i < m/2$, for all $i$. This implies that $-am/2 < ac_i < am/2$ for all entries of $ac \in aC$. Therefore

$$L_{am}(ac) = \sum_{i=1}^{n} |ac_i| = a \sum_{i=1}^{n} |c_i| = aL_m(c).$$

Notice that the map $C \to aC$, with $c \mapsto ac$, is a bijection. Therefore the result follows.  

**Corollary 4.2.** Let $C_1$ and $C_2$ be linear codes of length $n$ over $\mathbb{Z}_m$ and $a$ be a positive integer. If $\text{lwe}_{C_1} = \text{lwe}_{C_2}$, then $\text{lwe}_{aC_1} = \text{lwe}_{aC_2}$.

The next result examines the count of low weight codewords in dual codes of the form $(aC)^\perp$.

**Lemma 4.3.** Let $C$ be a linear code of length $n$ over $\mathbb{Z}_m$. Suppose that $a$ and $b$ are integers with $1 \leq a \leq b$. Then $A_q((aC)^\perp) = A_q((bC)^\perp)$ for $q < am/2$. In particular, $A_q(C^+) = A_q((bC)^\perp)$ for $q < m/2$.  

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Proof. Recall that we view $\mathbb{Z}_{am} = \{i : -am/2 < i \leq am/2\}$, and similarly for $\mathbb{Z}_{bm}$. Notice that a vector $y$ in $\mathbb{Z}_{am}^n$ can then be thought of as a vector in $\mathbb{Z}_{bm}^n$ with the same entries and the same Lee weight. Conversely, for $z \in \mathbb{Z}_{bm}^n$, under the hypothesis that $L(z) < am/2$, we can view $z$ as a vector in $\mathbb{Z}_{am}^n$ with the same entries and the same Lee weight. In other words, for $0 \leq q < am/2$, the sets $\{y \in \mathbb{Z}_{am}^n : L(y) = q\}$ and $\{z \in \mathbb{Z}_{bm}^n : L(z) = q\}$ are equal, if we ignore the modulus and think of the elements as just integer vectors.

Now let $0 \leq q < am/2$ and let $y$ be such that $-am/2 < y_i < am/2$ with $\sum_{i=1}^n |y_i| = q$, i.e., $l_{am}(y) = l_{bm}(y) = q$. We then have the following list of equivalent statements:

- $y \in (a\mathcal{C})^\perp$
- $\sum_{i=1}^n y_i(ac_i) \equiv 0 \mod am$, for all $c \in \mathcal{C}$
- $am|\sum_{i=1}^n y_i(ac_i)$, for all $c \in \mathcal{C}$
- $m|\sum_{i=1}^n y_i(c)$, for all $c \in \mathcal{C}$
- $bm|\sum_{i=1}^n y_i(bc_i)$, for all $c \in \mathcal{C}$
- $y \in (b\mathcal{C})^\perp$

This shows that $A_q(a\mathcal{C})^\perp = A_q(b\mathcal{C})^\perp$. \hfill \Box

**Corollary 4.4.** Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be linear codes of length $n$ over $\mathbb{Z}_m$. Let $b$ and $q$ positive integers with $q < m/2$. If $A_q(\mathcal{C}_1^\perp) \neq A_q(\mathcal{C}_2^\perp)$, then $A_q((b\mathcal{C}_1)^\perp) \neq A_q((b\mathcal{C}_2)^\perp)$.

**Proof of Theorem 1.4.** Suppose that $m \geq 5$. We will consider cases based on minimal factors of $m$ that are also greater than or equal to 5. For instance, if $m$ is not divisible by any prime $p$, $p \geq 5$, then $m$ is divisible only by powers of 2 and powers of 3, say $m = 2^r3^s$. The smallest such numbers $2^r3^s$ that are themselves greater than or equal to 5 are 6, 8 and 9, as seen in the following diagram.

![Diagram showing cases for $m$ and $q$.](attachment:diagram.png)

Sections 2 and 3 provided examples of linear codes $\mathcal{C}_1, \mathcal{C}_2$ over $\mathbb{Z}_m$ satisfying $lwe_{\mathcal{C}_1} = lwe_{\mathcal{C}_2}$ and $A_q(\mathcal{C}_1^\perp) \neq A_q(\mathcal{C}_2^\perp)$ for the cases $m = 5, 6, 8, 9$ and $m = p$ prime, $p \geq 7$. To extend these examples from $\mathbb{Z}_m$ to $\mathbb{Z}_{am}$ we need to verify that the numbers $m$ and $q$ satisfy the hypotheses of Corollary 4.4. We summarize the various cases in the following chart:

<table>
<thead>
<tr>
<th>$m$</th>
<th>5</th>
<th>6</th>
<th>8</th>
<th>9</th>
<th>$p$, $p \geq 7$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$q : A_q(\mathcal{C}_1^\perp) \neq A_q(\mathcal{C}_2^\perp)$</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>1</td>
<td>3</td>
</tr>
</tbody>
</table>

Since $q < m/2$ in all of these cases, Corollary 1.4 implies that the examples provided in Sections 2 and 3 yield examples with $lwe_{a\mathcal{C}_1} = lwe_{a\mathcal{C}_2}$ and $lwe_{a\mathcal{C}_1^\perp} \neq lwe_{a\mathcal{C}_2^\perp}$ over $\mathbb{Z}_{am}$ for any positive integer $a$. Since any $m \geq 5$ is necessarily a multiple of $m = 5, 6, 8, 9$ or $m = p$ prime, $p \geq 7$, the result follows. \hfill \Box
5. Future work

It is natural to consider the same problem for other weights on $\mathbb{Z}_m$, specifically the Euclidean weight and the homogeneous weight. The Euclidean weight $e$ is defined on $\mathbb{Z}_m$ by $e(a) = l(a)^2$ for $a \in \mathbb{Z}_m$.

The Euclidean weight coincides with the Lee weight and the Hamming weight over $\mathbb{Z}_2$ and $\mathbb{Z}_3$, thus the MacWilliams identities hold for the Euclidean weight enumerator in these cases. The following example shows the failure of the MacWilliams identities for the Euclidean weight enumerator over $\mathbb{Z}_4$.

Example 5.1. Let $m = 4$. Let

$$
G_1 = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 2 \\
1 & 1 & 0 & 0 & 1 & 1 & 2 & 2 & 3 & 3 & 1
\end{bmatrix},
$$

$$
G_2 = \begin{bmatrix}
0 & 0 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 2 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 2 & 2 & 2 & 0
\end{bmatrix}.
$$

Then $\text{ewe}_{C_4^1}(X,Y) = \text{ewe}_{C_4^2}(X,Y) = X^{48} + 12X^{32}Y^{16} + 3X^{16}Y^{32}$ and $\text{ewe}_{C_4^1} \neq \text{ewe}_{C_4^2}$. In fact $A_1(C^1_4) \neq A_1(C^2_4)$; because of the zero column in $G_2$, there are codewords of Euclidean weight 1 in $C^1_4$ but not in $C^2_4$.

The homogeneous weight over $\mathbb{Z}_m$ was defined by Constantinescu and Heise [2]. We will not give the general definition here; rather we simply state the values of the homogeneous weight for specific $m$ as needed.

For $m = p$, a prime, the homogeneous weight on $\mathbb{Z}_p$ is just a scaled version of the Hamming weight. For $m = 4$, the homogeneous weight equals the Lee weight. For $m = 4$, the homogeneous weight equals the Lee weight. In those cases, the MacWilliams identities hold.

Here are two examples where the MacWilliams identities fail for the homogeneous weight enumerator.

Example 5.2. Let $m = 6$. The homogeneous weight $w$ has the following values

$$
\begin{array}{c|ccccccc}
a & 0 & 1 & 2 & 3 & 4 & 5 \\
w(a) & 0 & 1 & 3 & 4 & 3 & 1
\end{array}
$$

Set

$$
G_1 = [1 \ 1 \ 1], \quad G_2 = [1 \ 3 \ 3].
$$

Then $\text{howe}_{C_6^1}(X,Y) = \text{howe}_{C_6^2}(X,Y) = X^{12} + 2X^9Y^3 + 2X^3Y^9 + Y^{12}$, but $A_2(C^1_6) = 6$ and $A_2(C^2_6) = 4$.

Example 5.3. This example is due to the referee of [3, Example 5.6]. Let $m = 8$, and use the following values for $w$:

$$
\begin{array}{c|ccccccc}
a & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
w(a) & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1
\end{array}
$$

Set

$$
G_1 = [1 \ 1 \ 4], \quad G_2 = [2 \ 2 \ 4 \ 4 \ 0].
$$

Then $\text{howe}_{C_8^1}(X,Y) = \text{howe}_{C_8^2}(X,Y) = X^6 + 2X^4Y^2 + 5X^2Y^4$, but $A_2(C^1_8) = 7$ and $A_2(C^2_8) = 23$.

To our knowledge, the validity of the MacWilliams identities for the Euclidean weight enumerator or the homogeneous weight enumerator over $\mathbb{Z}_m$ is still open in general. Except for the cases described above where the MacWilliams identities are known to hold, we expect the identities to fail.

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