Isometry Groups of Additive Codes

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5. Isometries of additive codes

- Additive codes as linear codes over modules
- Failure of EP
- Monomial and isometry groups
- Examples
- Criteria in terms of multiplicity functions
- Structure of ker $W$
- Building codes with prescribed groups
- EP for short codes
- Extreme examples
Additive $\mathbb{F}_4$-codes

- There has been interest in additive codes with alphabet $A = \mathbb{F}_4$.
- Such codes are the same as $R$-linear codes over $A$ with $R = \mathbb{F}_2$ and $A = \mathbb{F}_4$, regarding $\mathbb{F}_4$ as an $\mathbb{F}_2$-vector space of dimension 2.
- Generalize to case of $R = M_{k \times k}(\mathbb{F}_q)$ and $A = M_{k \times \ell}(\mathbb{F}_q)$. Information module will be $M = M_{k \times m}(\mathbb{F}_q)$. Use Hamming weight $\text{wt}$ on $A$.
- Call this the matrix module context.
Failure of EP

- Recall that EP for Hamming weight fails in the matrix module context when $k < \ell$ and $k < m$.
- In terms of the $W$-map:

$$W : F_0(\mathcal{O}^\#, \mathbb{Q}) \rightarrow F_0(\mathcal{O}, \mathbb{Q})$$

is not injective for some information module $M$. 
Isometry group

- General set-up: ring $R$, alphabet $A$, weight $w$ on $A$.
- Let $C \subseteq A^n$ be an $R$-linear code.
- Consider linear isometries $f : C \rightarrow C$; i.e., $w(cf) = w(c)$, for all $c \in C$.
- When $C$ is given as the image of a parametrized code $\Lambda : M \rightarrow A^n$, we define the isometry group:

\[ \text{Isom}(C) = \{ g \in GL_R(M) : \text{there exists a linear isometry } f : C \rightarrow C \text{ such that } g\Lambda = \Lambda f \} \]

- View isometries on $M$ rather than $C$. 
Monomial group

- Recall that the weight $w$ on $A$ has a right symmetry group $G_{rt}$.
- For linear code $C \subseteq A^n$, define the monomial group

$$\mathcal{M}(C) = \{ T : A^n \rightarrow A^n, G_{rt}\text{-monomial transformation, with } CT = C \}.$$
Restriction map

- Any \( T \in \mathcal{M}(C) \), when restricted to \( C \), gives an isometry on \( C \). By viewing the isometry on \( M \), we get a group homomorphism

\[
\text{restr} : \mathcal{M}(C) \to \text{Isom}(C).
\]

- Denote \( \ker \text{restr} = \mathcal{M}_0(C) \). Think of repeated columns in a generator matrix.

- Denote image of \( \mathcal{M}(C) \) under restr by \( r\mathcal{M}(C) \).

- If EP holds, then restr is surjective:

\[ r\mathcal{M}(C) = \text{Isom}(C). \]
Main question

- When EP fails, restr will not be surjective for some linear codes $C$ or information modules $M$.
- Then $r\mathcal{M}(C) \subseteq \text{Isom}(C) \subseteq GL_R(M)$.
- Which subgroups of $GL_R(M)$ can occur as $r\mathcal{M}(C)$ and $\text{Isom}(C)$?
Example 1 (a)

Additive code over $\mathbb{F}_4 = \mathbb{F}_2[\omega]/(\omega^2 + \omega + 1)$ with generator matrix $G_1$ and list of codewords. $M = \mathbb{F}_2^3$.

$$G_1 = \begin{bmatrix}
1 & \omega & 0 \\
\omega & 1 & 0 \\
1 & 0 & 1
\end{bmatrix}$$
Example 1 (b)

Consider three elements of $GL_R(M) = GL(3, \mathbb{F}_2)$:

\[
\begin{align*}
  f_1 &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\
  f_2 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \\
  f_3 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.
\end{align*}
\]

- $f_1, f_2$ generate $rM(C)$, a Klein 4-group. But $f_1, f_3$ generate Isom$(C)$, a dihedral group of order 8. ($f_2 = f_1f_3^2$.)
- Magma found only the cyclic 2-group generated by $f_1f_2$. 

Example 2 (a)

- Additive code over $\mathbb{F}_4$ with generator matrix $G_2$ and list of codewords. Again, $M = \mathbb{F}_2^3$.

$$G_2 = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & \omega & \omega & \omega \\
\omega & \omega & 1 & 0 & \omega^2 & \omega^2
\end{bmatrix},$$

$$\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & \\
0 & 1 & 1 & 1 & 1 & 1 & \\
1 & 0 & 1 & \omega & \omega & \omega & \\
1 & 1 & 0 & \omega^2 & \omega^2 & \\
\omega & \omega & 1 & 0 & \omega^2 & \\
\omega^2 & \omega & 0 & \omega & 1 & \\
\omega^2 & \omega^2 & 1 & \omega^2 & 0
\end{array}$$
Example 2 (b)

- Consider three elements of \( GL_R(M) = GL(3, \mathbb{F}_2) \):
  
  \[
  f_4 = \begin{bmatrix}
  0 & 0 & 1 \\
  0 & 1 & 0 \\
  1 & 0 & 0
  \end{bmatrix}, \quad f_5 = \begin{bmatrix}
  1 & 0 & 0 \\
  0 & 1 & 0 \\
  1 & 1 & 1
  \end{bmatrix}, \quad f_6 = \begin{bmatrix}
  1 & 0 & 0 \\
  1 & 1 & 1 \\
  0 & 0 & 1
  \end{bmatrix}.
  \]

- These elements generate \( r\mathcal{M}(C) \cong \Sigma_4 \), the symmetric group on 4 elements, while \( \text{Isom}(C) = GL(3, \mathbb{F}_2) \), the simple group of order 168.

- Magma found only a cyclic 4-group generated by \( f = f_4 f_5 f_6 f_4 f_5 f_4 f_6 \).
Closure for group actions

- Some of the hypotheses of the main result involve a notion of \textbf{closure} with respect to a group action.
- This idea goes back at least to Wielandt, 1964.
- Suppose a finite group $G$ acts on a set $X$.
- A subgroup $H \subseteq G$ partitions $X$ into $H$-orbits.
- Define the \textbf{closure} of $H$ with respect to the action:

$$\bar{H} = \{ g \in G : g \cdot \text{orb}_H(x) = \text{orb}_H(x), \ x \in X \}.$$ 

- Subgroup $H \subseteq G$ is \textbf{closed} with respect to the action if $\bar{H} = H$. 
Aside: stabilizer subgroups

- Let $G$ act on $X$.
- The **stabilizer subgroup** of $x \in X$ is

$$\text{Stab}_G(x) = \{g \in G : g \cdot x = x\}.$$
Aside: not every subgroup is a stabilizer subgroup of a given action

- Let $G = \Sigma_3$, the symmetric group on three objects, acting on $X = \{1, 2, 3\}$.
- Then $\text{Stab}_{\Sigma_3}(1) = \langle (2, 3) \rangle$, the cyclic 2-subgroup generated by the transposition $(2, 3)$.
- The cyclic 3-subgroup $\langle (1, 2, 3) \rangle$ is not a stabilizer subgroup for this action. Neither are $\{\text{id}_X\}$ and $\Sigma_3$. 
Aside: closure and stabilizers

- Let $G$ act on $X$.
- Let $F(X, \mathbb{C})$ be the vector space of all complex-valued functions on $X$. There is an induced action of $G$ on $F(X, \mathbb{C})$.
- A subgroup $H \subseteq G$ is closed with respect to the action on $X$ if and only if $H$ is a stabilizer subgroup for the action on $F(X, \mathbb{C})$.
- Proof is an exercise: separate $H$-orbits.
- End of aside.
Closure conditions

- Usual set-up: ring $R$, alphabet $A$, weight $w$, information module $M$. Orbit spaces $\mathcal{O}$ and $\mathcal{O}^\#$.

$$\mathcal{O} = G_{lt} \backslash M: \text{GL}_R(M) \text{ acts on the right, and on the left of } F_0(\mathcal{O}, \mathbb{Q}).$$

$$\mathcal{O}^\# = \text{Hom}_R(M, A)/G_{rt}: \text{GL}_R(M) \text{ acts on the left, and on the right of } F_0(\mathcal{O}^\#, \mathbb{Q}): (\eta f)([\lambda]) = \eta([f \lambda]).$$

- For $H_1 \subseteq H_2 \subseteq \text{GL}_R(M)$, will want $H_1$ to be closed for the $\mathcal{O}^\#$-action and $H_2$ closed for the $\mathcal{O}$-action.

- “Not every subgroup gets to be an isometry group.”
Statement of main result

Theorem

Matrix module context with $k < \ell < m$. For any choice of subgroups $H_1 \subseteq H_2 \subseteq \text{GL}_R(M)$ with $H_1$ closed for the $O^\#$-action and $H_2$ closed for the $O$-action, there exists a linear code $C$ modeled on $M$ such that $H_1 = r\mathcal{M}(C)$ and $H_2 = \text{Isom}(C)$.

Corollary

Same matrix module context. There exists a linear code $C$ modeled on $M$ with $r\mathcal{M}(C) = \{\mathbb{F}_q^\times \cdot \text{id}_M\}$ and $\text{Isom}(C) = \text{GL}_R(M)$. 
Aside: generalized context

- Suppose a finite group $G$ acts on two finite sets $X$ and $Y$.
- Suppose $f : X \to Y$ is a $G$-equivariant map; i.e., $f(g \cdot x) = g \cdot f(x)$, for all $g \in G$ and $x \in X$.
- Equivariance implies: for any $x \in X$,

$$\text{Stab}_G(x) \subseteq \text{Stab}_G(f(x)).$$

- If $g \cdot x = x$, then $g \cdot f(x) = f(g \cdot x) = f(x)$.
Now assume $f : X \to Y$ is injective.

Then, for any $x \in X$,

$$\text{Stab}_G(x) = \text{Stab}_G(f(x)).$$

If $g \cdot f(x) = f(x)$, then $f(g \cdot x) = g \cdot f(x) = f(x)$.

$f$ injective implies $g \cdot x = x$. 
Aside: general $f$

- General $f : X \to Y$, so $\text{Stab}_G(x) \subseteq \text{Stab}_G(f(x))$.
- Let $S_X = \{\text{Stab}_G(x) : x \in X\}$ and $S_Y$ similarly.
- Given $H_1 \in S_X$ and $H_2 \in S_Y$, with $H_1 \subseteq H_2 \subseteq G$, does there exist $x \in X$ with $H_1 = \text{Stab}_G(x)$ and $H_2 = \text{Stab}_G(f(x))$?
- That is, can we achieve *pairs* of stabilizer groups?
- The various conditions stated are clearly necessary.
- One more: $H_2 \in S_{f(X)}$. Redundant if $f$ is onto.
- End of aside.
Statement of main result (again)

**Theorem**

Matrix module context with $k < \ell < m$. For any choice of subgroups $H_1 \subseteq H_2 \subseteq \text{GL}_R(M)$ with $H_1$ closed for the $\mathcal{O}^\#$-action and $H_2$ closed for the $\mathcal{O}$-action, there exists a linear code $C$ modeled on $M$ such that $H_1 = r \mathcal{M}(C)$ and $H_2 = \text{Isom}(C)$.

**Corollary**

Same matrix module context. There exists a linear code $C$ modeled on $M$ with $r \mathcal{M}(C) = \{ \mathbb{F}_q^\times \cdot \text{id}_M \}$ and $\text{Isom}(C) = \text{GL}_R(M)$. 
Using multiplicity functions

- Up to $G_{rt}$-monomial transformations, a parametrized code $\Lambda : M \to A^n$ is determined by its multiplicity function $\eta_\Lambda \in F_0(\mathcal{O}^\#, \mathbb{N})$.

- Recall the right action of $GL_R(M)$ on $F_0(\mathcal{O}^\#, \mathbb{Q})$: $(\eta f)([\lambda]) = \eta([f \lambda])$. Left action via $\eta f^{-1}$.

- $W : F_0(\mathcal{O}^\#, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q})$ has $W(f \eta) = f W(\eta)$.

- For $f \in GL_R(M)$, $f \in r\mathcal{M}(\eta)$ if and only if $\eta f = \eta$.

- For $f \in GL_R(M)$, $f \in \text{Isom}(\eta)$ if and only if $\eta f - \eta \in \ker W$. 
Interpretation as stabilizer subgroups

- $W$ map satisfies $W(f \eta) = fW(\eta)$.
- For $f \in \text{GL}_R(M)$, $f \in rM(\eta)$ if and only if $\eta f = \eta$.
- That is, $rM(\eta)$ is the stabilizer subgroup of $\eta$ for the action of $\text{GL}_R(M)$ on $F_0(\mathcal{O}^\#, \mathbb{Q})$.
- For $f \in \text{GL}_R(M)$, $f \in \text{Isom}(\eta)$ if and only if $\eta f - \eta \in \ker W$ if and only if $W(\eta f) = W(\eta)$.
- That is, $\text{Isom}(\eta)$ is the stabilizer subgroup of $W(\eta)$ for the action of $\text{GL}_R(M)$ on $F_0(\mathcal{O}, \mathbb{Q})$.
Structure of $\ker W$ (a)

- In the matrix module context, $O^\#$ is the set of CRE matrices of size $m \times \ell$, while $O$ is the set of RRE matrices of size $k \times m$.

- Remember $k < \ell < m$. By dimension counting,

$$\ker W \geq \sum_{i=k+1}^{\ell} \left[ \begin{array}{c} m \\ i \end{array} \right]_q,$$

using $q$-binomial coefficients.
Structure of $\ker W$ (b)

- The orbit space $O^\#$ is partitioned by rank.
- By explicit constructions, one produces independent elements $\eta[\lambda] \in \ker W$. For each $i = k + 1, \ldots, \ell$, one produces $\binom{m_i}{q}$ of them, each $\eta[\lambda]$ supported on $[\lambda]$ of rank $i$ and on specific elements of smaller rank. ("Triangular.") This produces as many independent elements of $\ker W$ as the sum in (1).
- Separately, one shows that $W$ is surjective, so there is equality in (1), and we have an explicit basis for $\ker W$. This part is somewhat technical.
Aside: EP for short codes

- Serhii Dyshko (Toulon) has shown that EP holds even when \( k < \ell \), provided \( n \) is sufficiently small \( (n \leq q \text{ when } k = 1) \).
- Elements of \( \ker W \) affect the length of the code.
- The exact details of this need to be better understood.
- End of aside.
Idea of proof (a)

- Elements $[x] \in \mathcal{O}$ have a well-defined rank, $\text{rk}[x]$. The $GL_R(M)$-action preserves this rank.
- Pick a function $w$ on $\mathcal{O}$ that (1) is constant on and separates the $H_2$-orbits on $\mathcal{O}$ and (2) is an increasing function of $\text{rk}[x]$.
- Because $W$ is surjective, there exists $\eta$ with $W(\eta) = w$. A priori, $\eta$ has rational values.
- Can modify $\eta$ to have non-negative integer values and still satisfy (1) and (2).
Idea of proof (b)

- Replace $\eta$ by an averaged version so that $\eta$ is also constant on the $H_2$-orbits on $O^\#$. This does not change $W(\eta)$. Clear denominators of $\eta$, which scales everything.

- At this point, $\eta$ has non-negative integer values, is constant on $H_2$-orbits on $O^\#$, and $W(\eta)$ is constant on and separates $H_2$-orbits on $O$. 
Idea of proof (c)

- Claim $r\mathcal{M}(\eta) = \text{Isom}(\eta) = H_2$.
- From $\eta$ constant on $H_2$-orbits on $\mathcal{O}^\#$, $H_2 \subseteq r\mathcal{M}(\eta)$.
- We always have $r\mathcal{M}(\eta) \subseteq \text{Isom}(\eta)$.
- Suppose $f \in \text{Isom}(\eta)$. Because $w = W(\eta)$ separates $H_2$-orbits on $\mathcal{O}$, $w(xf) = w(x)$ implies $f \in \tilde{H}_2$. The closure hypothesis implies $f \in H_2$. 
Idea of proof (d)

- Modify $\eta$ using $\eta[\lambda] \in \ker W$ to separate $H_1$-orbits on $O^\#$ (rank-by-rank, from rank $\ell$ down to rank $k + 1$).
- Because of "triangular" form of $\eta[\lambda]$, a change at rank $i$ does not disturb changes at higher ranks.
- The final $\eta$ preserves $H_1$-orbits on $O^\#$, so $H_1 \subseteq rM(\eta)$. Conversely, any $f \in rM(\eta)$ preserves $H_1$-orbits on $O^\#$ ($\eta$ separates), so $f \in H_1$. Closure implies $f \in H_1$.
- Because modifications were made by $\eta[\lambda] \in \ker W$, $W(\eta)$ has not changed. We still have $\text{Isom}(\eta) = H_2$. 
Other alphabets

- Most of the result carries over to any alphabet with non-cyclic socle, such as non-Frobenius rings.
- Get $rM(\eta) \subseteq H_1$ only, but still have $H_2 = \text{Isom}(\eta)$.
- This is enough to get the extreme cases.
Extreme example (a)

- \( R = \mathbb{F}_2, A = \mathbb{F}_4, M = \mathbb{F}_2^3 \). Multiplicities as indicated. Length \( n = 28 \).

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- All codewords have weight 22, so \( \text{Isom}(C) = GL(3, \mathbb{F}_2) \), while \( rM(C) = \{ \text{id}_M \} \).
Extreme example (b)

- Additive code over $\mathbb{F}_9 = \mathbb{F}_3[\omega]/(\omega^2 - \omega - 1)$.

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Extreme example (b) continued

- Code has length $n = 86$; all codewords have weight 72.
- $\text{Isom}(C) = \text{GL}(3, \mathbb{F}_3)$, of order 11,232.
- $r\mathcal{M}(C) = \{ \pm \text{id}_M \}$ is minimum possible.
The first non-trivial case is, as in earlier examples, $R = \mathbb{F}_2$, $A = \mathbb{F}_4$, and $M = \mathbb{F}_2^3$. This is the case of $k = 1 < \ell = 2 < m = 3$ over $\mathbb{F}_2$.

Let $G = GL_R(M) = GL(3, \mathbb{F}_2)$; $G$ is a simple group of size $|G| = 168$.

$\dim F_0(\mathcal{O}^\#, \mathbb{Q}) = 14$; $\dim F_0(\mathcal{O}, \mathbb{Q}) = 7$. 

More detailed examples
Subgroup lattice of $GL(3, \mathbb{F}_2)$

- Up to conjugacy, the group $G = GL(3, \mathbb{F}_2)$ contains the following subgroups (via Magma): $G$, two $\Sigma_4$, $SD_{21}$, two $A_4$, $D_8$, $\Sigma_3$, two Klein $V_4$, $C_7$, $C_4$, $C_3$, $C_2$, $\{\text{id}_G\}$.
- Which of these subgroups appear as $r\mathcal{M}(\eta)$ or $\text{Isom}(\eta)$?
Proof of main result

Recall criteria

- For $f \in G$, $f \in rM(\eta)$ if and only if $\eta f = \eta$.
- For $f \in G$, $f \in Isom(\eta)$ if and only if $\eta f - \eta \in \ker W$ if and only if $W(\eta f) = W(\eta)$.
- For a fixed $f \in G$, these are linear equations in the 14 entries of $\eta$.
- Amenable to computer-assisted computations (Maple).
Some results of computations

- Suppose $f \in G$ has order 7. If $f \in rM(\eta)$ for some $\eta$, then $rM(\eta) = G$. Similarly, if $f \in Isom(\eta)$ for some $\eta$, then $Isom(\eta) = G$.

- Suppose $f \in G$ has order 4. If $f \in rM(\eta)$ (resp. $f \in Isom(\eta)$), then $D_8 \subseteq rM(\eta)$ (resp. $D_8 \subseteq Isom(\eta)$).

- Suppose $f \in G$ has order 3. If $f \in rM(\eta)$ (resp. $f \in Isom(\eta)$), then $\Sigma_3 \subseteq rM(\eta)$ (resp. $\Sigma_3 \subseteq Isom(\eta)$).
Ruling out subgroups

- The containment relations in the subgroup lattice for $G$ then imply that the following subgroups cannot be $r\mathcal{M}(\eta)$ or $\text{Isom}(\eta)$ for any $\eta$: $SD_{21}$, either $A_4$, $C_7$, $C_4$, and $C_3$.
- What’s left? $G$, two $\Sigma_4$, $D_8$, $\Sigma_3$, two Klein $V_4$, $C_2$ and $\{\text{id}_G\}$.
- There are about 40 possible containment pairs.
- Earlier examples realized $r\mathcal{M}(\eta) \subseteq \text{Isom}(\eta)$ for $V_4 \subset D_8$, $\Sigma_4 \subset G$, and $\{\text{id}_G\} \subset G$. 
### More examples (a)

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<th>$r\mathcal{M}$</th>
<th>Isom</th>
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</table>
### More examples (b)

<table>
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<th>Multiplicity function $\eta$</th>
<th>$n$</th>
<th>$r\mathcal{M}$</th>
<th>Isom</th>
<th>$r\mathcal{M}'$</th>
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<tr>
<td>$0 \ 1 \ 1 \ 1 \ 0 \ 1 \ 1 \ \omega \ \omega \ \omega \ \omega$</td>
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<tr>
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<tr>
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</tbody>
</table>
More examples (c)

- In the tables, $r\mathcal{M}'$ is the subgroup of $r\mathcal{M}$ obtained by Magma.
- It appears that Magma uses $GL_{F_4}(F_4)$, not the more general $GL_{F_2}(F_4)$, when calculating monomial transformations.
Thank you

- Once more, let me thank the organizers of the workshop for their hospitality and work.
- And thanks to you the audience for your kind attention.