Isometry Groups of Additive Codes

Jay A. Wood

Department of Mathematics
Western Michigan University
http://homepages.wmich.edu/∼jwood/

On the Algebraic and Geometric Classifications of Projective Varieties
University of Messina
June 23, 2016

5. Isometries of additive codes

- Additive codes as linear codes over modules
- Failure of EP
- Monomial and isometry groups
- Examples
- Criteria in terms of multiplicity functions
- Structure of ker W
- Building codes with prescribed groups
- EP for short codes
- Extreme examples

Additive $\mathbb{F}_4$-codes

- There has been interest in additive codes with alphabet $A = \mathbb{F}_4$.
- Such codes are the same as R-linear codes over A with $R = \mathbb{F}_2$ and $A = \mathbb{F}_4$, regarding $\mathbb{F}_4$ as an $\mathbb{F}_2$-vector space of dimension 2.
- Generalize to case of $R = M_{k \times k}(\mathbb{F}_q)$ and $A = M_{k \times \ell}(\mathbb{F}_q)$. Information module will be $M = M_{k \times m}(\mathbb{F}_q)$. Use Hamming weight wt on A.
- Call this the **matrix module context**.

Failure of EP

- Recall that EP for Hamming weight fails in the matrix module context when $k < \ell$ and $k < m$.
- In terms of the W-map:

$$W : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q})$$

is not injective for some information module M.

Isometry group

- General set-up: ring R, alphabet A, weight w on A.
- Let $C \subseteq A^n$ be an R-linear code.
- Consider linear isometries $f : C \to C$; i.e., $w(cf) = w(c)$, for all $c \in C$.
- When C is given as the image of a parametrized code $\Lambda : M \to A^n$, we define the **isometry group**:

$$\begin{aligned}\mathrm{Isom}(C) = \{g \in GL_R(M) : & \text{ there exists a linear} \\ & \text{isometry } f : C \to C \text{ such that } g\Lambda = \Lambda f\}.\end{aligned}$$

- View isometries on M rather than C.

Monomial group

- Recall that the weight w on A has a right symmetry group G_{rt}.
- For linear code $C \subseteq A^n$, define the **monomial group**

$$\mathcal{M}(C) = \{T : A^n \to A^n, G_{\mathrm{rt}}\text{-monomial transformation, with } CT = C\}.$$

Restriction map

- Any $T \in \mathcal{M}(C)$, when restricted to C, gives an isometry on C. By viewing the isometry on M, we get a group homomorphism

$$\text{restr} : \mathcal{M}(C) \to \text{Isom}(C).$$

- Denote $\ker \text{restr} = \mathcal{M}_0(C)$. Think of repeated columns in a generator matrix.
- Denote image of $\mathcal{M}(C)$ under restr by $\mathrm{r}\mathcal{M}(C)$.
- If EP holds, then restr is surjective: $\mathrm{r}\mathcal{M}(C) = \text{Isom}(C)$.

Main question

- When EP fails, restr will not be surjective for some linear codes C or information modules M.
- Then $\mathrm{r}\mathcal{M}(C) \subseteq \mathrm{Isom}(C) \subseteq GL_R(M)$.
- Which subgroups of $GL_R(M)$ can occur as $\mathrm{r}\mathcal{M}(C)$ and $\mathrm{Isom}(C)$?

Example 1 (a)

- Additive code over $\mathbb{F}_4 = \mathbb{F}_2[\omega]/(\omega^2+\omega+1)$ with generator matrix G_1 and list of codewords. $M = \mathbb{F}_2^3$.

$$G_1 = \begin{bmatrix} 1 & \omega & 0 \\ \omega & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \begin{matrix} 0 & 0 & 0 \\ 1 & \omega & 0 \\ \omega & 1 & 0 \\ \omega^2 & \omega^2 & 0 \\ 1 & 0 & 1 \\ 0 & \omega & 1 \\ \omega^2 & 1 & 1 \\ \omega & \omega^2 & 1 \end{matrix}$$

Example 1 (b)

- Consider three elements of $GL_R(M) = GL(3, \mathbb{F}_2)$:

$$f_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

- f_1, f_2 generate $\mathrm{r}\mathcal{M}(C)$, a Klein 4-group. But f_1, f_3 generate $\mathrm{Isom}(C)$, a dihedral group of order 8. ($f_2 = f_1 f_3^2$.)
- Magma found only the cyclic 2-group generated by $f_1 f_2$.

Example 2 (a)

- Additive code over $\mathbb{F}_4$ with generator matrix G_2 and list of codewords. Again, $M = \mathbb{F}_2^3$.

$$G_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega \\ \omega & \omega & 1 & 0 & \omega^2 \end{bmatrix},$$

$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega \\ 1 & 1 & 0 & \omega^2 & \omega^2 \\ \omega & \omega & 1 & 0 & \omega^2 \\ \omega & \omega^2 & 0 & 1 & \omega \\ \omega^2 & \omega & 0 & \omega & 1 \\ \omega^2 & \omega^2 & 1 & \omega^2 & 0 \end{array}$$

Example 2 (b)

- Consider three elements of $GL_R(M) = GL(3, \mathbb{F}_2)$:

$$f_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad f_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad f_6 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

- These elements generate $\mathrm{r}\mathcal{M}(C) \cong \Sigma_4$, the symmetric group on 4 elements, while $\mathrm{Isom}(C) = GL(3, \mathbb{F}_2)$, the simple group of order 168.
- Magma found only a cyclic 4-group generated by $f = f_4 f_5 f_6 f_4 f_5 f_4 f_6$.

Closure for group actions

- Some of the hypotheses of the main result involve a notion of **closure** with respect to a group action.
- This idea goes back at least to Wielandt, 1964.
- Suppose a finite group G acts on a set X.
- A subgroup $H \subseteq G$ partitions X into H-orbits.
- Define the **closure** of H with respect to the action:

$$\bar{H} = \{g \in G : g \cdot \mathrm{orb}_H(x) = \mathrm{orb}_H(x), x \in X\}.$$

- Subgroup $H \subseteq G$ is **closed** with respect to the action if $\bar{H} = H$.

Aside: stabilizer subgroups

- Let G act on X.
- The **stabilizer subgroup** of $x \in X$ is

$$\mathrm{Stab}_G(x) = \{g \in G : g \cdot x = x\}.$$

Aside: not every subgroup is a stabilizer subgroup of a given action

- Let $G = \Sigma_3$, the symmetric group on three objects, acting on $X = \{1, 2, 3\}$.
- Then $\mathrm{Stab}_{\Sigma_3}(1) = \langle (2,3) \rangle$, the cyclic 2-subgroup generated by the transposition $(2, 3)$.
- The cyclic 3-subgroup $\langle (1,2,3) \rangle$ is not a stabilizer subgroup for this action. Neither are $\{\mathrm{id}_X\}$ and Σ_3.

Aside: closure and stabilizers

- Let G act on X.
- Let $F(X, \mathbb{C})$ be the vector space of all complex-valued functions on X. There is an induced action of G on $F(X, \mathbb{C})$.
- A subgroup $H \subseteq G$ is closed with respect to the action on X if and only if H is a stabilizer subgroup for the action on $F(X, \mathbb{C})$.
- Proof is an exercise: separate H-orbits.
- End of aside.

Closure conditions

- Usual set-up: ring R, alphabet A, weight w, information module M. Orbit spaces $\mathcal{O}$ and $\mathcal{O}^\sharp$.
- $\mathcal{O} = G_{\mathrm{lt}} \backslash M$: $GL_R(M)$ acts on the right, and on the left of $F_0(\mathcal{O}, \mathbb{Q})$.
- $\mathcal{O}^\sharp = \mathrm{Hom}_R(M, A)/G_{\mathrm{rt}}$: $GL_R(M)$ acts on the left, and on the right of $F_0(\mathcal{O}^\sharp, \mathbb{Q})$: $(\eta f)([\lambda]) = \eta([f\lambda])$.
- For $H_1 \subseteq H_2 \subseteq GL_R(M)$, will want H_1 to be closed for the $\mathcal{O}^\sharp$-action and H_2 closed for the $\mathcal{O}$-action.
- "Not every subgroup gets to be an isometry group."

Statement of main result

Theorem

Matrix module context with $k < \ell < m$. *For any choice of subgroups* $H_1 \subseteq H_2 \subseteq GL_R(M)$ *with* H_1 *closed for the* $\mathcal{O}^\sharp$*-action and* H_2 *closed for the* $\mathcal{O}$*-action, there exists a linear code* C *modeled on* M *such that* $H_1 = \mathrm{r}\mathcal{M}(C)$ *and* $H_2 = \mathrm{Isom}(C)$.

Corollary

Same matrix module context. There exists a linear code C *modeled on* M *with* $\mathrm{r}\mathcal{M}(C) = \{\mathbb{F}_q^\times \cdot \mathrm{id}_M\}$ *and* $\mathrm{Isom}(C) = GL_R(M)$.

Aside: generalized context

- Suppose a finite group G acts on two finite sets X and Y.
- Suppose $f : X \to Y$ is a G-equivariant map; i.e., $f(g \cdot x) = g \cdot f(x)$, for all $g \in G$ and $x \in X$.
- Equivariance implies: for any $x \in X$,

$$\mathrm{Stab}_G(x) \subseteq \mathrm{Stab}_G(f(x)).$$

- If $g \cdot x = x$, then $g \cdot f(x) = f(g \cdot x) = f(x)$.

Aside: assume injectivity

- Now assume $f : X \to Y$ is injective.
- Then, for any $x \in X$,

$$\mathrm{Stab}_G(x) = \mathrm{Stab}_G(f(x)).$$

- If $g \cdot f(x) = f(x)$, then $f(g \cdot x) = g \cdot f(x) = f(x)$.
- f injective implies $g \cdot x = x$.

Aside: general f

- General $f : X \to Y$, so $\mathrm{Stab}_G(x) \subseteq \mathrm{Stab}_G(f(x))$.
- Let $\mathcal{S}_X = \{\mathrm{Stab}_G(x) : x \in X\}$ and $\mathcal{S}_Y$ similarly.
- Given $H_1 \in \mathcal{S}_X$ and $H_2 \in \mathcal{S}_Y$, with $H_1 \subseteq H_2 \subseteq G$, does there exist $x \in X$ with $H_1 = \mathrm{Stab}_G(x)$ and $H_2 = \mathrm{Stab}_G(f(x))$?
- That is, can we achieve *pairs* of stabilizer groups?
- The various conditions stated are clearly necessary.
- One more: $H_2 \in \mathcal{S}_{f(X)}$. Redundant if f is onto.
- End of aside.

Statement of main result (again)

Theorem

Matrix module context with $k < \ell < m$. For any choice of subgroups $H_1 \subseteq H_2 \subseteq GL_R(M)$ with H_1 closed for the $\mathcal{O}^\sharp$-action and H_2 closed for the $\mathcal{O}$-action, there exists a linear code C modeled on M such that $H_1 = \mathrm{r}\mathcal{M}(C)$ and $H_2 = \mathrm{Isom}(C)$.

Corollary

Same matrix module context. There exists a linear code C modeled on M with $\mathrm{r}\mathcal{M}(C) = \{\mathbb{F}_q^\times \cdot \mathrm{id}_M\}$ and $\mathrm{Isom}(C) = GL_R(M)$.

Using multiplicity functions

- Up to G_{rt}-monomial transformations, a parametrized code $\Lambda : M \to A^n$ is determined by its multiplicity function $\eta_\Lambda \in F_0(\mathcal{O}^\sharp, \mathbb{N})$.
- Recall the right action of $GL_R(M)$ on $F_0(\mathcal{O}^\sharp, \mathbb{Q})$: $(\eta f)([\lambda]) = \eta([f\lambda])$. Left action via ηf^{-1}.
- $W : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q})$ has $W(f\eta) = fW(\eta)$.
- For $f \in GL_R(M)$, $f \in \mathrm{r}\mathcal{M}(\eta)$ if and only if $\eta f = \eta$.
- For $f \in GL_R(M)$, $f \in \mathrm{Isom}(\eta)$ if and only if $\eta f - \eta \in \ker W$.

Interpretation as stabilizer subgroups

- W map satisfies $W(f\eta) = fW(\eta)$.
- For $f \in GL_R(M)$, $f \in \mathrm{r}\mathcal{M}(\eta)$ if and only if $\eta f = \eta$.
- That is, $\mathrm{r}\mathcal{M}(\eta)$ is the stabilizer subgroup of η for the action of $GL_R(M)$ on $F_0(\mathcal{O}^\sharp, \mathbb{Q})$.
- For $f \in GL_R(M)$, $f \in \mathrm{Isom}(\eta)$ if and only if $\eta f - \eta \in \ker W$ if and only if $W(\eta f) = W(\eta)$.
- That is, $\mathrm{Isom}(\eta)$ is the stabilizer subgroup of $W(\eta)$ for the action of $GL_R(M)$ on $F_0(\mathcal{O}, \mathbb{Q})$.

Structure of ker W (a)

- In the matrix module context, $\mathcal{O}^\sharp$ is the set of CRE matrices of size $m \times \ell$, while $\mathcal{O}$ is the set of RRE matrices of size $k \times m$.
- Remember $k < \ell < m$. By dimension counting,

$$\ker W \geq \sum_{i=k+1}^{\ell} \begin{bmatrix} m \\ i \end{bmatrix}_q , \tag{1}$$

using q-binomial coefficients.

Structure of ker W (b)

- The orbit space $\mathcal{O}^\sharp$ is partitioned by rank.
- By explicit constructions, one produces independent elements $\eta_{[\lambda]} \in \ker W$. For each $i = k+1, \ldots, \ell$, one produces $\left[{m \atop i} \right]_q$ of them, each $\eta_{[\lambda]}$ supported on $[\lambda]$ of rank i and on specific elements of smaller rank. ("Triangular.") This produces as many independent elements of ker W as the sum in (1).
- Separately, one shows that W is surjective, so there is equality in (1), and we have an explicit basis for ker W. This part is somewhat technical.

Aside: EP for short codes

- Serhii Dyshko (Toulon) has shown that EP holds even when $k < \ell$, **provided** n is sufficiently small ($n \leq q$ when $k = 1$).
- Elements of ker W affect the length of the code.
- The exact details of this need to be better understood.
- End of aside.

Idea of proof (a)

- Elements $[x] \in \mathcal{O}$ have a well-defined rank, $\mathrm{rk}[x]$. The $GL_R(M)$-action preserves this rank.
- Pick a function w on $\mathcal{O}$ that (1) is constant on and separates the H_2-orbits on $\mathcal{O}$ and (2) is an increasing function of $\mathrm{rk}[x]$.
- Because W is surjective, there exists η with $W(\eta) = w$. A priori, η has rational values.
- Can modify η to have non-negative integer values and still satisfy (1) and (2).

Idea of proof (b)

- Replace η by an averaged version so that η is also constant on the H_2-orbits on $\mathcal{O}^\sharp$. This does not change $W(\eta)$. Clear denominators of η, which scales everything.
- At this point, η has non-negative integer values, is constant on H_2-orbits on $\mathcal{O}^\sharp$, and $W(\eta)$ is constant on and separates H_2-orbits on $\mathcal{O}$.

Idea of proof (c)

- Claim $\mathrm{r}\mathcal{M}(\eta) = \mathrm{Isom}(\eta) = H_2$.
- From η constant on H_2-orbits on $\mathcal{O}^\sharp$, $H_2 \subseteq \mathrm{r}\mathcal{M}(\eta)$.
- We always have $\mathrm{r}\mathcal{M}(\eta) \subseteq \mathrm{Isom}(\eta)$.
- Suppose $f \in \mathrm{Isom}(\eta)$. Because $w = W(\eta)$ separates H_2-orbits on $\mathcal{O}$, $w(xf) = w(x)$ implies $f \in \bar{H}_2$. The closure hypothesis implies $f \in H_2$.

Idea of proof (d)

- Modify η using $\eta_{[\lambda]} \in \ker W$ to separate H_1-orbits on $\mathcal{O}^\sharp$ (rank-by-rank, from rank ℓ down to rank $k+1$).
- Because of "triangular" form of $\eta_{[\lambda]}$, a change at rank i does not disturb changes at higher ranks.
- The final η preserves H_1-orbits on $\mathcal{O}^\sharp$, so $H_1 \subseteq \mathrm{r}\mathcal{M}(\eta)$. Conversely, any $f \in \mathrm{r}\mathcal{M}(\eta)$ preserves H_1-orbits on $\mathcal{O}^\sharp$ (η separates), so $f \in \overline{H}_1$. Closure implies $f \in H_1$.
- Because modifications were made by $\eta_{[\lambda]} \in \ker W$, $W(\eta)$ has not changed. We still have $\mathrm{Isom}(\eta) = H_2$.

Other alphabets

- Most of the result carries over to any alphabet with non-cyclic socle, such as non-Frobenius rings.
- Get $\mathrm{r}\mathcal{M}(\eta) \subseteq H_1$ only, but still have $H_2 = \mathsf{Isom}(\eta)$.
- This is enough to get the extreme cases.

Extreme example (a)

- $R = \mathbb{F}_2$, $A = \mathbb{F}_4$, $M = \mathbb{F}_2^3$. Multiplicities as indicated. Length $n = 28$.

multiplicity	1	4	2	2	4	1	3	5	6
	1	0	0	1	1	1	1	1	1
G	0	1	1	ω	ω	ω	ω	0	1
	1	0	1	0	ω	1	ω^2	ω	ω

- All codewords have weight 22, so $\mathrm{Isom}(C) = GL(3, \mathbb{F}_2)$, while $\mathrm{r}\mathcal{M}(C) = \{\mathrm{id}_M\}$.

Extreme example (b)

- Additive code over $\mathbb{F}_9 = \mathbb{F}_3[\omega]/(\omega^2 - \omega - 1)$.

$$
\begin{array}{c|cccccccccccc}
\text{mult.} & 5 & 3 & 6 & 1 & 1 & 1 & 2 & 2 & 2 & 4 & 3 & 2 \\
\hline
 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
G_3 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & -1 & -1 & -1 \\
 & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 1 & -1
\end{array}
$$

$$
\begin{array}{|ccccccccccc}
6 & 3 & 7 & 8 & 9 & 6 & 4 & 5 & 2 & 3 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & \omega & \omega & \omega & \omega & \omega & \omega & \omega & \omega & \omega \\
\omega & \omega & 0 & 1 & -1 & \omega & \omega+1 & \omega-1 & -\omega & -\omega+1 & -\omega-1
\end{array}
$$

Extreme example (b) continued

- Code has length $n = 86$; all codewords have weight 72.
- $\mathsf{Isom}(C) = GL(3, \mathbb{F}_3)$, of order $11,232$.
- $\mathrm{r}\mathcal{M}(C) = \{\pm\,\mathrm{id}_M\}$ is minimum possible.

More detailed examples

- The first non-trivial case is, as in earlier examples, $R = \mathbb{F}_2$, $A = \mathbb{F}_4$, and $M = \mathbb{F}_2^3$. This is the case of $k = 1 < \ell = 2 < m = 3$ over $\mathbb{F}_2$.
- Let $G = GL_R(M) = GL(3, \mathbb{F}_2)$; G is a simple group of size $|G| = 168$.
- $\dim F_0(\mathcal{O}^\sharp, \mathbb{Q}) = 14$; $\dim F_0(\mathcal{O}, \mathbb{Q}) = 7$.

Subgroup lattice of $GL(3, \mathbb{F}_2)$

- Up to conjugacy, the group $G = GL(3, \mathbb{F}_2)$ contains the following subgroups (via Magma): G, two Σ_4, SD_{21}, two A_4, D_8, Σ_3, two Klein V_4, C_7, C_4, C_3, C_2, $\{\mathrm{id}_G\}$.
- Which of these subgroups appear as $\mathrm{r}\mathcal{M}(\eta)$ or $\mathrm{Isom}(\eta)$?

Recall criteria

- For $f \in G$, $f \in \mathrm{r}\mathcal{M}(\eta)$ if and only if $\eta f = \eta$.
- For $f \in G$, $f \in \mathsf{Isom}(\eta)$ if and only if $\eta f - \eta \in \ker W$ if and only if $W(\eta f) = W(\eta)$.
- For a fixed $f \in G$, these are linear equations in the 14 entries of η.
- Amenable to computer-assisted computations (Maple).

Some results of computations

- Suppose $f \in G$ has order 7. If $f \in \mathrm{r}\mathcal{M}(\eta)$ for some η, then $\mathrm{r}\mathcal{M}(\eta) = G$. Similarly, if $f \in \mathrm{Isom}(\eta)$ for some η, then $\mathrm{Isom}(\eta) = G$.
- Suppose $f \in G$ has order 4. If $f \in \mathrm{r}\mathcal{M}(\eta)$ (resp. $f \in \mathrm{Isom}(\eta)$), then $D_8 \subseteq \mathrm{r}\mathcal{M}(\eta)$ (resp. $D_8 \subseteq \mathrm{Isom}(\eta)$).
- Suppose $f \in G$ has order 3. If $f \in \mathrm{r}\mathcal{M}(\eta)$ (resp. $f \in \mathrm{Isom}(\eta)$), then $\Sigma_3 \subseteq \mathrm{r}\mathcal{M}(\eta)$ (resp. $\Sigma_3 \subseteq \mathrm{Isom}(\eta)$).

Ruling out subgroups

- The containment relations in the subgroup lattice for G then imply that the following subgroups cannot be $\mathrm{r}\mathcal{M}(\eta)$ or $\mathrm{Isom}(\eta)$ for any η: SD_{21}, either A_4, C_7, C_4, and C_3.
- What's left? G, two Σ_4, D_8, Σ_3, two Klein V_4, C_2 and $\{\mathrm{id}_G\}$.
- There are about 40 possible containment pairs.
- Earlier examples realized $\mathrm{r}\mathcal{M}(\eta) \subseteq \mathrm{Isom}(\eta)$ for $V_4 \subset D_8$, $\Sigma_4 \subset G$, and $\{\mathrm{id}_G\} \subset G$.

More examples (a)

Multiplicity function η									n	r$\mathcal{M}$	Isom	r$\mathcal{M}'$
0	1	1	1	1	1	1	1	1				
1	0	0	0	1	1	ω	ω	ω				
ω	0	1	ω	0	1	0	ω	ω^2				
2		1	2	1		2			8	C_2	V_4	I
1	1	1			1			1	5	C_2	Σ_3	I
2		1	1	1		1			6	C_2	D_8	I
1					1			1	3	V_4	D_8	C_2
1	1				1			1	4	D_8	Σ_4	C_2
1							1	1	3	Σ_4	Σ_4	Σ_3

More examples (b)

Multiplicity function η											n	r$\mathcal{M}$	Isom	r$\mathcal{M}'$
0	0	0	0	1	1	1	1	1	1	1				
0	1	1	1	0	1	1	ω	ω	ω	ω				
1	0	1	ω	ω	0	ω	0	1	ω	ω^2				
			2			1		1		2	6	V_4	V_4	I
			1			1		1	1	1	5	D_8	D_8	I
1	1			1	1			1		1	6	Σ_3	Σ_4	C_2
		1		1		1	1	1			5	Σ_4	G	C_4

More examples (c)

- In the tables, r$\mathcal{M}'$ is the subgroup of r$\mathcal{M}$ obtained by Magma.
- It appears that Magma uses $GL_{\mathbb{F}_4}(\mathbb{F}_4)$, not the more general $GL_{\mathbb{F}_2}(\mathbb{F}_4)$, when calculating monomial transformations.

Thank you

- Once more, let me thank the organizers of the workshop for their hospitality and work.
- And thanks to you the audience for your kind attention.