Foundational Results on Linear Codes over Rings and Modules

Jay A. Wood

Department of Mathematics
Western Michigan University
http://sites.google.com/a/wmich.edu/jaywood

Centennial
Universidad Michoacana de San Nicolás de Hidalgo
Instituto de Física y Matemáticas
Morelia, Michoacán
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4. The extension problem for Hamming weight

- Extension property (EP)
- EP for Hamming weight over Frobenius bimodules via linear independence of characters
- Generalization for module alphabets
- Axiomatic viewpoint
- Parametrized codes and multiplicity functions
- Failure of EP for landscape matrix modules
- Converse of extension theorem: EP implies Frobenius
Notation

- Let \( R \) be a finite associative ring with 1.
- Let \( A \) be a finite unital left \( R \)-module: the alphabet.
- Let \( w : A \to \mathbb{Q} \) be a weight: \( w(0) = 0 \). Extend to \( A^n \) by
  \[
  w(a_1, \ldots, a_n) = \sum_{i=1}^{n} w(a_i).
  \]
Symmetry groups

Define the **symmetry groups** of $w$:

$$G_{lt} = \{ u \in \mathcal{U}(R) : w(ua) = w(a), \ a \in A \},$$
$$G_{rt} = \{ \phi \in \text{GL}_R(A) : w(a\phi) = w(a), \ a \in A \}.$$  

- $\mathcal{U}(R)$ is the group of units of $R$, and $\text{GL}_R(A)$ is the group of invertible $R$-linear homomorphisms $A \to A$.
- I will usually write homomorphisms of left modules on the right side; $f : A \to A$, $(ra)f = r(af)$.  

Monomial transformations

- For a subgroup $G \subseteq \text{GL}_R(A)$, a $G$-monomial transformation of $A^n$ is an invertible $R$-linear homomorphism $T : A^n \rightarrow A^n$ of the form

$$ (a_1, a_2, \ldots, a_n) T = (a_{\sigma(1)} \phi_1, a_{\sigma(2)} \phi_2, \ldots, a_{\sigma(n)} \phi_n), $$

for $(a_1, a_2, \ldots, a_n) \in A^n$.

- Here, $\sigma$ is a permutation of $\{1, 2, \ldots, n\}$ and $\phi_i \in G$ for $i = 1, 2, \ldots, n$. 
Isometries

- Let $C_1, C_2 \subseteq A^n$ be two linear codes. An $R$-linear isomorphism $f : C_1 \rightarrow C_2$ is a linear isometry with respect to $w$ if $w(xf) = w(x)$ for all $x \in C_1$.
- Every $G_{rt}$-monomial transformation is an isometry from $A^n$ to itself.
Extension property (EP)

- Given ring $R$, alphabet $A$, and weight $w$ on $A$.
- The alphabet $A$ has the **extension property** (EP) with respect to $w$ if the following holds: For any left linear codes $C_1, C_2 \subseteq A^n$, if $f : C_1 \rightarrow C_2$ is a linear isometry, then $f$ extends to a $G_{rt}$-monomial transformation $A^n \rightarrow A^n$.
- That is, there exists a $G_{rt}$-monomial transformation $T : A^n \rightarrow A^n$ such that $xT = xf$ for all $x \in C_1$. 
Slightly different point of view

- Linear codes are often presented by generator matrices. A generator matrix serves as a linear encoder from an information space to a message space.

- If \( f : C_1 \rightarrow C_2 \) is a linear isometry, then \( C_1 \) and \( C_2 \) are isomorphic as \( R \)-modules. Let \( M \) be a left \( R \)-module isomorphic to \( C_1 \) and \( C_2 \). Call \( M \) the information module.

- Then \( C_1 \) and \( C_2 \) are the images of \( R \)-linear homomorphisms \( \Lambda : M \rightarrow A^n \) and \( N : M \rightarrow A^n \), respectively. Then, \( N = \Lambda f \): inputs on left!
Coordinate functionals

- \( C_1 \) was given by \( \Lambda : M \to A^n \). Write the individual components as \( \Lambda = (\lambda_1, \ldots, \lambda_n) \), with \( \lambda_i \in \text{Hom}_R(M, A) \). Call the \( \lambda_i \) coordinate functionals.

- Similarly, \( N = (\nu_1, \ldots, \nu_n) \), \( \nu_i \in \text{Hom}_R(M, A) \).

- The isometry \( f \) extends to a \( G_{rt} \)-monomial transformation if there exists a permutation \( \sigma \) and \( \phi_i \in G_{rt} \) such that \( \nu_i = \lambda_{\sigma(i)} \phi_i \) for all \( i = 1, \ldots, n \).
Case of $\hat{R}$

Theorem

For any finite ring $R$, $A = \hat{R}$ has EP with respect to the Hamming weight.

- It follows that $A = R$ itself has EP with respect to the Hamming weight when $R$ is Frobenius.
- The Frobenius ring case came first (1999).
- The more general $A = \hat{R}$ case is due to Greferath, Nechaev, and Wisbauer (2004).
Techniques

- For any alphabet $A$, the summation formulas for characters imply that the Hamming weight $wt$ satisfies

$$wt(a) = 1 - \frac{1}{|A|} \sum_{\pi \in \hat{A}} \pi(a), \quad a \in A.$$  

- Characters are linearly independent over $\mathbb{C}$.
- Recursive argument using maximal elements in a finite poset.
Symmetry groups for the Hamming weight

- Consider the Hamming weight $wt$ on $A = \hat{R}$, which is an $(R, R)$-bimodule.
- Both symmetry groups $G_{lt}$ and $G_{rt}$ equal $U(R)$. 
Posets

- Given a set $S$, a (non-strict) **partial order** $\preceq$ on $S$ is reflexive, antisymmetric, and transitive. The pair $(S, \preceq)$ is a **partially ordered set** or **poset**.

- Example. Let $X$ be a nonempty set. Then $S = P(X)$, the set of all subsets of $X$, with set inclusion, i.e., $U \preceq V$ when $U \subseteq V$, is a poset.

- Example. Let $B$ be a finite right $R$-module. Then $S = \{bR : b \in B\}$ is the poset of all cyclic right $R$-submodules of $B$ under set inclusion.

- Fact: $b_1R = b_2R$ if and only if $b_1 = b_2u$, where $u \in \mathcal{U}(R)$. 

Proof of Theorem, (a)

- $R, \Lambda = \hat{R}$, with Hamming weight. $C_1, C_2 \subseteq \hat{R}^n$, with $f : C_1 \rightarrow C_2$ linear isometry.
- $\hat{R}$ has a generating character: $\rho : \hat{R} \rightarrow \mathbb{C}$, $\rho(\pi) = \pi(1)$ for $\pi \in \hat{R}$. (Evaluate at $1 \in R$.) Every $\pi \in \hat{R}$ has the form $\pi = r \rho$ for some unique $r \in R$.
- $C_1$ is image of $\Lambda : M \rightarrow \hat{R}^n$; $C_2$ is image of $N : M \rightarrow \hat{R}^n$. $N = \Lambda f$.
- Isometry: $\text{wt}(x\Lambda) = \text{wt}(xN)$, for all $x \in M$. 

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Proof (b)

- Hamming weight as character sum:

\[
\sum_{i=1}^{n} \sum_{r \in R} r \rho(x \lambda_i) = \sum_{j=1}^{n} \sum_{s \in R} s \rho(x \nu_j), \quad x \in M.
\]

- That is,

\[
\sum_{i=1}^{n} \sum_{r \in R} \rho(x \lambda_i r) = \sum_{j=1}^{n} \sum_{s \in R} \rho(x \nu_j s), \quad x \in M.
\]

- This is an equation of characters on $M$. 

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The extension problem for Hamming weight
Proof (c)

- Let $B = \text{Hom}_R(M, \hat{R})$, a right $R$-module. Poset $S = \{\lambda R : \lambda \in \text{Hom}_R(M, \hat{R})\}$ under $\subseteq$.
- Among the $\lambda_i R, \nu_j R$, choose one that is maximal for $\subseteq$. Say, $\nu_1 R$.
- Let $j = 1$ and $s = 1$ on the right side of the character equation.
- By linear independence of characters, there exists $i$ and $r \in R$ so that $\rho(x\lambda_ir) = \rho(x\nu_1)$ for all $x \in M$.
- Thus $\rho(x(\nu_1 - \lambda_ir)) = 1$ for all $x \in M$. I.e., $M(\nu_1 - \lambda_ir) \subseteq \ker \rho$. 
Proof (d)

By \( \rho \) a generating character, \( \nu_1 = \lambda_i r \). Thus, \( \nu_1 R \subseteq \lambda_i R \).

By maximality of \( \nu_1 R \), \( \nu_1 R = \lambda_i R \). Thus, \( \nu_1 = \lambda_i u_1 \), for some \( u_1 \in \mathcal{U}(R) \).

Then inner sums agree:
\[
\sum_{r \in R} \rho(x \lambda_i r) = \sum_{s \in R} \rho(x \nu_1 s), \quad x \in M.
\]

Set \( \sigma(1) = i \). Subtract inner sums to reduce the size of the outer sums by 1. Proceed by induction.
Generalize to module alphabets

- For ring $R$, alphabet $A$, and Hamming weight $wt$, EP holds if $A$: (1) is pseudo-injective and (2) has a cyclic socle (embeds into $\hat{R}$).

- Pseudo-injective means injective with respect to submodules. That is, if $B$ is a submodule of $A$ and $h : B \rightarrow A$ is any injective module homomorphism, then $h$ extends to $\tilde{h} : A \rightarrow A$.

- Main idea: use $\hat{R}$-case to get $GL_R(\hat{R})$-monomial extension. Use pseudo-injectivity to show existence of $GL_R(A)$-monomial extension.
Axiomatic viewpoint

- Consider linear codes up to monomial equivalence. What matters?
- Actually, I want to consider parametrized codes up to monomial equivalence.
- Usual set-up: ring $\mathbb{R}$, alphabet $A$, weight $w$ on $A$.
- A **parametrized code** is a finite left $\mathbb{R}$-module $M$ and an $\mathbb{R}$-linear homomorphism $\Lambda : M \to A^n$. 
Scale classes

- The right symmetry group $G_{rt}$ acts on $\text{Hom}_R(M, A)$ on the right: $\lambda \mapsto \lambda \phi$.
- Call the orbit space $O^\# = \text{Hom}_R(M, A)/G_{rt}$. Denote orbit/“scale class” of $\lambda$ by $[\lambda]$.
- Up to $G_{rt}$-monomial equivalence, a parametrized code $\Lambda : M \to A^n$ is completely determined by the number of coordinate functionals $\lambda_i$ belonging to the various classes $[\lambda] \in O^\#$. 
Multiplicity functions

- Let $F(O^\#, \mathbb{N})$ denote the set of functions $\eta : O^\# \to \mathbb{N}$. Call these multiplicity functions.

- Given a parametrized code $\Lambda : M \to A^n$, define its multiplicity function $\eta_\Lambda$ by

$$\eta_\Lambda([\lambda]) = |\{i : \lambda_i \in [\lambda]\}|.$$

- Other authors: multisets, value function (Chen, et al.), projective systems, etc.

- No zero columns: $F_0(O^\#, \mathbb{N}) = \{\eta : \eta([0]) = 0\}$. 
Weights of elements

- Given $\Lambda : M \rightarrow A^n$, consider the weights $w(x\Lambda)$ for $x \in M$.
- The weights $w(x\Lambda)$, $x \in M$, depend only on $\eta_\Lambda$, not $\Lambda$ itself: $G_{rt}$-monomial transformations are isometries. In fact:

\[ w(x\Lambda) = \sum_{[\lambda] \in \mathcal{O}^\#} w(x\lambda) \eta_\Lambda([\lambda]), \quad x \in M. \]
Invariance under $G_{lt}$

- If $u \in G_{lt}$, then $w((ux)\Lambda) = w(u(x\Lambda)) = w(x\Lambda)$, for all $x \in M$.
- $G_{lt}$ acts on $M$ on the left: $x \mapsto ux$, $x \in M$. Denote orbit space by $\mathcal{O} = G_{lt} \backslash M$.
- $w(0\Lambda) = w(0) = 0$.
- Denote $F_0(\mathcal{O}, \mathbb{Q}) = \{f : \mathcal{O} \to \mathbb{Q}, f(0) = 0\}$. 
Well-defined $W$ map

We get a well-defined map

$$W : F_0(\mathcal{O}^\#, \mathbb{N}) \to F_0(\mathcal{O}, \mathbb{Q}),$$

with

$$W(\eta)(x) = \sum_{[\lambda] \in \mathcal{O}^\#} w(x \lambda) \eta([\lambda]),$$

for $x \in \mathcal{O}$, $\eta \in F_0(\mathcal{O}^\#, \mathbb{N})$. 
Completion over $\mathbb{Q}$

- $F_0(\mathcal{O}^\#, \mathbb{N})$ is an additive semi-group, and $F_0(\mathcal{O}, \mathbb{Q})$ is a $\mathbb{Q}$-vector space. The map $W$ is additive.
- The addition in $F_0(\mathcal{O}^\#, \mathbb{N})$ corresponds to concatenation of generator matrices.
- By tensoring over $\mathbb{Q}$, we get a $\mathbb{Q}$-linear transformation of $\mathbb{Q}$-vector spaces:

$$W : F_0(\mathcal{O}^\#, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q}).$$
Re-interpretation of EP

- An alphabet $A$ has EP with respect to a $\mathbb{Q}$-valued weight $w$ if and only if the linear map
  
  $$W : F_0(\mathcal{O}^\# , \mathbb{Q}) \rightarrow F_0(\mathcal{O} , \mathbb{Q})$$

  is injective for all information modules $M$.


Matrix modules and Hamming weight

- What does $W$ look like for matrix module alphabets with the Hamming weight?

- Let $R = M_{k \times k}(\mathbb{F}_q)$, $A = M_{k \times \ell}(\mathbb{F}_q)$, with Hamming weight $wt$.

- Symmetry groups: $G_{lt} = U(R) = \text{GL}(k, \mathbb{F}_q)$; $G_{rt} = \text{GL}_R(A) = \text{GL}(\ell, \mathbb{F}_q)$. 

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Orbit spaces

- For $M = M_{k \times m}(F_q)$, $\text{Hom}_R(M, A) = M_{m \times \ell}(F_q)$.
- Then $O = G_{\text{lt}} \backslash M = \text{GL}(k, F_q) \backslash M_{k \times m}(F_q)$, which is represented by the set of row reduced echelon (RRE) matrices of size $k \times m$.
- And
  $O^\# = \text{Hom}_R(M, A) / G_{\text{rt}} = M_{m \times \ell}(F_q) / \text{GL}(\ell, F_q)$, which is represented by the set of column reduced echelon (CRE) matrices of size $m \times \ell$. 

The extension problem for Hamming weight
Dimension counting

- First note that \( \dim_{\mathbb{Q}} F_0(\mathcal{O}, \mathbb{Q}) = |\mathcal{O}| - 1 \) and \( \dim_{\mathbb{Q}} F_0(\mathcal{O}^\#, \mathbb{Q}) = |\mathcal{O}^\#| - 1 \).
- So, \( \dim_{\mathbb{Q}} F_0(\mathcal{O}, \mathbb{Q}) \) is the number of nonzero RRE matrices of size \( k \times m \).
- And \( \dim_{\mathbb{Q}} F_0(\mathcal{O}^\#, \mathbb{Q}) \) is the number of nonzero CRE matrices of size \( m \times \ell \).
- If \( k < \ell \) and \( k < m \), there are more of the CRE matrices than the RRE matrices; i.e.,

\[
\dim_{\mathbb{Q}} F_0(\mathcal{O}^\#, \mathbb{Q}) > \dim_{\mathbb{Q}} F_0(\mathcal{O}, \mathbb{Q}).
\]
- This says that EP fails when \( k < \ell \). ("Landscape")
Converse of EP for Hamming weight

- We claim: if an alphabet $A$ has EP for the Hamming weight, then $A$ (1) is pseudo-injective and (2) has a cyclic socle.
- Likewise: if a ring $R$ has EP for the Hamming weight, then $R$ is Frobenius (which means $\text{Soc}(R)$ is cyclic).
Proof

- If Soc($A$) is not cyclic (same idea for $R$), then Soc($A$) contains a matrix module of the form $A' = M_{k \times \ell}(F_q)$ with $k < \ell$.
- There exist counter-examples to EP over $A'$.
- Regard these codes as codes over $A$: $A' \subseteq \text{Soc}(A) \subseteq A$.
- They are also counter-examples over $A$.
- Pseudo-injectivity is equivalent to the length 1 case of EP (Dinh, López-Permouth).
Summary

- A finite ring $R$ has EP for the Hamming weight iff $R$ is Frobenius.
- A finite alphabet $A$ over $R$ has EP for the Hamming weight iff $A$ is pseudo-injective and $\text{Soc}(A)$ is cyclic.