Isometry Groups of Additive Codes over Finite Fields

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In memoriam:
James Wood Jr., 1922–2015
Robert W. Moore, 1933–2016

Abstract. When $C \subseteq \mathbb{F}^n$ is a linear code over a finite field $\mathbb{F}$, every linear Hamming isometry of C to itself is the restriction of a linear Hamming isometry of $\mathbb{F}^n$ to itself, i.e., a monomial transformation. This is no longer the case for additive codes over non-prime fields. Every monomial transformation mapping C to itself is an additive Hamming isometry, but there may exist additive Hamming isometries that are not monomial transformations.

The monomial transformations mapping C to itself form a group $\mathrm{rMon}(C)$, and the additive Hamming isometries form a larger group $\mathrm{Isom}(C)$: $\mathrm{rMon}(C) \subseteq \mathrm{Isom}(C)$. The main result says that these two subgroups can be as different as possible: for any two subgroups $H_1 \subseteq H_2$, subject to some natural necessary conditions, there exists an additive code C such that $\mathrm{rMon}(C) = H_1$ and $\mathrm{Isom}(C) = H_2$.

1. Introduction

This paper studies the monomial transformations and Hamming isometries of linear codes and the extent to which they can be different. For many code alphabets, including the classical case of linear codes over finite fields, there is no difference. But for other alphabets, including the additive codes of the title, there can be a significant difference. For example, there exist additive codes over finite fields with

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only scalar multiples of the identity as monomial transformations but with a maximally large group of Hamming isometries.

In classical coding theory, linear codes are linear subspaces C of $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is a finite field of prime power order q. A monomial transformation of C is a linear automorphism $T : \mathbb{F}_q^n \to \mathbb{F}_q^n$ of a special form that maps C to itself. The special form is the composite of a permutation of the coordinate positions with an invertible diagonal transformation. The restriction of T to C is then a linear isometry of C with respect to the Hamming weight. A noted theorem of MacWilliams gives the converse: any linear Hamming isometry extends to a monomial transformation [**17**, **18**].

An additive code over a finite field is simply an additive subgroup C of $\mathbb{F}_q^n$. A monomial transformation is now a composite of a permutation and a diagonal transformation consisting of additive automorphisms of $\mathbb{F}_q$. Once again, any monomial transformation mapping C to itself, when restricted to C, is an additive Hamming isometry of C. As long as q is not itself prime, the extension theorem of MacWilliams fails [**27**]. It is the failure of the extension theorem that allows for the existence of additive codes over finite fields with large groups of isometries but small groups of monomial transformations.

Linear codes and additive codes can be studied in a common context: linear codes over finite modules [**14**]. In this context, there is a finite ring R and a finite R-module A. It is A that forms the alphabet for the linear codes. An R-linear code over A is an R-submodule C in A^n. For codes over finite fields, linear codes have $R = A = \mathbb{F}_q$, while additive codes have $R = \mathbb{F}_p$ and $A = \mathbb{F}_q$, where q is a power of the prime p. Another concrete example of linear codes over modules is the *matrix module context*: $R = M_{k\times k}(\mathbb{F}_q)$ and $A = M_{k\times\ell}(\mathbb{F}_q)$, the ring (resp., left R-module) of all $k \times k$ (resp., $k \times \ell$) matrices over $\mathbb{F}_q$. Note that additive codes over finite fields are the case where $k = 1$ and ℓ satisfies $q = p^\ell$. The matrix module context is both concrete enough to allow for detailed analysis and general enough to be widely applicable.

For linear codes C over an R-module A, a monomial transformation is a composite of a permutation and a diagonal transformation consisting of R-automorphisms of A. Any monomial transformation mapping C to itself, when resticted to C, is an R-linear Hamming isometry of C. Whether or not the converse holds, i.e., whether or not the alphabet A has the *extension property* for the Hamming weight, is now crucial.

In the matrix module context, $A = M_{k\times\ell}(\mathbb{F}_q)$ has the extension property for the Hamming weight if and only if $k \geq \ell$ [**27**]. So, when $k \geq \ell$, the extension property holds and all isometries extend to monomial transformations. But when $k < \ell$, the extension property fails, and the main theorem, Theorem 5.1, says that for any two subgroups $H_1 \subseteq H_2 \subseteq \mathrm{GL}(m, \mathbb{F}_q)$, subject to some necessary closure conditions, there exists an R-linear code C isomorphic to $M_{k\times m}(\mathbb{F}_q)$ whose group of isometries is H_2 and whose group of monomial transformations is H_1. In particular, by taking $H_1 = \{\alpha I_m : \alpha \in \mathbb{F}_q^\times\}$ and $H_2 = \mathrm{GL}(m, \mathbb{F}_q)$, there exist linear codes with a maximal group of isometries and a minimal group of monomial transformations, Corollary 5.3.

For a general alphabet A, a key condition for the extension property is whether or not the socle of A is a cyclic (one-generator) R-module. When the socle of A is not cyclic, the socle contains some submodule isomorphic to $M_{k\times\ell}(\mathbb{F}_q)$ with $k < \ell$ [27]. This allows the results proved in the matrix module context to be applied in the case of a general alphabet A. The resulting theorem, Theorem 8.1, is almost as strong as Theorem 5.1: the isometry group is still H_2, but the group of monomial transformations is merely contained in H_1, not necessarily equal to H_1. This is strong enough to imply the existence of linear codes with a maximal group of isometries and a minimal group of monomial transformations, Corollary 8.2.

Here is a short guide to the paper. Section 2 provides background on linear codes defined over modules. Section 3 interprets a linear code C in terms of a multiplicity function η that gives information equivalent to a generator matrix for C. By associating to η the list of all the Hamming weights of codewords of C, one defines a linear transformation W of certain finite-dimensional rational vector spaces. This interpretation allows one to describe several key ideas: a function f is a restriction of a monomial transformation of C if and only if $\eta f = \eta$; a function f is an isometry of C if and only if $\eta f - \eta \in \ker W$; and the extension property holds if and only if W is injective. When the extension property holds, we see immediately that isometries are the same as restrictions of monomial transformations.

Section 4 describes the closure conditions needed for the statement of the main theorem, Theorem 5.1. The difficulty is this: not every group can be the isometry group or the monomial group of a code. The closure conditions describe some necessary group-action-theoretic conditions that arise in the main theorem.

The main theorem appears in Section 5, together with an outline of its proof. One of the key technical steps is to understand the mapping W of Section 3 in the matrix module context when the extension property fails. This occurs in Section 6, where we prove that W is surjective, and a specific basis of $\ker W$ is determined. The proof of the main theorem is in Section 7. The surjectivity of W allows one to produce a code whose isometry group is H_2. The specific basis of $\ker W$

then allows one to modify the multiplicity function of the code without changing the isometry group. By introducing enough asymmetry, the monomial group reduces to H_1.

The application of the main theorem to the case of a general alphabet with non-cyclic socle appears in Section 8. The paper concludes in Section 9 with examples of additive codes over $\mathbb{F}_4$ and $\mathbb{F}_9$. Appendix A gives examples of additive codes of binary dimension 3 over $\mathbb{F}_4$ that achieve all the possible containments $H_1 \subseteq H_2 \subseteq \mathrm{GL}(3, \mathbb{F}_2)$. Appendix B discusses general additive codes.

Convention 1.1. When writing homomorphisms of left R-modules, inputs will be written on the left and homomorphisms on the right. For example, if $\lambda : M \to A$ is a homomorphism of left R-modules, then the homomorphism property of λ is expressed by $(r_1x_1 + r_2x_2)\lambda = r_1(x_1\lambda) + r_2(x_2\lambda)$, for $r_1, r_2 \in R$, $x_1, x_2 \in M$.

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2. Preliminaries

Although originally inspired by additive codes over $\mathbb{F}_4$, the results in this paper are naturally situated in the more general context of linear codes defined over module alphabets. This section defines many of the relevant concepts.

Let R be a finite, associative ring with 1; we do not assume that R is commutative. Let A be a finite, unital left R-module. The module A will serve as the alphabet for R-linear codes. A (left) *R-linear code* of length n over the alphabet A is a left R-submodule $C \subseteq A^n$.

Example 2.1. If $R = \mathbb{F}_2$ is the binary field and $A = \mathbb{F}_4$, which is a vector space over $\mathbb{F}_2$, then an R-linear code over A is exactly an additive $\mathbb{F}_4$-code, i.e., an additive subgroup of $\mathbb{F}_4^n$.

Example 2.2. If A is any finite abelian group, written additively, then an *additive code* over A is any subgroup C of A^n. Any such additive code can also be viewed as an R-linear code over A, as follows. The abelian group A is a $\mathbb{Z}$-module. Every element of A has finite order dividing the order $|A|$ of A. Let e be the exponent of A, i.e., the least common multiple of the orders of the elements of A; e divides $|A|$. Then $ea = 0$ for every $a \in A$, and, setting $R = \mathbb{Z}/e\mathbb{Z}$, we see that A is an R-module. Also see Appendix B. I thank the referees for suggesting this example.

Suppose the alphabet A is equipped with a *weight* w, i.e., a rational-valued function $w : A \to \mathbb{Q}$ such that $w(0) = 0$. (Some authors impose additional conditions on w, such as the triangle inequality and strict positivity. One could also allow w to take values in a larger field.) The weight is extended to A^n via $w(x) = \sum w(x_i)$, where $x = (x_1, x_2, \ldots, x_n) \in A^n$.

Given a weight w on A, there are two *symmetry groups* (left and right) associated to w:

$$\begin{aligned} \mathrm{Sym}_{\mathrm{lt}} &:= \{u \in \mathcal{U}(R) : w(ua) = w(a), \text{ for all } a \in A\}, \\ \mathrm{Sym}_{\mathrm{rt}} &:= \{\phi \in \mathrm{GL}_R(A) : w(a) = w(a\phi), \text{ for all } a \in A\}. \end{aligned}$$

Here, $\mathcal{U}(R)$ denotes the group of units (invertible elements) of the ring R, and $\mathrm{GL}_R(A)$ denotes the group of invertible R-linear homomorphisms of the left R-module A to itself.

A *monomial transformation* of A^n is an invertible R-linear homomorphism $T : A^n \to A^n$ of the form

$$(x_1, x_2, \ldots, x_n)T = (x_{\tau(1)}\phi_1, x_{\tau(2)}\phi_2, \ldots, x_{\tau(n)}\phi_n),$$

for all $(x_1, x_2, \ldots, x_n) \in A^n$, where τ is a permutation of $\{1, 2, \ldots, n\}$ and $\phi_1, \phi_2, \ldots, \phi_n$ are elements of $\mathrm{GL}_R(A)$. If, in addition, we require that the ϕ_i belong to the right symmetry group $\mathrm{Sym}_{\mathrm{rt}}$, then we say that T is a $\mathrm{Sym}_{\mathrm{rt}}$-*monomial transformation* of A^n. The reader will verify that a $\mathrm{Sym}_{\mathrm{rt}}$-monomial transformation T preserves the weight w: $w(xT) = w(x)$ for all $x \in A^n$.

Suppose $C \subset A^n$ is an R-linear code. Define the *monomial group* of C to be

$$\mathrm{Mon}(C) := \{T : T \text{ is } \mathrm{Sym}_{\mathrm{rt}}\text{-monomial and } CT = C\}.$$

Because monomial transformations are invertible and the codes are finite, $CT = C$ is equivalent to $CT \subseteq C$. In addition, define the *isometry group* of C to be

$$\mathrm{Isom}(C) := \{f \in \mathrm{GL}_R(C) : w(xf) = w(x), \text{ for all } x \in C\}.$$

As remarked above, every $\mathrm{Sym}_{\mathrm{rt}}$-monomial transformation preserves the weight w, so that restriction defines a group homomorphism:

$$\mathrm{restr} : \mathrm{Mon}(C) \to \mathrm{Isom}(C), \quad T \mapsto T|_C\,.$$

Denote the kernel of this homomorphism by $\mathrm{Mon}_0(C)$, so that

$$\mathrm{Mon}_0(C) = \{\mathrm{Sym}_{\mathrm{rt}}\text{-monomial } T : T|_C = \mathrm{id}_C\}.$$

In summary, we have a left exact sequence of groups

$$1 \to \mathrm{Mon}_0(C) \to \mathrm{Mon}(C) \to \mathrm{Isom}(C). \tag{2.1}$$

We will denote the image $\mathrm{restr}(\mathrm{Mon}(C))$ of $\mathrm{Mon}(C)$ under the restriction map by $\mathrm{rMon}(C)$, so that

$$\mathrm{rMon}(C) \subseteq \mathrm{Isom}(C) \subseteq \mathrm{GL}_R(C). \tag{2.2}$$

In the main theorem, Theorem 5.1, we will see how different the two groups $\mathrm{rMon}(C) \subseteq \mathrm{Isom}(C)$ can be in certain contexts.

3. Parametrized Codes and Multiplicity Functions

In this section, we define parametrized codes, examine their properties carefully, and interpret the groups in the exact sequence (2.1) in terms of multiplicity functions. As in the previous section, the coefficient ring R is a finite ring with 1, and the alphabet A is a finite, unital left R-module. We assume the alphabet A has a weight w satisfying $w(0) = 0$, with right symmetry group $\mathrm{Sym}_{\mathrm{rt}}$.

Let M be a finite, unital left R-module. A *parametrized code* of length n modeled on M is an injective homomorphism $\Lambda : M \to A^n$ of left R-modules. The image $C = M\Lambda$ of Λ is then an R-linear code over A of length n. Write the component functionals of Λ as $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, where each $\lambda_i \in \mathrm{Hom}_R(M, A)$; $\mathrm{Hom}_R(M, A)$ is the group of all left R-module homomorphisms from M to A. Then, a parametrized code modeled on M is an element of $\mathrm{Hom}_R(M, A^n) = \mathrm{Hom}_R(M, A)^n$.

Remark 3.1. Generator matrices provide examples of parametrized codes. If R is a finite field $\mathbb{F}$ and A also equals $\mathbb{F}$, then by choosing a basis for the vector space M, Λ can be expressed as an $m \times n$ matrix over $\mathbb{F}$, where $m = \dim M$. As is customary in coding theory, the elements of M (information bits) are written as row vectors of size m. The generator matrix is written to the right of the information bits, and matrix multiplication gives the map from M to $\mathbb{F}^n$. The columns of the generator matrix define coordinate functionals from M to $\mathbb{F}$.

Thus, in the general module context, the homomorphism $\Lambda : M \to A^n$ generalizes the notion of a generator matrix for a linear code. This allows Λ to be viewed as a linear encoder, with the module M playing the role of the information space. We call M the *information module*.

As a first step towards interpreting (2.1) in terms of parametrized codes, we describe parametrized codes up to the action of monomial transformations in terms of multiplicity functions.

Recall that a $\mathrm{Sym}_{\mathrm{rt}}$-monomial transformation $T : A^n \to A^n$ has the form

$$(a_1, a_2, \ldots, a_n)T = (a_{\tau(1)}\phi_1, a_{\tau(2)}\phi_2, \ldots, a_{\tau(n)}\phi_n), \tag{3.1}$$

where τ is a permutation of $\{1, 2, \ldots, n\}$ and the $\phi_i \in \mathrm{Sym}_{\mathrm{rt}}$. If we compose a monomial transformation $T : A^n \to A^n$ and a parametrized code $\Lambda : M \to A^n$ modeled on M, we get another parametrized code modeled on M, namely $\Lambda T : M \to A^n$. Examining the components of ΛT, we see that $\Lambda T = (\lambda_{\tau(1)}\phi_1, \lambda_{\tau(2)}\phi_2, \ldots, \lambda_{\tau(n)}\phi_n)$, where $\lambda_{\tau(i)}\phi_i$ is function composition. That is, T takes $(\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathrm{Hom}_R(M, A)^n$ to $(\lambda_{\tau(1)}\phi_1, \lambda_{\tau(2)}\phi_2, \ldots, \lambda_{\tau(n)}\phi_n)$.

We introduce a new space in order to capture the invariants in this situation. Being a subgroup of $\mathrm{GL}_R(A)$, $\mathrm{Sym}_{\mathrm{rt}}$ acts on A on the right: $\phi \in \mathrm{Sym}_{\mathrm{rt}}$ sends $a \in A$ to $a\phi \in A$. This action of $\mathrm{Sym}_{\mathrm{rt}}$ on A induces a right action of $\mathrm{Sym}_{\mathrm{rt}}$ on $\mathrm{Hom}_R(M, A)$: $\phi \in \mathrm{Sym}_{\mathrm{rt}}$ sends $\lambda \in \mathrm{Hom}_R(M, A)$ to $\lambda\phi \in \mathrm{Hom}_R(M, A)$. Denote by $\mathcal{O}^\sharp$ the set of all orbits of $\mathrm{Sym}_{\mathrm{rt}}$ acting on $\mathrm{Hom}_R(M, A)$. An orbit can be viewed as an R-module homomorphism $\lambda : M \to A$ modulo composition with ('scaling' by) an element of $\mathrm{Sym}_{\mathrm{rt}}$. We will often denote the $\mathrm{Sym}_{\mathrm{rt}}$-orbit of $\lambda \in \mathrm{Hom}_R(M, A)$ by $[\lambda]$ and the stabilizer subgroup of λ by $\mathrm{Stab}_{\mathrm{Sym}_{\mathrm{rt}}}(\lambda)$.

In the language of generator matrices over finite fields, this allows columns to be scaled by invertible field elements that do not change the weight (the elements of $\mathrm{Sym}_{\mathrm{rt}}$).

In going from $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ to $(\lambda_{\tau(1)}\phi_1, \lambda_{\tau(2)}\phi_2, \ldots, \lambda_{\tau(n)}\phi_n)$, we see that the order of terms can change but that the number of component functionals belonging to a given $\mathrm{Sym}_{\mathrm{rt}}$-orbit does not change. We next introduce a multiplicity function that counts the number of component functionals in a given $\mathrm{Sym}_{\mathrm{rt}}$-orbit.

Let $F(\mathcal{O}^\sharp, \mathbb{N}) = \{\eta : \mathcal{O}^\sharp \to \mathbb{N}\}$ be the set of all $\mathbb{N}$-valued functions on $\mathcal{O}^\sharp$; $\mathbb{N}$ is the set of nonnegative integers. There is a map $\mathrm{Hom}_R(M, A)^n \to F(\mathcal{O}^\sharp, \mathbb{N})$ that sends $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathrm{Hom}_R(M, A)^n$ to the function $\eta_\Lambda([\lambda]) = |\{i : \lambda_i \in [\lambda]\}|$. We say that η_Λ is the *multiplicity function* associated to the parametrized code $\Lambda : M \to A^n$. The discussion above has proved the following proposition.

Proposition 3.2. *Suppose $\Lambda_i : M \to A^n$, $i = 1, 2$, are two parametrized codes modeled on M. Then there exists a $\mathrm{Sym}_{\mathrm{rt}}$-monomial transformation T on A^n with $\Lambda_1 T = \Lambda_2$ if and only if $\eta_{\Lambda_1} = \eta_{\Lambda_2}$.*

Remark 3.3. Other authors have used geometric language to describe linear codes up to monomial equivalence, when $A = R$ and $\mathrm{Sym}_{\mathrm{rt}} = \mathcal{U}(R)$, in terms of multisets in a projective geometry. See, for example, [5, 13, 21].

Example 3.4. To illustrate the terminology just introduced, consider the following simple example. Let $R = \mathbb{F}_2$ and $A = \mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$, with $\omega^2 = 1 + \omega$; $\mathbb{F}_4$ is a vector space over $\mathbb{F}_2$ of dimension 2. Let M be a vector space over $\mathbb{F}_2$ of dimension 2; by fixing a basis of M, the elements of M will be viewed as row vectors over $\mathbb{F}_2$ of length 2. A functional $\lambda \in \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_4)$ can then be written as a column vector over $\mathbb{F}_4$ of length 2. There are $4^2 = 16$ such column vectors; i.e., $|\mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_4)| = 16$. If $x = (a, b) \in M$ and $\lambda = (1, \omega)^{\mathsf{T}} \in \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_4)$, then

$$x\lambda = (a, b) \begin{pmatrix} 1 \\ \omega \end{pmatrix} = a + b\omega \in \mathbb{F}_4. \tag{3.2}$$

Equip $\mathbb{F}_4$ with the Hamming weight wt, so that $\mathrm{wt}(0) = 0$ and $\mathrm{wt}(1) = \mathrm{wt}(\omega) = \mathrm{wt}(\omega^2) = 1$. Then $\mathrm{Sym}_{\mathrm{rt}} = \mathrm{GL}_{\mathbb{F}_2}(\mathbb{F}_4)$ is the group of 2×2 invertible matrices over $\mathbb{F}_2$; this group is isomorphic to the symmetric group Σ_3, acting as permutations on the set $\{1, \omega, \omega^2\}$. Let $\phi \in \mathrm{Sym}_{\mathrm{rt}}$ be the element that fixes 1 and tranposes ω and ω^2. If $x \in M$ and $\lambda \in \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_4)$ are as above, then $x\lambda\phi = a + b\omega^2$.

As noted, $|\mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_4)| = 16$. The right action of $\mathrm{Sym}_{\mathrm{rt}}$ on $\mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_4)$ has 5 orbits, i.e., $|\mathcal{O}^\sharp| = 5$. This is seen most easily by viewing elements of $\mathbb{F}_4$ as ordered pairs over $\mathbb{F}_2$ via a choice of basis, say $\{1, \omega\}$. Then elements $\lambda \in \mathrm{Hom}_{\mathbb{F}_2}(M, \mathbb{F}_4)$ are 2×2 matrices over $\mathbb{F}_2$. The right action by $\mathrm{Sym}_{\mathrm{rt}} = \mathrm{GL}_{\mathbb{F}_2}(\mathbb{F}_4)$ then has orbits represented by column reduced echelon matrices. The orbits themselves are listed below, both as 2×2 matrices over $\mathbb{F}_2$ and as column vectors over $\mathbb{F}_4$:

$$O_0 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\},$$
$$O_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} \omega \\ 0 \end{pmatrix}, \begin{pmatrix} \omega^2 \\ 0 \end{pmatrix} \right\},$$
$$O_2 = \left\{ \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} \omega \\ \omega \end{pmatrix}, \begin{pmatrix} \omega^2 \\ \omega^2 \end{pmatrix} \right\},$$
$$O_3 = \left\{ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ \omega \end{pmatrix}, \begin{pmatrix} 0 \\ \omega^2 \end{pmatrix} \right\},$$
$$\begin{aligned} O_4 &= \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} 1 \\ \omega \end{pmatrix}, \begin{pmatrix} \omega^2 \\ \omega \end{pmatrix}, \begin{pmatrix} 1 \\ \omega^2 \end{pmatrix}, \begin{pmatrix} \omega \\ \omega^2 \end{pmatrix}, \begin{pmatrix} \omega^2 \\ 1 \end{pmatrix}, \begin{pmatrix} \omega \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

Note that (3.2) can now be written as

$$x\lambda = (a, b) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (a, b) = a + b\omega. \tag{3.3}$$

We have already seen how $\mathrm{Sym}_{\mathrm{rt}}$-orbits play a role in understanding parametrized codes. Now we bring in the group $\mathcal{G} = \mathrm{GL}_R(M)$ of all invertible R-homomorphisms of the information module M to itself.

REMARK 3.5. Given Convention 1.1, the group $\mathcal{G}$ acts on M on the right. This induces a left action of $\mathcal{G}$ on $\mathrm{Hom}_R(M, A)$, with $f \in \mathcal{G}$ taking $\lambda \in \mathrm{Hom}_R(M, A)$ to $f\lambda$, function composition with inputs on the left. Note that this action of $\mathcal{G}$ commutes with the right action of $\mathrm{Sym}_{\mathrm{rt}}$ on $\mathrm{Hom}_R(M, A)$, so that the left action of $\mathcal{G}$ passes to a well-defined left action on $\mathcal{O}^\sharp$: $f[\lambda] = [f\lambda]$. Finally, there is an induced right action of $\mathcal{G}$ on $F(\mathcal{O}^\sharp, \mathbb{N})$, denoted ηf, with $(\eta f)([\lambda]) = \eta(f[\lambda]) = \eta([f\lambda])$. If $\Lambda : M \to A^n$ is a parametrized code modeled on M, then $\Lambda' = f\Lambda$ is another parametrized code modeled on M. Observe that $\eta_{\Lambda'} f = \eta_\Lambda$.

EXAMPLE 3.6. Continuing the notation of Example 3.4, the group $\mathcal{G} = \mathrm{GL}(2, \mathbb{F}_2)$ acts on row vectors on the right. For x and λ as in Example 3.4 and $f \in \mathcal{G}$ below, we see that

$$\text{for } f = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}: \quad xf = (b, a + b), \quad f\lambda = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Note in this example how f acts on the orbits $O_i \in \mathcal{O}^\sharp$: $fO_0 = O_0$, $fO_1 = O_3$, $fO_2 = O_1$, $fO_3 = O_2$, and $fO_4 = O_4$. In general, the six elements of $\mathcal{G}$ always fix O_0 and O_4. Their actions on the other orbits are summarized next.

$$\begin{array}{c|ccc}
f & fO_1 & fO_2 & fO_3 \\ \hline
\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & O_1 & O_2 & O_3 \\
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} & O_1 & O_3 & O_2 \\
\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} & O_2 & O_1 & O_3 \\
\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} & O_3 & O_1 & O_2 \\
\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} & O_2 & O_3 & O_1 \\
\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & O_3 & O_2 & O_1
\end{array} .$$

If a parametrized code is given by Λ below and f as above, then $\Lambda' = f\Lambda$ is

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad \Lambda' = f\Lambda = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Their respective multiplicity functions are

$$\begin{array}{c|ccccc}
 & O_0 & O_1 & O_2 & O_3 & O_4 \\ \hline
\eta_\Lambda & 0 & 1 & 2 & 3 & 0 \\
\eta_{\Lambda'} & 0 & 2 & 3 & 1 & 0
\end{array} .$$

We now interpret $\mathrm{Mon}(C)$ and $\mathrm{Mon}_0(C)$ from (2.1) in terms of parametrized codes. If $T \in \mathrm{Mon}(C)$, we have the following commutative diagram:

$$\begin{array}{ccc}
C & \xrightarrow{\text{inclusion}} & A^n \\
\downarrow{\scriptstyle T|_C} & & \downarrow{\scriptstyle T} \\
C & \xrightarrow{\text{inclusion}} & A^n
\end{array}$$

In writing this diagram, we are tacitly viewing C as a parametrized code, with $M = C$ (as modules) and Λ equaling the inclusion. Separating the roles of M and C gives the diagram:

$$\begin{array}{ccccc}
M & \xrightarrow{\Lambda} & C = M\Lambda & \xrightarrow{\text{inclusion}} & A^n \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle T|_C} & & \downarrow{\scriptstyle T} \\
M & \xrightarrow{\Lambda} & C = M\Lambda & \xrightarrow{\text{inclusion}} & A^n
\end{array}$$

We modify the definitions of $\mathrm{Mon}(C)$ and $\mathrm{Mon}_0(C)$ to account for the language of parametrized codes. Given a parametrized code $\Lambda : M \to A^n$, define

$$\mathrm{Mon}(\Lambda) = \{\mathrm{Sym}_{\mathrm{rt}}\text{-monomial } T\text{: } \Lambda T = f\Lambda \text{ for some } f \in \mathcal{G}\}.$$

Note that $f \in \mathcal{G}$ is unique, as it equals $T|_C$. This allows us to define a restriction homomorphism

$$\mathrm{restr} : \mathrm{Mon}(\Lambda) \to \mathcal{G}; \quad T \mapsto f.$$

Define $\mathrm{Mon}_0(\Lambda) = \ker \mathrm{restr}$, and write $\mathrm{restr}(\mathrm{Mon}(\Lambda))$ as $\mathrm{rMon}(\Lambda)$.

By using multiplicity functions, we can characterize $\mathrm{rMon}(\Lambda)$.

Proposition 3.7. *Let $\Lambda : M \to A^n$ be a parametrized code. An element $f \in \mathcal{G}$ belongs to $\mathrm{rMon}(\Lambda)$ if and only if $\eta_\Lambda f = \eta_\Lambda$.*

Proof. Given $f \in \mathcal{G}$, let $\Lambda' = f\Lambda$. Now apply Proposition 3.2 and the fact that $\eta_{\Lambda'} f = \eta_\Lambda$. □

Example 3.8. Continuing the notation of Example 3.6, suppose that $f \in \mathcal{G}$ and Λ are as given below. Then $\Lambda' = f\Lambda$ is

$$f = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \end{pmatrix}; \quad \Lambda' = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}.$$

Notice that Λ and Λ' have the same multiplicity function and that they are permutation equivalent.

For another example, define Λ'' with its multiplicity function $\eta_{\Lambda''}$:

$$\Lambda'' = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & \omega \end{pmatrix}; \qquad \begin{array}{c|ccccc} & O_0 & O_1 & O_2 & O_3 & O_4 \\ \hline \eta_{\Lambda''} & 0 & 1 & 2 & 3 & 1 \end{array}.$$

Because the multiplicities of O_1, O_2, O_3 are all different, it follows from Example 3.6 that $\mathrm{rMon}(\Lambda'')$ consists of the identity only.

Proposition 3.9. *Let $\Lambda : M \to A^n$ be a parametrized code, and let $T : A^n \to A^n$ be a $\mathrm{Sym}_{\mathrm{rt}}$-monomial transformation. Then $\mathrm{Mon}(\Lambda T) = T^{-1}\mathrm{Mon}(\Lambda)T$ and $\mathrm{Mon}_0(\Lambda T) = T^{-1}\mathrm{Mon}_0(\Lambda)T$.*

Proof. If $T_0 \in \mathrm{Mon}(\Lambda)$, then there exists $f \in \mathcal{G}$ with $\Lambda T_0 = f\Lambda$. By composing on the right with T and manipulating, we see

$$f(\Lambda T) = (f\Lambda)T = (\Lambda T_0)T = (\Lambda T)(T^{-1}T_0T).$$

Thus $T^{-1}T_0T \in \mathrm{Mon}(\Lambda T)$, using the same $f \in \mathcal{G}$. The reverse inclusion and the $\mathrm{Mon}_0(\Lambda)$ case are similar. □

Corollary 3.10. *Suppose $\Lambda_i : M \to A^n$, $i = 1, 2$, are two parametrized codes modeled on M. If $\eta_{\Lambda_1} = \eta_{\Lambda_2}$, then $\mathrm{rMon}(\Lambda_1) = \mathrm{rMon}(\Lambda_2)$. That is, $\mathrm{rMon}(\Lambda)$ depends only on η_Λ.*

Proof. By Proposition 3.2, there is a $\mathrm{Sym}_{\mathrm{rt}}$-monomial transformation T such that $\Lambda_2 = \Lambda_1 T$. Proposition 3.9 applies; its proof shows that $T_0 \in \mathrm{Mon}(\Lambda_1)$ and $T^{-1}T_0T \in \mathrm{Mon}(\Lambda_2)$ have the same $f \in \mathcal{G}$. □

Proposition 3.11. *Let $\Lambda : M \to A^n$ be a parametrized code, with multiplicity function η_Λ. Then the number of elements in $\mathrm{Mon}_0(\Lambda)$ is*

$$|\mathrm{Mon}_0(\Lambda)| = \prod_{[\lambda]\in\mathcal{O}^\sharp} |\mathrm{Stab}_{\mathrm{Sym}_{\mathrm{rt}}}(\lambda)|^{\eta_\Lambda([\lambda])}\eta_\Lambda([\lambda])!\,.$$

Proof. By Proposition 3.2 we may replace $\Lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ by a parametrized code that is 'locally constant.' That is, we may assume Λ has the property: if $[\lambda_i] = [\lambda_j]$, then $\lambda_i = \lambda_j$. By permuting equal component functionals and scaling each λ by elements of $\mathrm{Stab}_{\mathrm{Sym}_{\mathrm{rt}}}(\lambda)$, we get the stated number of elements. □

We now turn our attention to isometries.

Let $\Lambda : M \to A^n$ be a parametrized code. For any $x \in M$, the weight $w(x\Lambda)$ is

$$w(x\Lambda) = \sum_{i=1}^{n} w(x\lambda_i).$$

Lemma 3.12. *If $\Lambda' = \Lambda T$, where T is a $\mathrm{Sym}_{\mathrm{rt}}$-monomial transformation, then $w(x\Lambda') = w(x\Lambda)$, for all $x \in M$.*

Proof. Compute, using the form of T from (3.1):

$$w(x\Lambda') = \sum_{i=1}^{n} w(x\lambda'_i) = \sum_{i=1}^{n} w(x\lambda_{\tau(i)}\phi_i) = \sum_{i=1}^{n} w(x\lambda_{\tau(i)}) = \sum_{i=1}^{n} w(x\lambda_i) = w(x\Lambda),$$

where we make use of the definition of $\mathrm{Sym}_{\mathrm{rt}}$ to eliminate the ϕ_i and note that the permutation τ does not affect the sum. □

Lemma 3.12 shows that a $\mathrm{Sym}_{\mathrm{rt}}$-monomial transformation preserves the weight w. It also shows that the function $M \to \mathbb{Q}$, $x \mapsto w(x\Lambda)$, depends only on the multiplicity function η_Λ. We formalize this idea next.

Define $W : F(\mathcal{O}^\sharp, \mathbb{N}) \to F(M, \mathbb{Q})$ by

$$W(\eta)(x) = \sum_{[\lambda]\in\mathcal{O}^\sharp} w(x\lambda)\eta([\lambda]), \quad x \in M. \tag{3.4}$$

It is evident that $w(x\Lambda) = W(\eta_\Lambda)(x)$, for all $x \in M$. Also note that $w(ux\Lambda) = w(x\Lambda)$, for all $u \in \mathrm{Sym}_{\mathrm{lt}}$ and $x \in M$, by the definition of the left symmetry group $\mathrm{Sym}_{\mathrm{lt}}$. The group $\mathrm{Sym}_{\mathrm{lt}}$ acts on M, on the left, via the left module structure. Denote by $\mathcal{O}$ the orbit space

$\mathcal{O} = \mathrm{Sym}_{\mathrm{lt}} \backslash M$ of this action and the $\mathrm{Sym}_{\mathrm{lt}}$-orbit of $x \in M$ by $[x]$. The following lemma is now clear.

LEMMA 3.13. *The map $W : F(\mathcal{O}^\sharp, \mathbb{N}) \to F(\mathcal{O}, \mathbb{Q})$ is a well-defined additive homomorphism: $W(\eta + \eta') = W(\eta) + W(\eta')$.*

REMARK 3.14. As in Remark 3.5, the group $\mathcal{G} = \mathrm{GL}_R(M)$ acts on M on the right. This action commutes with the left action of $\mathrm{Sym}_{\mathrm{lt}}$ on M, so there is a well-defined right action of $\mathcal{G}$ on the orbit space $\mathcal{O}$: $[x]f = [xf]$ for $[x] \in \mathcal{O}$ and $f \in \mathcal{G}$. In turn, there is a well-defined left action by $\mathcal{G}$ on $F(\mathcal{O}, \mathbb{Q})$: $fv([x]) = v([x]f) = v([xf])$ for $f \in \mathcal{G}$, $v \in F(\mathcal{O}, \mathbb{Q})$, and $[x] \in \mathcal{O}$.

The next result is an exercise for the reader.

PROPOSITION 3.15. *The map $W : F(\mathcal{O}^\sharp, \mathbb{N}) \to F(M, \mathbb{Q})$ is $\mathcal{G}$-equivariant:*

$$W(\eta f^{-1}) = fW(\eta),$$

for $f \in \mathcal{G}$ and $\eta \in F(\mathcal{O}^\sharp, \mathbb{N})$.

We turn our attention to the isometry group of a parametrized code at last. Let $\Lambda : M \to A^n$ be a parametrized code. Define

$$\mathrm{Isom}(\Lambda) = \{f \in \mathcal{G} : W(\eta_\Lambda)(xf) = W(\eta_\Lambda)(x), \text{ for all } x \in M\}.$$

From Lemma 3.12 we see that $\mathrm{Isom}(\Lambda)$ depends only on the multiplicity function η_Λ and that restriction maps $\mathrm{Mon}(\Lambda)$ into $\mathrm{Isom}(\Lambda)$. We now state the counterpart to Proposition 3.7: determining when $f \in \mathcal{G}$ belongs to $\mathrm{Isom}(\Lambda)$.

PROPOSITION 3.16. *Let $\Lambda : M \to A^n$ be a parametrized code. An element $f \in \mathcal{G}$ belongs to $\mathrm{Isom}(\Lambda)$ if and only if $\eta_\Lambda f - \eta_\Lambda \in \ker W$.*

PROOF. This follows from a computation showing that $f \in \mathrm{Isom}(\Lambda)$ is equivalent to $W(\eta_\Lambda f) = W(\eta_\Lambda)$. This involves a reindexing argument and the fact that $f \in \mathrm{Isom}(\Lambda)$ if and only if $f^{-1} \in \mathrm{Isom}(\Lambda)$. The details are left to the reader. □

COROLLARY 3.17. *If $\ker W = 0$, then $\mathrm{rMon}(\Lambda) = \mathrm{Isom}(\Lambda)$ for any parametrized code $\Lambda : M \to A^n$.*

It will be important in Theorem 6.13 and other results that we generalize $F(\mathcal{O}^\sharp, \mathbb{N})$ and the map $W : F(\mathcal{O}^\sharp, \mathbb{N}) \to F(\mathcal{O}, \mathbb{Q})$ to the case of rational coefficients. Let $F(\mathcal{O}^\sharp, \mathbb{Q})$ be the set of all functions from $\mathcal{O}^\sharp$ to the rational numbers $\mathbb{Q}$; $F(\mathcal{O}^\sharp, \mathbb{Q})$ is a $\mathbb{Q}$-vector space of dimension $|\mathcal{O}^\sharp|$. Then define $W : F(\mathcal{O}^\sharp, \mathbb{Q}) \to F(\mathcal{O}, \mathbb{Q})$ by the same formula (3.4). It will also be convenient for later use to restrict the domain of W. Let

$$F_0(\mathcal{O}^\sharp, \mathbb{Q}) = \{\eta \in F(\mathcal{O}^\sharp, \mathbb{Q}) : \eta([0]) = 0\};$$

i.e., those η whose value on the zero functional is zero. Viewed in terms of generator matrices, this means there are no zero columns. Because we assume the weight w satisfies $w(0) = 0$, we see that $W(\eta)(0) = 0$, for any η. Thus, W takes values in

$$F_0(\mathcal{O}, \mathbb{Q}) = \{v \in F(\mathcal{O}, \mathbb{Q}) : v(0) = 0\}.$$

Because every $f \in \mathcal{G}$ satisfies $0f = 0$, both $F_0(\mathcal{O}, \mathbb{Q})$ and $F_0(\mathcal{O}^\sharp, \mathbb{Q})$ are invariant under the action of $\mathcal{G}$. The main feature of W follows.

Proposition 3.18. *The function $W : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q})$ is a linear transformation of $\mathbb{Q}$-vector spaces that is $\mathcal{G}$-equivariant:*

$$W(\eta f^{-1}) = fW(\eta),$$

for $f \in \mathcal{G}$ and $\eta \in F(\mathcal{O}^\sharp, \mathbb{Q})$.

Remark 3.19. Lemma 3.12 says that any $\mathrm{Sym}_{\mathrm{rt}}$-monomial transformation induces an isometry. This in turn was used to show that the map W is well-defined. The converse of Lemma 3.12 is called the extension property. To be formal, an alphabet A has the *extension property* with respect to a weight w if the following property holds: for any two linear codes $C_1, C_2 \subseteq A^n$, if $f : C_1 \to C_2$ is an isomorphism that preserves the weight w (i.e., $w(xf) = w(x)$ for all $x \in C_1$), then f extends to a $\mathrm{Sym}_{\mathrm{rt}}$-monomial transformation of A^n.

We translate the extension property into the context of parametrized codes. Let the information module M be C_1 itself. Let $\Lambda_1 : M \to A^n$ be the inclusion $C_1 \subseteq A^n$, and let $\Lambda_2 : M \to A^n$ equal f followed by the inclusion $C_2 \subseteq A^n$. Let η_{Λ_1} and η_{Λ_2} be the respective multiplicity functions. The weight-preservation property $w(xf) = w(x)$ then takes the form $w(x\Lambda_1) = w(x\Lambda_2)$ for all $x \in M$, i.e., $W(\eta_{\Lambda_1}) = W(\eta_{\Lambda_2})$. If the extension property holds, then there exists a $\mathrm{Sym}_{\mathrm{rt}}$-monomial transformation T of A^n such that $\Lambda_1 T = \Lambda_2$, i.e., $\eta_{\Lambda_1} = \eta_{\Lambda_2}$, by Proposition 3.2. In summary, the extension property holds if $W(\eta_{\Lambda_1}) = W(\eta_{\Lambda_2})$ implies $\eta_{\Lambda_1} = \eta_{\Lambda_2}$, i.e., if W is injective.

Proposition 3.20. *An alphabet A has the extension property with respect to a weight w if and only if the function $W : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q})$ is injective for every information module M.*

Remark 3.21. MacWilliams proved that finite fields have the extension property with respect to the Hamming weight in [**17, 18**]. A finite ring has the extension property with respect to the Hamming weight if ([**23**]) and only if ([**26**]) the ring is Frobenius. These results also apply to the homogeneous weight by work of Greferath and Schmidt [**12**]. Greferath, Nechaev, and Wisbauer [**11**] proved that a

Frobenius bimodule alphabet has the extension property with respect to the homogeneous and the Hamming weights. A general alphabet A has the extension property with respect to the homogeneous or the Hamming weight if and only if A is pseudo-injective and its socle is cyclic [27].

The literature also contains results on the extension property with respect to the Hamming weight for *non-linear* codes [19]. When the extension property fails to hold, Dyshko [7] has proved results similar to Theorem 5.1.

For weights on linear codes other than the homogeneous or the Hamming weight, our knowledge is fragmentary. There is a general criterion over Frobenius rings in [24], but it is often hard to apply. There has been recent progress for weights with maximal symmetry groups over products of chain rings [10] and over principal ideal rings [9]. The extension property for egalitarian weights holds over $\mathbb{Z}/m\mathbb{Z}$ [4] and over module alphabets having a cyclic socle [16]. For the Lee and Euclidean weights, recent work has shown that the extension property holds over the local rings $\mathbb{Z}/p^k\mathbb{Z}$, p prime, [8, 15], but we do not know if the extension property holds over a general $\mathbb{Z}/m\mathbb{Z}$. (Added in revision: Recently, Dyshko proved that the extension property holds for the Lee weight over any $\mathbb{Z}/m\mathbb{Z}$ [6].)

Matrix module context with the Hamming weight. We close this section with an example that will be the focus of later sections.

Example 3.22. We will refer to the following setting as the *matrix module context with the Hamming weight*. (For the homogeneous weight, see Remark 6.17.) Fix a finite field $\mathbb{F}_q$. Let R be $M_{k\times k}(\mathbb{F}_q)$, the ring of $k \times k$ matrices over $\mathbb{F}_q$, let the alphabet A be $M_{k\times\ell}(\mathbb{F}_q)$, and let the information module M be $M_{k\times m}(\mathbb{F}_q)$. (Because the ring R is simple, all nonzero finite modules over R have the form $M_{k\times t}(\mathbb{F}_q)$ for some positive integer t.) The left R-module structures are given by matrix multiplication. Let $w = \mathrm{wt}$ be the Hamming weight on A, so that $\mathrm{wt}(0) = 0$ and $\mathrm{wt}(a) = 1$ if $a \neq 0$. Then $\mathrm{Sym}_{\mathrm{lt}} = \mathcal{U} = \mathrm{GL}(k, \mathbb{F}_q)$ and $\mathrm{Sym}_{\mathrm{rt}} = \mathrm{GL}_R(A) = \mathrm{GL}(\ell, \mathbb{F}_q)$; $\mathrm{Sym}_{\mathrm{lt}}$ acts on M by left matrix multiplication, and $\mathrm{Sym}_{\mathrm{rt}}$ acts on A by right matrix multiplication.

One sees that $\mathrm{Hom}_R(M, A) = M_{m\times\ell}(\mathbb{F}_q)$, acting on elements of M by right matrix multiplication. Then $\mathcal{O}^\sharp$ consists of the right orbits of $\mathrm{Sym}_{\mathrm{rt}} = \mathrm{GL}(\ell, \mathbb{F}_q)$ acting on $\mathrm{Hom}_R(M, A) = M_{m\times\ell}(\mathbb{F}_q)$. Thus $\mathcal{O}^\sharp$ can be identitfied with the set of column reduced echelon matrices of size $m \times \ell$ over $\mathbb{F}_q$. Similarly, $\mathcal{O}$ consists of the left orbits of $\mathrm{Sym}_{\mathrm{lt}} = \mathrm{GL}(k, \mathbb{F}_q)$ acting on $M = M_{k\times m}(\mathbb{F}_q)$. Thus $\mathcal{O}$ can be identified with the set of row reduced echelon matrices of size $k \times m$ over $\mathbb{F}_q$. The group

$\mathcal{G} = \mathrm{GL}_R(M)$ is just $\mathrm{GL}(m, \mathbb{F}_q)$ acting on $M = M_{k\times m}(\mathbb{F}_q)$ by right matrix multiplication. The $\mathcal{G}$-action on $\mathrm{Hom}_R(M, A) = M_{m\times\ell}(\mathbb{F}_q)$ is by left matrix multiplication.

View $x \in M = M_{k\times m}(\mathbb{F}_q)$ and $\lambda \in \mathrm{Hom}_R(M, A) = M_{m\times\ell}(\mathbb{F}_q)$ as $\mathbb{F}_q$-linear transformations with inputs on the right. Thus $x : \mathbb{F}_q^m \to \mathbb{F}_q^k$ and $\lambda : \mathbb{F}_q^\ell \to \mathbb{F}_q^m$; $x\lambda$ is the composite, $x\lambda : \mathbb{F}_q^\ell \to \mathbb{F}_q^k$. Every element $x \in M$ and $\lambda \in \mathrm{Hom}_R(M, A)$ has a well-defined rank (denoted $\mathrm{rk}\, x$ or $\mathrm{rk}\, \lambda$) as a matrix over the field $\mathbb{F}_q$.

The following lemma summarizes a number of facts about the orbit spaces $\mathcal{O}$ and $\mathcal{O}^\sharp$ in the matrix module context. Recall that $\mathcal{G}$-actions on $\mathcal{O}$ and $\mathcal{O}^\sharp$ were defined in Remarks 3.5 and 3.14.

Lemma 3.23. *The following are true in the matrix module context with the Hamming weight:*

1. *$\mathcal{O}^\sharp$ can be identified with the set of linear subspaces of $\mathbb{F}_q^m$ of dimension at most ℓ, via $[\lambda] \leftrightarrow \mathrm{Im}\, \lambda$.*
2. *$\mathcal{O}$ can be identified with the set of linear subsaces of $\mathbb{F}_q^m$ of dimension at least $m - k$, via $[x] \leftrightarrow \ker x$.*
3. *For $x \in M$ and $u \in \mathrm{Sym}_{\mathrm{lt}}$, $\mathrm{rk}\, ux = \mathrm{rk}\, x$. Thus, $\mathrm{rk}[x]$, $[x] \in \mathcal{O}$, is well-defined.*
4. *For $\lambda \in \mathrm{Hom}_R(M, A)$ and $\phi \in \mathrm{Sym}_{\mathrm{rt}}$, $\mathrm{rk}\, \lambda\phi = \mathrm{rk}\, \lambda$. Thus, $\mathrm{rk}[\lambda]$, $[\lambda] \in \mathcal{O}^\sharp$, is well-defined.*
5. *The group $\mathcal{G}$ acts transitively on the sets $\mathcal{O}_a = \{[x] \in \mathcal{O} : \mathrm{rk}[x] = a\}$ and $\mathcal{O}_b^\sharp = \{[\lambda] \in \mathcal{O}^\sharp : \mathrm{rk}[\lambda] = b\}$.*

Proof. Any linear transformation $\nu : \mathbb{F}_q^\ell \to \mathbb{F}_q^m$ belonging to $[\lambda] \in \mathcal{O}^\sharp$ has the same image $\mathrm{Im}\, \nu = \mathrm{Im}\, \lambda$. Similarly, any linear transformation $y : \mathbb{F}_q^m \to \mathbb{F}_q^k$ belonging to $[x] \in \mathcal{O}$ has the same kernel $\ker y = \ker x$. The claims about rank follow from u and ϕ being invertible. The claims about transitivity of the $\mathcal{G}$-actions are proved by viewing orbits as subspaces and mapping a basis of one subspace to that of another of the same dimension. $\square$

The *q-binomial coefficient* $\begin{bmatrix} m \\ b \end{bmatrix}_q$ is defined for integers $0 \le b \le m$:

$$\begin{bmatrix} m \\ b \end{bmatrix}_q = \frac{(q^m - 1)(q^{m-1} - 1) \cdots (q^{m-b+1} - 1)}{(q^b - 1)(q^{b-1} - 1) \cdots (q - 1)}.$$

When $b > m$, $\begin{bmatrix} m \\ b \end{bmatrix}_q = 0$. We will make use of the fact that $\begin{bmatrix} m \\ b \end{bmatrix}_q$ equals the number of linear subspaces of dimension b in a vector space of dimension m over $\mathbb{F}_q$ (see, for example, [1, Chapter 3]).

Lemma 3.24. *Assume the matrix module context with the Hamming weight. Define multiplicity functions η_b on $\mathcal{O}^\sharp$, $b = 1, 2, \ldots, \min\{m, \ell\}$:*

$$\eta_b([\lambda]) = \begin{cases} 1, & \operatorname{rk}[\lambda] = b, \\ 0, & \operatorname{rk}[\lambda] \neq b. \end{cases}$$

Then for any $x \in M$,

$$W(\eta_b)(x) = \begin{bmatrix} m \\ b \end{bmatrix}_q - \begin{bmatrix} m - \operatorname{rk} x \\ b \end{bmatrix}_q .$$

Proof. By the definitions of W and η_b, we have

$$\begin{aligned} W(\eta_b)(x) &= \sum_{[\lambda] \in \mathcal{O}^\sharp} \operatorname{wt}(x\lambda)\eta_b([\lambda]) = \sum_{\operatorname{rk}[\lambda]=b} \operatorname{wt}(x\lambda) \\ &= \sum_{\substack{\operatorname{rk}[\lambda]=b \\ x\lambda \neq 0}} 1 = \sum_{\operatorname{rk}[\lambda]=b} 1 - \sum_{\substack{\operatorname{rk}[\lambda]=b \\ x\lambda = 0}} 1. \end{aligned}$$

The sum at the end of the first line is exactly the number of orbits $[\lambda]$ with $\operatorname{rk}[\lambda] = b$ such that $x\lambda \neq 0$. In turn (second line), this number equals the total number of $[\lambda]$ with $\operatorname{rk}[\lambda] = b$ minus the number of those with $x\lambda = 0$. We compute each of these latter numbers separately.

As in Lemma 3.23, we identify $[\lambda]$ with $\operatorname{Im} \lambda$. Of course, $\dim \operatorname{Im} \lambda = \operatorname{rk} \lambda = b$. Thus, the total number of $[\lambda]$ with $\operatorname{rk}[\lambda] = b$ is the number of linear subspaces of dimension b in $\mathbb{F}_q^m$. That number is $\left[{m \atop b} \right]_q$.

The equation $x\lambda = 0$ holds when $\operatorname{Im} \lambda \subseteq \ker x$. By the rank-nullity theorem, $\dim \ker x = \dim \mathbb{F}_q^m - \operatorname{rk} x = m - \operatorname{rk} x$. Then the number of $[\lambda]$ with $\operatorname{rk} \lambda = b$ and $x\lambda = 0$ is exactly the number of linear subspaces of dimension b in the vector space $\ker x$ of dimension $m - \operatorname{rk} x$. This number is $\left[{m - \operatorname{rk} x \atop b} \right]_q$. The formula follows. $\square$

The following lemma references the actions of $\mathcal{G}$ (and hence any subgroup $H \subseteq \mathcal{G}$) on $\mathcal{O}$ and $\mathcal{O}^\sharp$, Remarks 3.5 and 3.14. We make use of multiplicity functions with rational values, as discussed for Proposition 3.18.

Lemma 3.25. *Assume the matrix module context with the Hamming weight, and let H be a subgroup of $\mathcal{G}$. Suppose a multiplicity function η has the property that $W(\eta)$ is constant on H-orbits in $\mathcal{O}$. Then there exists a multiplicity function η' such that η' is constant on H-orbits in $\mathcal{O}^\sharp$ and $W(\eta') = W(\eta)$.*

Proof. The idea is to average η over H-orbits in $\mathcal{O}^\sharp$. Define

$$\eta'(\lambda) = \frac{1}{|H|} \sum_{h \in H} \eta(h\lambda).$$

Clearly, η' is constant on H-orbits: $\eta'(h\lambda) = \eta'(\lambda)$ for $h \in H$. Then a re-indexing computation shows that $W(\eta') = W(\eta)$. □

4. Closure

This section describes the notion of the closure of a subgroup with respect to a group action. This notion will be needed in the statement and proof of the main theorem because closure describes some natural necessary conditions a subgroup must satisfy in order to be a monomial group or an isometry group of a linear code. The necessary conditions are summarized in Proposition 4.7. There is a substantial literature on closure dating back at least to Wielandt's 2-closure theory, [22]. The notion of closure described here should really be called 1-*closure*, but the abbreviated form is used for convenience.

Let G be a finite group acting on a finite set X on the left; there is a similar treatment for right actions. The action of $g \in G$ on $x \in X$ will be denoted by $gx \in X$. Let H be a subgroup of G. For any $x \in X$, let $\operatorname{orb}_H(x) = \{hx : h \in H\}$ denote the H-orbit of x. Observe that $h \operatorname{orb}_H(x) = \operatorname{orb}_H(x)$, for any $x \in X$ and $h \in H$.

Define the *closure* $\bar{H}$ of H with respect to the action of G on X by

$$\bar{H} = \{g \in G : g \operatorname{orb}_H(x) = \operatorname{orb}_H(x), \text{ for all } x \in X\}.$$

Of course, $H \subseteq \bar{H}$. It follows from the definition that $\bar{H}$ is the largest subgroup of G with the same orbits as H. A subgroup H of G is *closed* with respect to the action of G on X if $H = \bar{H}$. We may denote $\bar{H}$ by $\operatorname{Cl}_X(H)$ when the set X is not obvious from context.

EXAMPLE 4.1. Let $X = \{1, 2, \ldots, n\}$ and let $G = \Sigma_n$, the symmetric group of all permutations of X. Let H be the cyclic n-subgroup of G generated by the n-cycle $(1, 2, \ldots, n)$, i.e., the permutation taking i to $i + 1$ (with n going to 1). Then H acts transitively on X, so that $\operatorname{orb}_H(x) = X$ for any $x \in X$. We then see that $\bar{H} = G$, so that H is not closed if $n > 2$.

In contrast, let C be the cyclic 2-group generated by the transposition $(1, 2)$. Any element of $\bar{C}$ must preserve the C-orbit $\{1, 2\}$ and fix the singleton C-orbits $\{i\}$, $i = 3, 4, \ldots, n$. Thus $\bar{C} = C$, and C is closed.

EXAMPLE 4.2. Assume the matrix module context with the Hamming weight, as in Example 3.22. Then $\mathcal{G} = \operatorname{GL}(m, \mathbb{F}_q)$ acts on $\mathcal{O} = \operatorname{GL}(k, \mathbb{F}_q)\backslash M_{k\times m}(\mathbb{F}_q)$ on the right and on $\mathcal{O}^\sharp = M_{m\times\ell}(\mathbb{F}_q)/\operatorname{GL}(\ell, \mathbb{F}_q)$ on the left.

Note that the closure (with respect to either action) of the trivial group $\{I_m\}$ is the group $\{\alpha I_m : \alpha \in \mathbb{F}_q^\times\}$ of all nonzero scalar multiples of the identity (the center of $\mathcal{G}$). Every closed subgroup of $\mathcal{G}$ with respect to either action must contain the center of $\mathcal{G}$.

Closed subgroups with respect to a group action are intimately related to stabilizer subgroups of an associated action on function spaces.

As above, suppose a finite group G acts on a finite set X on the left. Let S be any set, and denote the set of functions from X to S by $F(X,S) = \{\eta : X \to S\}$. Then G acts on $F(X,S)$ on the right by $(\eta g)(x) = \eta(gx)$, for $x \in X$. The *stabilizer subgroup* (or *isotropy subgroup*) of $\eta \in F(X,S)$ is $\mathrm{Stab}_G(\eta) = \{g \in G : \eta g = \eta\}$.

Proposition 4.3. *Suppose $H = \mathrm{Stab}_G(\eta)$ for some $\eta \in F(X,S)$. Then H is closed with respect to the action of G on X.*

Proof. Let $g \in \bar{H}$. Then $g\,\mathrm{orb}_H(x) = \mathrm{orb}_H(x)$ for all $x \in X$. This implies that, for any $x \in X$, $gx = hx$ for some $h \in H$ (with h depending upon g and x). Then, $(\eta g)(x) = \eta(gx) = \eta(hx) = (\eta h)(x) = \eta(x)$, because $H = \mathrm{Stab}_G(\eta)$. Thus $g \in \mathrm{Stab}_G(\eta) = H$, and H is closed with respect to the action on X. $\square$

Provided the set S is large enough, Proposition 4.3 has a converse.

Proposition 4.4. *Suppose H is closed with respect to the action of G on X, and suppose $|S| \geq |X/H|$. Then there exists $\eta \in F(X,S)$ such that $H = \mathrm{Stab}_G(\eta)$.*

Proof. The H-orbits partition X. Choose any function $\eta : X \to S$ that separates the H-orbits. That is, (i) η takes the same value on points in any given orbit: $\eta(hx) = \eta(x)$ for all $h \in H$ and $x \in X$; and (ii) η takes different values on different orbits: if $\mathrm{orb}_H(x) \neq \mathrm{orb}_H(y)$, then $\eta(x) \neq \eta(y)$. The size hypothesis $|S| \geq |X/H|$ guarantees the existence of such an η.

We claim $H = \mathrm{Stab}_G(\eta)$. Indeed, (i) says that $H \subseteq \mathrm{Stab}_G(\eta)$. Conversely, let $g \in \mathrm{Stab}_G(\eta)$. Then, for any $x \in X$, we have $\eta(gx) = (\eta g)(x) = \eta(x)$. By (ii), gx and x belong to the same H-orbit. Using hx in place of x, we see that, for a given $x \in X$, ghx, hx and x all belong to the same H-orbit. This implies that $g\,\mathrm{orb}_H(x) = \mathrm{orb}_H(x)$ for all $x \in X$, so that $g \in \bar{H}$. Because H is closed with respect to the action of G on X, we see that $g \in H$, as desired. $\square$

Suppose a finite group G acts on two sets X_1 and X_2. Then G acts on their disjoint union $X = X_1 \uplus X_2$.

Lemma 4.5. *Suppose G acts on X_1, X_2 and $X = X_1 \uplus X_2$. If $H \subseteq G$ is a subgroup of G, then $\mathrm{Cl}_X(H) \subseteq \mathrm{Cl}_{X_1}(H)$. If H is closed with respect to the action on X_1, then H is closed with respect to the action on X.*

PROOF. If $g \in G$ preserves the H-orbits in X, then, *a fortiori*, g preserves the H-orbits in X_1. Because $H \subseteq \mathrm{Cl}_X(H) \subseteq \mathrm{Cl}_{X_1}(H)$, if $H = \mathrm{Cl}_{X_1}(H)$, then $H = \mathrm{Cl}_X(H)$. $\square$

Suppose a finite group G acts on two sets F_1 and F_2. A function $\psi : F_1 \to F_2$ is *G-equivariant* if $\psi(g\eta) = g\psi(\eta)$, for all $g \in G$ and $\eta \in F_1$. Denote the stabilizer of $\eta_i \in F_i$ by $\mathrm{Stab}_G^{F_i}(\eta_i)$.

LEMMA 4.6. *Suppose G acts on F_1 and F_2, and suppose $\psi : F_1 \to F_2$ is G-equivariant. Then $\mathrm{Stab}_G^{F_1}(\eta) \subseteq \mathrm{Stab}_G^{F_2}(\psi(\eta))$ for all $\eta \in F_1$. If ψ is injective, then $\mathrm{Stab}_G^{F_1}(\eta) = \mathrm{Stab}_G^{F_2}(\psi(\eta))$ for all $\eta \in F_1$.*

PROOF. If $g \in G$ satisfies $g\eta = \eta$, then $g(\psi(\eta)) = \psi(g\eta) = \psi(\eta)$ by G-equivariance. The same equalities prove $\mathrm{Stab}_G^{F_2}(\eta) \subseteq \mathrm{Stab}_G^{F_1}(\psi(\eta))$ when ψ is injective. $\square$

The final result of this section applies the results above to give necessary conditions for a subgroup $H \subseteq \mathcal{G} = \mathrm{GL}_R(M)$ to be the monomial group or the isometry group of a linear code modeled on the R-module M. Said informally: not every subgroup of $\mathcal{G}$ gets to be an isometry group. Recall that several actions of $\mathcal{G}$ were defined in Remarks 3.5 and 3.14.

PROPOSITION 4.7. *Suppose $C \subseteq A^n$ is an R-linear code over the alphabet A modeled on the information module M. Then the subgroups $\mathrm{rMon}(C)$ and $\mathrm{Isom}(C)$ of $\mathcal{G} = \mathrm{GL}_R(M)$ satisfy:*

(1) $\mathrm{rMon}(C) \subseteq \mathrm{Isom}(C)$;
(2) $\mathrm{rMon}(C)$ *is closed with respect to the $\mathcal{G}$-action on $\mathcal{O}^\sharp$; and*
(3) $\mathrm{Isom}(C)$ *is closed with respect to the $\mathcal{G}$-action on $\mathcal{O}$.*

PROOF. The first statement is just (2.2). Suppose the linear code C is determined by the multiplicity function $\eta \in F_0(\mathcal{O}^\sharp, \mathbb{Q})$. Proposition 3.7 says that $\mathrm{rMon}(\eta)$ equals the stabilizer subgroup $\mathrm{Stab}_{\mathcal{G}}(\eta)$. In turn, Proposition 4.3 says that $\mathrm{rMon}(\eta) = \mathrm{Stab}_{\mathcal{G}}(\eta)$ is closed with respect to the action of $\mathcal{G}$ on $\mathcal{O}^\sharp$. For $\mathrm{Isom}(C)$, use Proposition 3.16 and the $\mathcal{G}$-equivariance of the map W, Proposition 3.18. $\square$

5. Statement of Main Theorem and Plan of Attack

The main theorem is a partial converse to Proposition 4.7 in that it addresses the relative sizes of the groups $\mathrm{Isom}(C)$ and $\mathrm{rMon}(C)$ in the matrix module context with the Hamming weight of Example 3.22.

The group $\mathcal{G} = \mathrm{GL}_R(M) = \mathrm{GL}(m, \mathbb{F}_q)$ acts on $\mathcal{O}$ and $\mathcal{O}^\sharp$, as in Remarks 3.5 and 3.14. In fact, Lemma 3.23 shows that $\mathcal{G}$ acts on $\mathcal{O}_b^\sharp$, those $[\lambda] \in \mathcal{O}^\sharp$ with $\mathrm{rk}[\lambda] = b$. Then $\mathcal{G}$ also acts on $X = \biguplus_{b=k+1}^{\ell} \mathcal{O}_b^\sharp$. Let

$H_1 \subseteq H_2 \subseteq \mathcal{G}$ be two subgroups of $\mathcal{G}$, where we assume H_1 is closed with respect to the action of $\mathcal{G}$ on X and H_2 is closed with respect to the action of $\mathcal{G}$ on $\mathcal{O}$. Then the main theorem says that, under a certain hypothesis on the alphabet A, we can find a linear code whose monomial transformations yield the smaller group and whose isometries yield the larger group. Using Lemma 4.5, we observe that the closure hypothesis on H_1 is stronger than the necessary condition on $\mathrm{rMon}(C)$ in Proposition 4.7. This is the best result that our method of proof allows.

THEOREM 5.1. *Assume the matrix module context with the Hamming weight. That is,* $R = M_{k\times k}(\mathbb{F}_q)$, $A = M_{k\times \ell}(\mathbb{F}_q)$, $M = M_{k\times m}(\mathbb{F}_q)$, *and* A *is equipped with the Hamming weight* wt.

I. Suppose $k \geq \ell$ *or* $\ell = m = k+1$. *Then,* $\mathrm{rMon}(\Lambda) = \mathrm{Isom}(\Lambda)$ *for any parametrized code* $\Lambda : M \to A^n$.

II. Suppose $k \geq \ell$ *or* $\ell = m = k+1$. *Let* H *be any subgroup of* $\mathcal{G} = \mathrm{GL}_R(M)$ *that is closed with respect to the action of* $\mathcal{G}$ *on* $\mathcal{O}^\sharp$. *Then, there exists a parametrized code* Λ *modeled on* M *such that*

$$\mathrm{rMon}(\Lambda) = \mathrm{Isom}(\Lambda) = H.$$

III. Suppose $k < \ell \leq m$, *excluding the case* $\ell = m = k+1$. *Let* H_1, H_2 *be any two subgroups of* $\mathcal{G} = \mathrm{GL}_R(M)$ *satisfying* $H_1 \subseteq H_2$, *with* H_1 *closed with respect to the action of* $\mathcal{G}$ *on* $X = \biguplus_{b=k+1}^{\ell} \mathcal{O}_b^\sharp$ *and* H_2 *closed with respect to the action of* $\mathcal{G}$ *on* $\mathcal{O}$. *Then, there exists a parametrized code* Λ *modeled on* M *such that*

$$\mathrm{rMon}(\Lambda) = H_1 \text{ and } \mathrm{Isom}(\Lambda) = H_2.$$

Theorem 5.1 also holds for the homogeneous weight: see Remark 6.17. The proof of Theorem 5.1 appears in Section 7.

WARNING 5.2. Theorem 5.1 does not say anything about the lengths of the codes produced. The lengths may be quite large, as Example 9.5 demonstrates.

COROLLARY 5.3. *Suppose* $k < \ell \leq m$, *excluding the case* $\ell = m = k+1$. *Let* $H_1 = \{\alpha I_m : \alpha \in \mathbb{F}_q^\times\}$ *and* $H_2 = \mathrm{GL}(m, \mathbb{F}_q)$. *Then there exists a parametrized code* Λ *such that*

$$\mathrm{rMon}(\Lambda) = \{\alpha I_m : \alpha \in \mathbb{F}_q^\times\} \text{ and } \mathrm{Isom}(\Lambda) = \mathrm{GL}(m, \mathbb{F}_q).$$

Outline of the proof of Theorem 5.1. Given a subgroup $H \subseteq \mathcal{G}$, it is easy to build a code Λ with $H \subseteq \mathrm{rMon}(\Lambda)$. Indeed, the group $\mathcal{G}$, and hence the subgroup H, acts on the orbit space $\mathcal{O}^\sharp$. Choose a multiplicity function $\eta \in F_0(\mathcal{O}^\sharp, \mathbb{N})$ that is constant on the H-orbits on $\mathcal{O}^\sharp$. Then $\eta(f[\lambda]) = \eta([\lambda])$ for all $f \in H$ and all $[\lambda] \in \mathcal{O}^\sharp$, i.e., $\eta f = \eta$

for all $f \in H$. By Proposition 3.7, $H \subseteq \mathrm{rMon}(\Lambda)$, where Λ is the code determined by η. Of course, this also implies $H \subseteq \mathrm{Isom}(\Lambda)$.

When one has two subgroups $H_1 \subseteq H_2 \subseteq \mathcal{G}$, one first finds a code Λ (in a manner similar to the above) such that $H_2 \subseteq \mathrm{rMon}(\Lambda) \subseteq \mathrm{Isom}(\Lambda)$. One then needs to modify the multiplicity function η_Λ of Λ in such a way that $\mathrm{rMon}(\Lambda)$ becomes smaller yet contains H_1, while $\mathrm{Isom}(\Lambda)$ remains unchanged. This is accomplished by adding terms to η_Λ that belong to $\ker W$ (so that $\mathrm{Isom}(\Lambda)$ remains unchanged), that are no longer constant on H_2-orbits (so that $\mathrm{rMon}(\Lambda)$ becomes smaller), but that are constant on H_1-orbits (so that $H_1 \subseteq \mathrm{rMon}(\Lambda)$). The closure hypotheses are used to prove equalities in the containments $H_2 \subseteq \mathrm{Isom}(\Lambda)$ and $H_1 \subseteq \mathrm{rMon}(\Lambda)$.

Key to making this argument work is an understanding of the form of the elements of $\ker W$ in order that η_Λ can be modified appropriately. It will also be necessary to show that the mapping W is surjective in order to utilize a closure argument to prove $H_2 = \mathrm{Isom}(\Lambda)$. Both of these topics, W being surjective and the form of a basis of $\ker W$, are addressed in the next section.

6. Analysis of the Mapping W

The objective of this section is to understand the linear map $W : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q})$ of Proposition 3.18 in the matrix module context with the Hamming weight. The case of primary interest will be $k < \ell \leq m$, but a few others cases will also be discussed. When $k < \ell \leq m$, Theorem 6.12 says that W is surjective, and Theorem 6.13 gives a basis for $\ker W$. Both of these results are needed for the proof of Theorem 5.1.

Lemma 6.1. *Assume the matrix module context with the Hamming weight, Example 3.22. Then*

$$\dim_{\mathbb{Q}} F_0(\mathcal{O}^\sharp, \mathbb{Q}) = |\mathcal{O}^\sharp| - 1 = \sum_{b=1}^{\min\{\ell,m\}} \begin{bmatrix} m \\ b \end{bmatrix}_q ,$$

$$\dim_{\mathbb{Q}} F_0(\mathcal{O}, \mathbb{Q}) = |\mathcal{O}| - 1 = \sum_{a=1}^{\min\{k,m\}} \begin{bmatrix} m \\ a \end{bmatrix}_q .$$

Proof. The two cases are similar. For the $\mathcal{O}$ case, $|\mathcal{O}| - 1$ counts the number of nonzero row reduced echelon matrices of size $k \times m$ over $\mathbb{F}_q$. Such row reduced echelon matrices of rank a correspond to linear subspaces of dimension a in $\mathbb{F}_q^m$, and the formula follows. □

Proposition 6.2. *Assume the matrix module context with the Hamming weight.*

(1) *If* $k \geq \ell$*, then* $\ker W = 0$.
(2) *If* $k < \min\{\ell, m\}$*, then* $\ker W \neq 0$.

Proof. Item (1) is equivalent to the extension theorem ([23, Theorem 6.3] for the case of Frobenius rings, e.g., when $k = \ell$; [27, Theorem 5.2] for the general case of $k \geq \ell$).

Item (2) follows because in this situation

$$\dim_{\mathbb{Q}} F_0(\mathcal{O}^\sharp, \mathbb{Q}) - \dim_{\mathbb{Q}} F_0(\mathcal{O}, \mathbb{Q}) = \sum_{b=k+1}^{\min\{\ell,m\}} \begin{bmatrix} m \\ b \end{bmatrix}_q > 0. \tag{6.1}$$

Thus, $\dim_{\mathbb{Q}} \ker W > 0$. (First shown in [27, §7.7].) □

We now describe a basis for $\ker W$ when $k < \ell \leq m$. In order to do so, we need to introduce the Möbius function associated to the partially ordered set of all linear subspaces of $\mathbb{F}_q^m$.

Denote by $\mathcal{L}(\mathbb{F}_q^m)$ the set of all linear subspaces of $\mathbb{F}_q^m$, partially ordered by set inclusion; i.e., $V_1 \leq V_2$ when $V_1 \subseteq V_2$. If $V_1 \leq V_2$, define the *interval* $[V_1, V_2]$ by $[V_1, V_2] = \{U \in \mathcal{L}(\mathbb{F}_q^m) : V_1 \leq U \leq V_2\}$. The *Möbius function* μ associated to $\mathcal{L}(\mathbb{F}_q^m)$ is a function $\mu : \mathcal{L}(\mathbb{F}_q^m) \times \mathcal{L}(\mathbb{F}_q^m) \to \mathbb{Z}$ that is uniquely determined by three properties (see [20]):

(1) $\mu(V_1, V_2) = 0$ if $V_1 \not\leq V_2$;
(2) $\mu(V, V) = 1$ for all $V \in \mathcal{L}(\mathbb{F}_q^m)$;
(3) If $V_1 < V_2$, i.e., $V_1 \leq V_2$ and $V_1 \neq V_2$, then

$$\sum_{U \in [V_1, V_2]} \mu(V_1, U) = 0.$$

For $\mathcal{L}(\mathbb{F}_q^m)$, $\mu(V_1, V_2) = (-1)^c q^{\binom{c}{2}}$, where $c = \dim V_2 - \dim V_1$. In particular, $\mu(0, V) = (-1)^d q^{\binom{d}{2}}$, where $d = \dim V$.

Suppose $k < \ell \leq m$. Recall from Lemma 3.23 that we identify $\mathcal{O}^\sharp$ with a subset of $\mathcal{L}(\mathbb{F}_q^m)$, via $[\lambda] \leftrightarrow \operatorname{Im} \lambda$. Similarly, an orbit $[x] \in \mathcal{O}$ corresponds to $\ker x$. For any $[\lambda] \in \mathcal{O}^\sharp$, define a multiplicity function $\eta_{[\lambda]} \in F_0(\mathcal{O}^\sharp, \mathbb{Q})$ by

$$\eta_{[\lambda]}([\nu]) = \begin{cases} 0, & \operatorname{Im} \nu \not\leq \operatorname{Im} \lambda \text{ or } \nu = 0, \\ \mu(0, \operatorname{Im} \nu), & \operatorname{Im} \nu \leq \operatorname{Im} \lambda. \end{cases} \tag{6.2}$$

Of course, $\mu(0, \operatorname{Im} \nu) = (-1)^{\operatorname{rk} \nu} q^{\binom{\operatorname{rk} \nu}{2}}$.

Recall that the group $\mathcal{G} = \mathrm{GL}_R(M)$ acts on $F_0(\mathcal{O}^\sharp, \mathbb{Q})$, as in Remark 3.5.

Lemma 6.3. *Let $f \in \mathcal{G}$. Then $\eta_{[f\lambda]} f = \eta_{[\lambda]}$.*

Proof. From the definitions we have

$$\begin{aligned} (\eta_{[f\lambda]} f)([\nu]) &= \eta_{[f\lambda]}([f\nu]) \\ &= \begin{cases} 0, & \operatorname{Im} f\nu \nleq \operatorname{Im} f\lambda \text{ or } f\nu = 0, \\ \mu(0, \operatorname{Im} f\nu), & \operatorname{Im} f\nu \leq \operatorname{Im} f\lambda. \end{cases} \end{aligned}$$

But $\operatorname{Im} f\nu \leq \operatorname{Im} f\lambda$ if and only if $\operatorname{Im} \nu \leq \operatorname{Im} \lambda$, because f is invertible. In addition, $\mu(0, \operatorname{Im} f\nu) = \mu(0, \operatorname{Im} \nu)$ because the values of μ depend only on rank. Thus, the last expression above equals $\eta_{[\lambda]}([\nu])$. ☐

Lemma 6.4. *For any nonzero $[\lambda] \in \mathcal{O}^\sharp$ and any $x \in M$,*

$$W(\eta_{[\lambda]})(x) = \begin{cases} 0, & \operatorname{Im} \lambda \cap \ker x \neq 0, \\ -1, & \operatorname{Im} \lambda \cap \ker x = 0. \end{cases}$$

Proof. As in Lemma 3.23, we view x as a linear transformation $\mathbb{F}_q^m \to \mathbb{F}_q^k$ and λ as a linear transformation $\mathbb{F}_q^\ell \to F_q^m$. Then

$$\begin{aligned} W(\eta_{[\lambda]})(x) &= \sum_{[\nu] \in \mathcal{O}^\sharp} \operatorname{wt}(x\nu) \eta_{[\lambda]}([\nu]) \\ &= \sum_{\operatorname{Im} \nu \leq \operatorname{Im} \lambda} \operatorname{wt}(x\nu) \mu(0, \operatorname{Im} \nu). \end{aligned}$$

We note that the value of $\eta_{[\lambda]}([0])$ has no effect on $W(\eta_{[\lambda]})(x)$; by setting $\eta_{[\lambda]}([0]) = 0$, we have $\eta_{[\lambda]} \in F_0(\mathcal{O}^\sharp, \mathbb{Q})$. We now split the sum into terms where $x\nu = 0$ and those where $x\nu \neq 0$. The sum with $x\nu = 0$ vanishes, as $\operatorname{wt}(x\nu) = 0$, so only the sum with $x\nu \neq 0$ remains. In that sum, $\operatorname{wt}(x\nu) = 1$.

$$\begin{aligned} W(\eta_{[\lambda]})(x) &= \sum_{\substack{\operatorname{Im} \nu \leq \operatorname{Im} \lambda \\ x\nu = 0}} \operatorname{wt}(x\nu) \mu(0, \operatorname{Im} \nu) + \sum_{\substack{\operatorname{Im} \nu \leq \operatorname{Im} \lambda \\ x\nu \neq 0}} \operatorname{wt}(x\nu) \mu(0, \operatorname{Im} \nu) \\ &= \sum_{\substack{\operatorname{Im} \nu \leq \operatorname{Im} \lambda \\ x\nu \neq 0}} \mu(0, \operatorname{Im} \nu) \\ &= \sum_{\operatorname{Im} \nu \leq \operatorname{Im} \lambda} \mu(0, \operatorname{Im} \nu) - \sum_{\substack{\operatorname{Im} \nu \leq \operatorname{Im} \lambda \\ x\nu = 0}} \mu(0, \operatorname{Im} \nu) \\ &= \sum_{\operatorname{Im} \nu \in [0, \operatorname{Im} \lambda]} \mu(0, \operatorname{Im} \nu) - \sum_{\operatorname{Im} \nu \in [0, \operatorname{Im} \lambda \cap \ker x]} \mu(0, \operatorname{Im} \nu) \end{aligned}$$

The last line makes use of the fact that $x\nu = 0$ if and only if $\operatorname{Im} \nu \leq \ker x$. The left sum vanishes by the properties of the Möbius function

μ, because $0 < \operatorname{Im}\lambda$ (the hypothesis that λ is nonzero). Similarly, the right sum will vanish if and only if $0 < \operatorname{Im}\lambda \cap \ker x$. When $\operatorname{Im}\lambda \cap \ker x = 0$, we see that $W(\eta_{[\lambda]})(x) = -1$. $\square$

COROLLARY 6.5. *If* $\operatorname{rk}[\lambda] > \operatorname{rk} x$*, then* $W(\eta_{[\lambda]})(x) = 0$*. In particular, if* $k+1 \leq \operatorname{rk}[\lambda] \leq \ell$*, then* $\eta_{[\lambda]} \in \ker W$*.*

PROOF. The first hypothesis implies that $\dim \operatorname{Im}\lambda + \dim \ker x > m$, so $\operatorname{Im}\lambda \cap \ker x \neq 0$. Now apply Lemma 6.4. Because $\operatorname{rk} x \leq k$ for all x, the second statement now follows from the first. $\square$

The vector space $F_0(\mathcal{O}^\sharp, \mathbb{Q})$ has a standard basis: the indicator functions of the nonzero orbits. That is, for any nonzero orbit $[\lambda] \in \mathcal{O}^\sharp$, define $\delta_{[\lambda]} \in F_0(\mathcal{O}^\sharp, \mathbb{Q})$ by

$$\delta_{[\lambda]}([\nu]) = \begin{cases} 1, & [\nu] = [\lambda], \\ 0, & [\nu] \neq [\lambda]. \end{cases}$$

We refer to this basis as $\mathcal{B}_1$. Let $\mathcal{B}_2 = \{\eta_{[\lambda]} : 0 < \operatorname{rk}[\lambda] \leq \ell\}$.

LEMMA 6.6. *The set* $\mathcal{B}_2$ *is a basis of* $F_0(\mathcal{O}^\sharp, \mathbb{Q})$*.*

PROOF. The two sets $\mathcal{B}_1$ and $\mathcal{B}_2$ have the same number of elements: $|\mathcal{O}^\sharp| - 1$. Order the nonzero orbits $[\lambda] \in \mathcal{O}^\sharp$ from lower values of $\operatorname{rk}[\lambda]$ to higher values. Express the elements of $\mathcal{B}_2$ in terms of the basis $\mathcal{B}_1$ of $F_0(\mathcal{O}^\sharp, \mathbb{Q})$. This defines a transition matrix P whose rows and columns are indexed by nonzero orbits $[\lambda] \in \mathcal{O}^\sharp$, and where the column indexed by $[\lambda]$ consists of the coefficients of $\eta_{[\lambda]}$ expressed as a linear combination of the $\delta_{[\nu]}$; i.e., the $([\nu], [\lambda])$-entry of P is $\eta_{[\lambda]}([\nu])$. Recall that we can identify $\mathcal{O}^\sharp$ with a subset of the partially ordered set $\mathcal{L}(\mathbb{F}_q^m)$. By using the partial order on $\mathcal{L}(\mathbb{F}_q^m)$, (6.2) shows that the matrix P is block upper-triangular, with diagonal blocks equal to diagonal matrices of the form $\mu(0, \operatorname{Im}\lambda) = (-1)^{\operatorname{rk}\lambda} q^{\binom{\operatorname{rk}\lambda}{2}}$ times an identity matrix of size $\begin{bmatrix} m \\ \operatorname{rk}[\lambda] \end{bmatrix}_q$. The matrix P is then invertible over $\mathbb{Q}$, and thus $\mathcal{B}_2$ is a basis of $F_0(\mathcal{O}^\sharp, \mathbb{Q})$. $\square$

We next define a matrix W_0 of size $(|\mathcal{O}| - 1) \times (|\mathcal{O}^\sharp| - 1)$. The rows of W_0 are indexed by nonzero orbits $[x] \in \mathcal{O}$, and the columns are indexed by nonzero $[\lambda] \in \mathcal{O}^\sharp$. The entry of W_0 at position $([x], [\lambda])$ is defined to be $\operatorname{wt}(x\lambda)$. Observe that the $([x], [\lambda])$-entry of the matrix product $W_0 P$ is exactly $W(\eta_{[\lambda]})(x)$. If we order the orbits $[x] \in \mathcal{O}$ from lower values of $\operatorname{rk} x$ to higher values and order the $[\lambda]$ as we did for P, then W_0P has the form

$$W_0P = \left[\begin{array}{ccccc|ccc} S_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & S_2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ * & * & S_3 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ * & * & * & \cdots & S_k & 0 & \cdots & 0 \end{array}\right], \tag{6.3}$$

where the column blocks to the left of the vertical line represent $[\lambda]$ with $\mathrm{rk}[\lambda] = 1, 2, \ldots, k$. To the right of the vertical line is the range $k + 1 \le \mathrm{rk}[\lambda] \le \ell$. The location of zeros follows from Corollary 6.5.

The matrices S_b, $b = 1, 2, \ldots, k$, are square of size $\left[{m \atop b} \right]_q$. If each S_b is invertible, then the matrix W_0P, and hence the matrix W_0, will have maximal rank $|\mathcal{O}| - 1$.

Proposition 6.7. *The matrices S_b, $b = 1, 2, \ldots, k$, are invertible over $\mathbb{Q}$.*

In order to prove Proposition 6.7, we will need a few lemmas. The lemmas will be phrased in terms of the linear geometry of $\mathbb{F}_q^m$. On $\mathbb{F}_q^m$, define the usual $\mathbb{F}_q$-valued *dot product* by

$$x \cdot y = \sum_{i=1}^{m} x_i y_i, \quad x = (x_1, x_2, \ldots, x_m), y = (y_1, y_2, \ldots, y_m) \in \mathbb{F}_q^m.$$

Given a linear subspace $Q \subseteq \mathbb{F}_q^m$, define $Q^\perp = \{y \in \mathbb{F}_q^m : x \cdot y = 0, \text{ all } x \in Q\}$. If $\dim Q = b$, then $\dim Q^\perp = m - b$. Also, $(Q^\perp)^\perp = Q$.

Lemma 6.8. *Suppose $P, Q \subseteq \mathbb{F}_q^m$ are linear subspaces with $\dim P = \dim Q$. Then $\dim(P \cap Q^\perp) = \dim(Q \cap P^\perp)$.*

Proof. We use the general fact that $(P \cap Q^\perp)^\perp = P^\perp + Q$ and compare dimensions:

$$m - \dim(P \cap Q^\perp) = m - \dim P + \dim Q - \dim(P^\perp \cap Q).$$

Using the hypothesis that $\dim P = \dim Q$, the result follows. □

Lemma 6.9. *Fix a subspace $Q \subseteq \mathbb{F}_q^m$ with $\dim Q = b$. Then*

$$|\{P \subseteq \mathbb{F}_q^m : \dim P = b \text{ and } P \cap Q^\perp = 0\}| = C_1(b),$$

where $C_1(b)$ is a positive constant that depends only on b, m, and q (not the choice of subspace Q).

Proof. Because $\dim Q = b$, $\dim Q^\perp = m - b$. Any vector space complement P (i.e., $P \oplus Q^\perp = \mathbb{F}_q^m$) will be an element of the given set. Being nonempty, the given set has a positive number of elements.

Given two subspaces Q, Q' of dimension b, there is an isomorphism $\mathbb{F}_q^m \to \mathbb{F}_q^m$ taking $Q^\perp$ to $Q'^\perp$. Because isomorphisms preserve incidence, the isomorphism carries the given set for Q to that for Q'. This implies that the cardinality of the set is independent of the choice of Q. $\square$

LEMMA 6.10. *Fix two subspaces $P, Q \subseteq \mathbb{F}_q^m$, both of dimension b, with $\dim(P^\perp \cap Q^\perp) = e \leq m - b$. Then*

$$|\{U \subseteq \mathbb{F}_q^m : \dim U = b, P \cap U^\perp = 0, \dim(U \cap Q^\perp) = d\}| = C_2(b, d, e),$$

where $C_2(b, d, e)$ is a constant depending only on b, d, e, m, and q (not the choice of subspaces P, Q). Moreover, when $d = m - b - e$, the constant $C_2(b, m - b - e, e)$ is positive, and when $d > m - b - e$, $C_2(b, d, e) = 0$.

PROOF. By Lemma 6.8, the given set can be expressed as

$$\{U \subseteq \mathbb{F}_q^m : \dim U = b, U \cap P^\perp = 0, \dim(U \cap Q^\perp) = d\}.$$

The independence of the cardinality of the choices of P and Q follows as in the proof of Lemma 6.9.

Notice that $m - b - e$ is the codimension of $P^\perp \cap Q^\perp$ inside $Q^\perp$. If $d > m - b - e$ and U satisfies $\dim(U \cap Q^\perp) = d$, then $U \cap (P^\perp \cap Q^\perp)$ is nonzero. Thus, the given set is empty, and $C_2(b, d, e) = 0$.

One calculates that $\dim(P^\perp + Q^\perp) = 2m - 2b - e \leq m$, so that the codimension of $P^\perp + Q^\perp$ inside $\mathbb{F}_q^m$ is $2b + e - m$. When $d = m - b - e$, there exists $U \subseteq \mathbb{F}_q^m$ of the form $U = U_1 \oplus U_2$, where U_1 is a complement of $P^\perp \cap Q^\perp$ inside $Q^\perp$ and U_2 is a complement of $P^\perp + Q^\perp$ inside $\mathbb{F}_q^m$. Then $\dim U_1 = d = m - b - e$ and $\dim U_2 = 2b + e - m$, so that $\dim U = b$. By construction, $U \cap P^\perp = 0$ and $U \cap Q^\perp = U_1$, which has dimension d. Thus the given set is nonempty, and $C(b, m - b - e, e)$ is positive. $\square$

The matrices S_b of Proposition 6.7 are particular examples of a matrix of a special form. Let S be a square matrix of size $\left[{m \atop b}\right]_q$, whose rows and columns are indexed by the b-dimensional linear subspaces of $\mathbb{F}_q^m$. The entries of S will be indeterminates s_d, $d = 0, 1, \ldots, b$. The special form of S is that the entry $S_{P,Q}$ at position (P, Q) is defined to be $S_{P,Q} = s_d$, where $d = \dim(P \cap Q^\perp)$. We will be concerned with the case where $s_0 = -1$ and $s_1 = s_2 = \cdots = s_b = 0$. Define another matrix T of the same special form with $T_{P,Q} = t_d$, where $d = \dim(P \cap Q^\perp)$.

LEMMA 6.11. *The matrix S has an inverse over $\mathbb{Q}$ of form T. That is, one can find values for the indeterminates $t_0, t_1, \ldots, t_b$ so that $T = S^{-1}$.*

Proof. We calculate the (P,Q)-entry of ST:

$$(ST)_{P,Q} = \sum_U S_{P,U} T_{U,Q} = \sum_U s_{\dim(P\cap U^\perp)} t_{\dim(U\cap Q^\perp)}$$
$$= -\sum_{U: P\cap U^\perp = 0} t_{\dim(U\cap Q^\perp)}$$

When $P = Q$, $(ST)_{P,P} = -t_0 \cdot |\{U : U \cap P^\perp = 0\}| = -t_0 \cdot C_1(b)$, independent of P, by Lemmas 6.8 and 6.9. Because $C_1(b) > 0$, we can set $t_0 = -1/C_1(b)$; then $(ST)_{P,P} = 1$ for all P.

For $P \neq Q$, assume $\dim(P^\perp \cap Q^\perp) = e$, where $e < m-b = \dim P^\perp = \dim Q^\perp$. Then

$$\begin{aligned}
-(ST)_{P,Q} &= \sum_{U: P\cap U^\perp = 0} t_{\dim(U\cap Q^\perp)} \\
&= \sum_{d=0}^{b} t_d \cdot |\{U : \dim(P\cap U^\perp) = 0, \dim(U\cap Q^\perp) = d\}| \\
&= \sum_{d=0}^{b} t_d \cdot C_2(b,d,e) = \sum_{d=0}^{m-b-e} t_d \cdot C_2(b,d,e),
\end{aligned}$$

where the last equality used the vanishing condition of Lemma 6.10.

Now solve the equations $\sum_{d=0}^{m-b-e} t_d \cdot C_2(b,d,e) = 0$ by induction on $m-b-e$. We have $t_0 = -1/C_1(b)$ already (when $m-b-e = 0$). When $m-b-e = 1$, we can solve for t_1 because its coefficient $C_2(b,1,e)$ is positive by Lemma 6.10. In general, if we already know $t_0, t_1, \ldots, t_{j-1}$, then we can solve the equation with $m-b-e = j$ for t_j because its coefficient $C_2(b,j,e)$ is positive. These t_j yield $ST = I$. □

Proof. (Proposition 6.7.) We show that the matrices S_b have the special form of the matrices in Lemma 6.11.

The matrix S_b has entries $W(\eta_{[\lambda]})(x)$ as $[x]$ varies over all orbits in $\mathcal{O}$ with $\operatorname{rk}[x] = b$ and $[\lambda]$ varies over all orbits in $\mathcal{O}^\sharp$ with $\operatorname{rk}[\lambda] = b$. By Lemma 3.23, this is the same as $\ker x$ varying over all subspaces of dimension $m-b$ and $\operatorname{Im}\lambda$ varying over all subspaces of dimension b. If we set $P = \operatorname{Im}\lambda$ and $Q = (\ker x)^\perp$, then both P and Q vary over subspaces of dimension b.

By Lemma 6.4, the value of $W(\eta_{[\lambda]})(x)$ depends upon $\operatorname{Im}\lambda \cap \ker x = \operatorname{Im}\lambda \cap ((\ker x)^\perp)^\perp = P \cap Q^\perp$. When $\dim(P\cap Q^\perp) = 0$, $W(\eta_{[\lambda]})(x) = -1$; when $\dim(P\cap Q^\perp) > 0$, $W(\eta_{[\lambda]})(x) = 0$. Thus S_b has the form claimed, and the result follows from Lemma 6.11. □

THEOREM 6.12. *Assume the matrix module context with the Hamming weight, and assume $k < \ell \leq m$. Then $W : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q})$ is surjective.*

PROOF. Refer to (6.3). The map W is represented by the matrix W_0. The matrix P and the matrices S_b are invertible, by Proposition 6.7. Thus W is surjective. $\square$

THEOREM 6.13. *Assume the matrix module context with the Hamming weight, and assume $k < \ell \leq m$. Then the set of all $\eta_{[\lambda]}$, $k+1 \leq \operatorname{rk}[\lambda] \leq \ell$, is a basis of $\ker W$.*

PROOF. We know from Corollary 6.5 that the $\eta_{[\lambda]}$, $k+1 \leq \operatorname{rk}[\lambda] \leq \ell$, belong to $\ker W$. We also know that they are linearly independent, because they are part of the basis $\mathcal{B}_2$, Lemma 6.6. By Theorem 6.12, W is surjective, so that $\dim \ker W = \dim F_0(\mathcal{O}^\sharp, \mathbb{Q}) - \dim F_0(\mathcal{O}, \mathbb{Q})$. Because orbits $[\lambda]$ correspond to linear subspaces $\operatorname{Im} \lambda \subseteq \mathbb{F}_q^m$, counting the number of orbits $[\lambda]$ with $k+1 \leq \operatorname{rk}[\lambda] \leq \ell$ yields exactly $\dim \ker W$; see (6.1). Thus we have a basis. $\square$

The dimension count is made explicit in this corollary.

COROLLARY 6.14. *Assume the matrix module context with the Hamming weight, and assume $k < \ell \leq m$. Then*

$$\dim \ker W = |\mathcal{O}^\sharp| - |\mathcal{O}| = \sum_{b=k+1}^{\ell} \begin{bmatrix} m \\ b \end{bmatrix}_q .$$

COROLLARY 6.15. *Assume the matrix module context with the Hamming weight, and assume $k+1 = \ell = m$. Then $\dim \ker W = 1$. Moreover, for any parametrized code $\Lambda : M \to A^n$, $\operatorname{rMon}(\Lambda) = \operatorname{Isom}(\Lambda)$.*

PROOF. The dimension statement follows from Corollary 6.14. The basis of $\ker W$ is $\eta_{[I_m]}$, where $I_m \in \operatorname{Hom}_R(M, A) = M_{m\times m}(\mathbb{F}_q)$ is the identity homomorphism (using $\ell = m$). Note that $[I_m]$ is the only element in $\mathcal{O}^\sharp$ of rank m. Then $f[I_m] = [fI_m] = [I_m]$ for any $f \in \mathcal{G}$.

As $\operatorname{rMon}(\Lambda) \subseteq \operatorname{Isom}(\Lambda)$, we prove the reverse inclusion. Suppose $f \in \operatorname{Isom}(\Lambda)$. By Proposition 3.16, $\eta_\Lambda f - \eta_\Lambda \in \ker W$. Thus $\eta_\Lambda f - \eta_\Lambda = C\eta_{[I_m]}$, for some $C \in \mathbb{Q}$. We evaluate both sides of this equation at $[\lambda] = [I_m]$, the only element of $\mathcal{O}^\sharp_m$:

$$C\mu(0, \mathbb{F}_q^m) = C\eta_{[I_m]}([I_m]) = \eta_\Lambda(f[I_m] - [I_m]) = 0.$$

Because $\mu(0, \mathbb{F}_q^m)$ is nonzero, we have $C = 0$. Thus $\eta_\Lambda f = \eta_\Lambda$, and $f \in \operatorname{rMon}(\Lambda)$ by Proposition 3.7. $\square$

REMARK 6.16. Corollary 6.15 proves the special case $k+1 = \ell = m$ in Theorem 5.1, Part I.

Remark 6.17. The *homogeneous weight* w_{Hom} has been defined for any module alphabet in [11, §4.1]. In the matrix module context, the left and right symmetry groups of w_{Hom} are the same as for the Hamming weight wt, so that the orbits spaces $\mathcal{O}$ and $\mathcal{O}^\sharp$ are also the same. By using w_{Hom} in place of wt, one again defines a homomorphism

$$W_{\mathrm{Hom}} : F_0(\mathcal{O}^\sharp, \mathbb{Q}) \to F_0(\mathcal{O}, \mathbb{Q})$$

of $\mathbb{Q}$-vector spaces. It follows easily from [11, Proposition 4.3] (which says that an injective homomorphism between linear codes preserves w_{Hom} if and only if it preserves wt) that $\ker W_{\mathrm{Hom}} = \ker W$. Thus, Theorems 6.12 and 6.13 and their consequences (including Corollary 6.15 and Theorem 5.1) hold for the homogeneous weight.

7. Proof of Main Theorem

This section is devoted to the proof of the main theorem, Theorem 5.1.

Proof of part I. When $k \geq \ell$, $\ker W = 0$, by Proposition 6.2. Then $\mathrm{rMon}(\Lambda) = \mathrm{Isom}(\Lambda)$, by Corollary 3.17. The case $k + 1 = \ell = m$ is proved in Corollary 6.15. □

Proof of part II. Suppose H is closed with respect to the action of $\mathcal{G}$ on $\mathcal{O}^\sharp$. Choose any $\eta \in F_0(\mathcal{O}^\sharp, \mathbb{N})$ that (1) is constant on H-orbits in $\mathcal{O}^\sharp$ and (2) takes different values on different H-orbits. (We say that η *separates H-orbits*.) Property (1) implies that $H \subseteq \mathrm{rMon}(\eta)$, by Proposition 3.7.

Let $f \in \mathrm{rMon}(\eta)$. Then $\eta f = \eta$, again by Proposition 3.7. This means that $\eta([\lambda]) = (\eta f)([\lambda]) = \eta([f\lambda])$ for all $[\lambda] \in \mathcal{O}^\sharp$. Because η separates H-orbits, $[f\lambda]$ and $[\lambda]$ must be in the same H-orbit. Thus, f preserves H-orbits, so that $f \in \bar{H}$. By the closure hypothesis, $f \in H$. Thus $H = \mathrm{rMon}(\eta)$. By Part I, $H = \mathrm{Isom}(\eta)$. □

Proof of part III. Step 1: Achieving $H_2 = \mathrm{Isom}(\Lambda)$. Pick a function $w \in F_0(\mathcal{O}, \mathbb{Q})$ with positive integer values that (1) separates the H_2-orbits in $\mathcal{O}$ and (2) has values that are an increasing function of $\mathrm{rk}[x]$, $[x] \in \mathcal{O}$. By virtue of W being surjective (Theorem 6.12), there exists a multiplicity function $\eta \in F_0(\mathcal{O}^\sharp, \mathbb{Q})$ with $w = W(\eta)$.

Because η takes rational values, it is possible that the set $N_\eta = \{[\lambda] \in \mathcal{O}^\sharp : \eta([\lambda]) < 0\}$ is nonempty. For any $[\lambda_0] \in N_\eta$, let $b = \mathrm{rk}[\lambda_0]$. We will make use of Lemma 3.24 to modify η. Add a sufficiently large integer multiple C of η_b to η, say $\eta' = \eta + C\eta_b$, so that $\eta'([\lambda_0])$ is nonnegative. By the structure of η_b, $\eta([\lambda]) \leq \eta'([\lambda])$ for all $[\lambda] \in \mathcal{O}^\sharp$, so that $N_{\eta'} \subseteq N_\eta$. Because $\eta'([\lambda_0]) \geq 0$, $[\lambda_0] \notin N_{\eta'}$, and the containment

$N_{\eta'} \subset N_\eta$ is strict. Because the values of $W(\eta_b)$ are positive and increasing as a function of $\mathrm{rk}[x]$, $W(\eta')$ will also satisfy properties (1) and (2) above. Continue to modify η in this manner (a finite process, as $\mathcal{O}^\sharp$, and therefore N_η, are finite), until all values are nonnegative. Again call the final modification η.

If η is not constant on H_2-orbits in $\mathcal{O}^\sharp$, then by Lemma 3.25 we can replace η by an averaged version which is constant on H_2-orbits in $\mathcal{O}^\sharp$ and has the same $W(\eta)$. By clearing denominators, i.e., by replacing η by a suitable positive integer multiple, we may assume that all the values of η are nonnegative integers, that η is constant on H_2-orbits in $\mathcal{O}^\sharp$, and that $W(\eta)$ separates H_2-orbits in $\mathcal{O}$.

We claim that $\mathrm{rMon}(\eta) = \mathrm{Isom}(\eta) = H_2$. Because η is constant on H_2-orbits in $\mathcal{O}^\sharp$, $H_2 \subseteq \mathrm{rMon}(\eta)$, by Proposition 3.7. Because we always have $\mathrm{rMon}(\eta) \subseteq \mathrm{Isom}(\eta)$, it will be enough to show that $\mathrm{Isom}(\eta) \subseteq H_2$. Suppose $f \in \mathrm{Isom}(\eta)$. Then $W(\eta)(x) = W(\eta)(xf)$ for all $x \in \mathcal{O}$. Because $W(\eta)$ separates H_2-orbits in $\mathcal{O}$, x and xf must belong to the same H_2-orbit. This means that f preserves the H_2-orbits in $\mathcal{O}$. Consequently, f belongs to the closure $\bar{H}_2$ of H_2 with respect to the action of $\mathcal{G}$ on $X = \mathcal{O}$. By the closure hypothesis on H_2, we have $f \in H_2$. Thus, $\mathrm{rMon}(\eta) = \mathrm{Isom}(\eta) = H_2$.

Step 2: Achieving $H_1 = \mathrm{rMon}(\Lambda)$. We will next modify the multiplicity function η by adding integer multiples of the $\eta_{[\lambda]}$ from Theorem 6.13 in such a way that the resulting multiplicity function

$$\eta' = \eta + \sum_{k+1 \leq \mathrm{rk}[\lambda] \leq \ell} c_{[\lambda]} \eta_{[\lambda]} \tag{7.1}$$

separates the H_1-orbits in $\mathcal{O}^\sharp$. The reason for restricting the range of summation is to not change the isometry group: $\eta' - \eta \in \ker W$, so that $\mathrm{Isom}(\eta') = \mathrm{Isom}(\eta) = H_2$.

Recall from Lemma 3.23 that $\mathrm{rk}[\lambda]$ is well-defined, depending only on the orbit $[\lambda]$ in $\mathcal{O}^\sharp$, and the action of $\mathcal{G}$ on $\mathcal{O}^\sharp$ preserves this rank function. We modify η inductively, working from $\mathrm{rk}[\lambda] = \ell$ down to $\mathrm{rk}[\lambda] = k + 1$. Consider an H_2-orbit O in $\mathcal{O}^\sharp$ of rank ℓ. Because $H_1 \subseteq H_2$, every H_2-orbit is a disjoint union of H_1-orbits. Say, $O = \biguplus_{i=1}^t \mathrm{orb}_{H_1}([\lambda_i])$. Choose different values for $c_{[\lambda_i]}$, and use the same value $c_{[\lambda_i]}$ for every $[\lambda] \in \mathrm{orb}_{H_1}([\lambda_i])$ in (7.1); we will write this as $c_{[\lambda]} = c_{\mathrm{orb}_{H_1}([\lambda])}$. Explicitly, we have

$$\eta' = \eta + \sum_{i=1}^{t} c_{\mathrm{orb}_{H_1}([\lambda_i])} \sum_{[\lambda] \in \mathrm{orb}_{H_1}([\lambda_i])} \eta_{[\lambda]}.$$

Note from (6.2), that if $\mathrm{rk}[\lambda] = \mathrm{rk}[\nu] = \ell$, then $\eta_{[\lambda]}([\nu])$ is nonzero if and only if $[\nu] = [\lambda]$. That is, for any $[\nu] \in \mathcal{O}^\sharp$ with $\mathrm{rk}[\nu] = \ell$, we have

$$\eta'([\nu]) = \begin{cases} \eta([\nu]), & [\nu] \notin O, \\ \eta([\nu]) + c_{\mathrm{orb}_{H_1}([\lambda_i])}, & [\nu] \in \mathrm{orb}_{H_1}([\lambda_i]). \end{cases}$$

Thus, the new η' separates the orbits $\mathrm{orb}_{H_1}([\lambda_i])$. We do the same process for other H_2-orbits of rank ℓ. We end up with an η' that separates the H_1-orbits of rank ℓ. But more is true.

By (6.2), for any $[\nu] \in \mathcal{O}^\sharp$, we have

$$\begin{aligned} \eta'([\nu]) &= \eta([\nu]) + \sum_{\mathrm{rk}[\lambda]=\ell} c_{\mathrm{orb}_{H_1}([\lambda])} \eta_{[\lambda]}([\nu]) \\ &= \eta([\nu]) + \sum_{\substack{\mathrm{rk}[\lambda]=\ell \\ \mathrm{Im}\,\nu \le \mathrm{Im}\,\lambda}} c_{\mathrm{orb}_{H_1}([\lambda])} \mu(0, \mathrm{Im}\,\nu). \end{aligned} \tag{7.2}$$

Take any $f \in H_1$. We claim that $\eta'([f\nu]) = \eta'([\nu])$. Certainly, $\eta([f\nu]) = \eta([\nu])$, because η is constant on H_2-orbits, hence also on H_1-orbits. As for the summation, these too are equal. Indeed, $\mathrm{orb}_{H_1}([f\lambda]) = \mathrm{orb}_{H_1}([\lambda])$, so that $c_{\mathrm{orb}_{H_1}([f\lambda])} = c_{\mathrm{orb}_{H_1}([\lambda])}$. Because f is invertible, $\mathrm{Im}\,\nu \le \mathrm{Im}\,\lambda$ if and only if $\mathrm{Im}\,f\nu \le \mathrm{Im}\,f\lambda$. And, finally, $\mu(0, \mathrm{Im}\,f\nu) = \mu(0,\nu)$ because $\mu(0, \mathrm{Im}\,\nu)$ depends only on $\mathrm{rk}[\nu] = \mathrm{rk}[f\nu]$. Thus, at this stage, η' is constant on H_1-orbits of all ranks and separates H_1-orbits of rank ℓ.

Technical comment 1: When choosing the values for $c_{\mathrm{orb}_{H_1}([\lambda])}$, they should be chosen sufficiently small relative to the values of η so that the resulting values for η' in (7.2) are still nonnegative; the trouble is that $\mu(0, \mathrm{Im}\,\nu)$ may be negative. This may require rescaling the values of η. As there are only a finite number of orbits and modifications, this is always possible to do.

Technical comment 2: It is possible that an H_2-orbit O is also an H_1-orbit. In that case there is no modification necessary. However, when $H_1 \neq H_2$, then there is some $b \geq k+1$ so that the H_1-orbits on $\mathcal{O}_b^\sharp$ differ from the H_2-orbits on $\mathcal{O}_b^\sharp$. Indeed, if all the orbits are the same for all $b \geq k+1$, then H_2 preserves the H_1-orbits on X, and $H_2 \subseteq \bar{H}_1$. By the closure hypothesis on H_1, we conclude $H_1 = H_2$. This situation arises when $k+1 = \ell = m$, which is why it is a special case.

For the inductive step, suppose η is constant on H_1-orbits of all ranks and separates H_1-orbits of rank $\geq b+1$. By essentially the same process as described above, we modify η using appropriate integer multiples of the $\eta_{[\lambda]}$ with $\mathrm{rk}[\lambda] = b$. The resulting η' is constant on H_1-orbits of all ranks and separates H_1-orbits of rank $\geq b$. By induction, we proceed until η' is constant on H_1-orbits of all ranks and separates H_1-orbits of rank $\geq k+1$. The final η' has the form in (7.1).

Because η' is constant on H_1-orbits, we have $H_1 \subseteq \mathrm{rMon}(\eta')$. Any $f \in \mathrm{rMon}(\eta')$ must preserve η', and, because η' separates H_1-orbits of rank $\geq k+1$, f must preserve the H_1-orbits of rank $\geq k+1$, i.e., the H_1 orbits on $X = \biguplus_{b=k+1}^{\ell} \mathcal{O}_b^\sharp$. Thus $f \in \bar{H}_1$. But $\bar{H}_1 = H_1$ by hypothesis, so $f \in H_1$. Thus $\mathrm{rMon}(\eta') = H_1$. Because $\eta' - \eta \in \ker W$, $\mathrm{Isom}(\eta') = \mathrm{Isom}(\eta) = H_2$. □

8. Application to Alphabets with Noncyclic Socle

Suppose R is a finite ring with 1 and A is a finite, unital left R-module. The alphabet A has the extension property with respect to the Hamming or homogeneous weight if and only if A is pseudo-injective and its socle $\mathrm{soc}(A)$ is a cyclic module [27, Theorems 5.2 and 6.2]. To see what happens when the extension property fails, we now assume that $\mathrm{soc}(A)$ is not cyclic.

The Wedderburn-Artin decomposition of $R/\mathrm{rad}(R)$ into a direct sum of simple rings has the form

$$R/\mathrm{rad}(R) \cong \bigoplus_{i=1}^{t} M_{k_i \times k_i}(\mathbb{F}_{q_i}), \tag{8.1}$$

for appropriate integers t, k_i, and prime powers q_i. The isomorphism classes of simple left R-modules are represented by the simple modules $T_i = M_{k_i \times 1}(\mathbb{F}_{q_i})$ of the summands $R_i = M_{k_i \times k_i}(\mathbb{F}_{q_i})$, regarded as R-modules via the projections $R \twoheadrightarrow R/\mathrm{rad}(R) \twoheadrightarrow R_i$. As a left R-module, $R/\mathrm{rad}(R)$ decomposes as

$${}_R(R/\mathrm{rad}(R)) \cong \bigoplus_{i=1}^{t} k_i T_i.$$

Given any finite left R-module A, its socle $\mathrm{soc}(A)$ is the left R-submodule of A generated by all the simple left submodules. Then $\mathrm{soc}(A)$ decomposes as a direct sum of the simple modules T_i:

$$\mathrm{soc}(A) = \bigoplus_{i=1}^{t} \ell_i T_i,$$

for appropriate nonnegative integers ℓ_i. The socle $\mathrm{soc}(A)$ is cyclic if and only if $k_i \geq \ell_i$ for all $i = 1, \ldots, t$.

Assume now that $\mathrm{soc}(A)$ is not cyclic. Then there exists some index i with $k_i < \ell_i$. Denote by $A_i \subseteq \mathrm{soc}(A)$ the direct summand $A_i = \ell_i T_i$. Recall the direct summand of R: $R_i = M_{k_i \times k_i}(\mathbb{F}_{q_i})$. As an R_i-module,

$A_i \cong M_{k_i \times \ell_i}(\mathbb{F}_{q_i})$, with $k_i < \ell_i$. Then R_i and A_i are part of a matrix module context. (See Appendix B for how this works for a general abelian group as alphabet.)

By applying Theorem 5.1 to R_i and A_i, we will be able to prove results for R and A. We do not get the full strength of Theorem 5.1, but we come close.

THEOREM 8.1. *Let R be a finite ring with 1 and A a finite, unital left R-module. Equip A with the Hamming weight. Assume* $\operatorname{soc}(A)$ *is not cyclic, so that* $\operatorname{soc}(A)$ *contains an R_i-submodule A_i, with $R_i = M_{k_i \times k_i}(\mathbb{F}_{q_i})$, $A_i \cong M_{k_i \times \ell_i}(\mathbb{F}_{q_i})$, and $k_i < \ell_i$.*

Let M be a left R_i-module $M \cong M_{k_i \times m}(\mathbb{F}_{q_i})$, with $m > \ell_i$. Let H_1, H_2 be any two subgroups of $\mathrm{GL}_{R_i}(M)$ satisfying $H_1 \subseteq H_2$, with H_1 closed with respect to the action of $\mathrm{GL}_{R_i}(M)$ on $X = \biguplus_{b=k_i+1}^{\ell_i} \mathcal{O}^\sharp_{i,b}$ and H_2 closed with respect to the action of $\mathrm{GL}_{R_i}(M)$ on $\mathcal{O}$. Then, there exists a parametrized code Λ over the alphabet A modeled on M regarded as an R-module, such that

$$\mathrm{rMon}(\Lambda) \subseteq H_1 \text{ and } \mathrm{Isom}(\Lambda) = H_2.$$

In the statement of the theorem, the orbit space $\mathcal{O}^\sharp_i$ is defined over R_i and A_i by $\mathcal{O}^\sharp_i = \mathrm{Hom}_{R_i}(M, A_i)/\mathrm{GL}_{R_i}(A_i)$.

PROOF. By Theorem 5.1 applied to R_i, A_i, and M, there exists a homomorphism of R_i-modules $\Lambda : M \to A_i^n$ that satisfies $\mathrm{Isom}(\Lambda) = H_2$ and $\mathrm{rMon}_i(\Lambda) := \mathrm{restr}(\mathrm{Mon}_i(\Lambda)) = H_1$. Here, $\mathrm{Mon}_i(\Lambda)$ denotes the monomial transformations defined over R_i and A_i that preserve Λ.

The idea is to view $\Lambda : M \to A_i^n$ (using A_i as the alphabet over R_i) instead as an R-linear code $\Lambda : M \to A_i^n \subseteq A^n$ (using A as the alphabet over R).

For any $x \in M$, the Hamming weight $\mathrm{wt}(x\Lambda)$ of $x\Lambda \in A_i^n \subset A^n$ is unambiguous. Whether an entry $x\lambda_j \in A_i \subseteq A$ is nonzero does not depend upon viewing the entry as an element of A_i or as an element of A. Because $\mathrm{GL}_{R_i}(M) = \mathrm{GL}_R(M)$, the group $\mathrm{Isom}(\Lambda)$ is also unambiguous; its meaning is independent of which alphabet, A_i or A, is being used. In either case, $\mathrm{Isom}(\Lambda) = H_2$.

As for monomial transformations, our aim is to show the containment $\mathrm{rMon}(\Lambda) \subseteq \mathrm{rMon}_i(\Lambda)$. It will then follow that $\mathrm{rMon}(\Lambda) \subseteq H_1$.

If we view $\Lambda : M \to A_i^n$ over the alphabet A_i, then Λ determines a multiplicity function $\eta_i \in \mathbb{F}(\mathcal{O}^\sharp_i, \mathbb{N})$. If we instead view $\Lambda : M \to A_i^n \subseteq A^n$ over the alphabet A, then Λ determines a multiplicity function $\eta \in F(\mathcal{O}^\sharp, \mathbb{N})$. The orbit spaces $\mathcal{O}^\sharp$ and $\mathcal{O}^\sharp_i$ are different: $\mathcal{O}^\sharp = \mathrm{Hom}_R(M, A)/\mathrm{GL}_R(A)$, while $\mathcal{O}^\sharp_i = \mathrm{Hom}_{R_i}(M, A_i)/\mathrm{GL}_{R_i}(A_i)$.

Because of the definition of M as a sum of copies of the simple module T_i, the image $M\lambda$ of M under any $\lambda \in \operatorname{Hom}_R(M, A)$ will also be a sum of copies of T_i. Thus $M\lambda \subseteq A_i \subseteq \operatorname{soc}(A)$, because A_i is the direct summand of $\operatorname{soc}(A)$ consisting of copies of T_i. This shows that $\operatorname{Hom}_R(M, A) = \operatorname{Hom}_{R_i}(M, A_i)$.

By a similar argument involving the simple module T_i, every element of $\operatorname{GL}_R(A)$ must map A_i to itself, thereby inducing a restriction homomorphism $\rho : \operatorname{GL}_R(A) \to \operatorname{GL}_{R_i}(A_i)$. Usually, ρ is neither injective nor surjective. Nonetheless, ρ induces a well-defined map $\mathcal{O}^\sharp \to \mathcal{O}_i^\sharp$. We then see that the multiplicity functions η and η_i are related by

$$\eta_i([\lambda]_i) = \sum_{[\nu] \in \rho^{-1}[\lambda]_i} \eta([\nu]), \quad [\lambda]_i \in \mathcal{O}_i^\sharp. \tag{8.2}$$

The notation $[\lambda]_i$ refers to the orbit of $\lambda \in \operatorname{Hom}_{R_i}(M, A_i)$ in $\mathcal{O}_i^\sharp$. The equation reflects the fact that different orbits in $\mathcal{O}^\sharp$ may have the same image in $\mathcal{O}_i^\sharp$ under the map ρ, and we 'sum over the fibers'.

We now prove $\operatorname{rMon}(\Lambda) \subseteq \operatorname{rMon}_i(\Lambda)$. Let $f \in \operatorname{rMon}(\Lambda)$. By Proposition 3.7, $\eta f = \eta$. By using (8.2), one verifies that $\eta_i f = \eta_i$, so that $f \in \operatorname{rMon}_i(\Lambda)$, again by Proposition 3.7. □

Corollary 8.2. *Assume the hypotheses of Theorem 8.1. Let* $H_1 = \{\alpha I_m : \alpha \in \mathbb{F}_{q_i}^\times\}$ *and* $H_2 = \operatorname{GL}(m, \mathbb{F}_{q_i})$. *Then there exists a parametrized code* Λ *such that*

$$\operatorname{rMon}(\Lambda) = \{\alpha I_m : \alpha \in \mathbb{F}_{q_i}^\times\} \text{ and } \operatorname{Isom}(\Lambda) = \operatorname{GL}(m, \mathbb{F}_{q_i}).$$

We conclude this section with an example of an alphabet A where $\rho : \operatorname{GL}_R(A) \to \operatorname{GL}_{R_i}(A_i)$ is not surjective and a code Λ where the containment $\operatorname{rMon}(\Lambda) \subseteq \operatorname{rMon}_i(\Lambda)$ is proper.

Example 8.3. Let R be the ring $R = \mathbb{F}_2[X, Y]/(X^2, XY, Y^2)$. The ring R is a commutative $\mathbb{F}_2$-algebra of order 8. It has a basis $\{1, X, Y\}$ as a vector space over $\mathbb{F}_2$, so that any element r of R has a unique representation in the form $r = a + bX + cY$, with a, b, and c in $\mathbb{F}_2$. The units are those elements with $a = 1$, and the radical $\operatorname{rad}(R)$ is the maximal ideal (X, Y) consisting of all elements with $a = 0$. This shows that R is a local ring with unique maximal ideal (X, Y); $R/\operatorname{rad}(R) = R/(X, Y) \cong \mathbb{F}_2$, so there is one isomorphism type of simple R-modules. The one-dimensional ideals RX, RY, and $R(X+Y)$ are all simple (and isomorphic). The socle of R is $\operatorname{soc}(R) = (X, Y) = RX \oplus RY$.

Let the alphabet A be the ring R itself, and let $A_1 = \operatorname{soc}(R) = \operatorname{soc}(A) = (X, Y)$. Because R is a ring with 1, any homomorphism $\varphi : A \to A$ of R-modules is given by multiplication by an element $s = \varphi(1) \in R$. Thus $\operatorname{GL}_R(A) \cong \mathcal{U}(R)$, the group of units of the ring

R. Note that any $\varphi \in \mathrm{GL}_R(A)$ maps A_1 back to itself, but also that φ acts as the identity on A_1.

In contrast, one checks that $\mathrm{GL}_R(A_1) \cong \mathrm{GL}(2, \mathbb{F}_2)$, thought of as invertible linear transformations of the 2-dimensional vector space spanned by X and Y. Thus the homomorphism $\rho : \mathrm{GL}_R(A) \to \mathrm{GL}_R(A_1)$ is the trivial homomorphism between two nontrivial groups.

By using (X, Y) as a code of length 1, we obtain an example showing that the containment $\mathrm{rMon}(\Lambda) \subseteq \mathrm{rMon}_i(\Lambda)$ in Theorem 8.1 is proper.

Proposition 8.4. *Let R be the ring $R = \mathbb{F}_2[X, Y]/(X^2, XY, Y^2)$, and let A equal R, equipped with the Hamming weight. Let $A_1 = \mathrm{soc}(R) = (X, Y) \subset R$. Consider $C = A_1$ as a linear code of length 1, both as a code with alphabet A and as a code with alphabet A_1. Then $\mathrm{rMon}(C) = \{\mathrm{id}_C\}$, while $\mathrm{rMon}_1(C) = \mathrm{Isom}(C) = \mathrm{GL}_R(C) \cong \mathrm{GL}(2, \mathbb{F}_2)$.*

9. Examples

In this final section we present some examples of linear codes in the matrix module context that have different subgroups $\mathrm{rMon}(C)$ and $\mathrm{Isom}(C)$. In all cases, the groups $\mathrm{rMon}(C)$ and $\mathrm{Isom}(C)$ have been verified by computations programmed by the author in the computer algebra system Maple that test subgroup membership for every element of $\mathrm{GL}_R(M)$ using Propositions 3.7 and 3.16. Also see Remark A.1.

Additive codes over $\mathbb{F}_4$. Additive codes over $\mathbb{F}_4$ correspond to the matrix module context with $R = \mathbb{F}_2$ and $A = \mathbb{F}_4$; $\mathbb{F}_4 \cong M_{1\times 2}(\mathbb{F}_2)$ as vector spaces over $\mathbb{F}_2$. The weight is the Hamming weight on $\mathbb{F}_4$, so that $\mathrm{Sym}_{\mathrm{rt}} = \mathrm{GL}_R(A) = \mathrm{GL}(2, \mathbb{F}_2) \cong \Sigma_3$, the symmetric group on 3 elements. Write the elements of $\mathbb{F}_4$ as $0, 1, \omega, \omega^2$, with $\omega^2 = \omega + 1$.

Example 9.1. We consider an additive code C_1 of dimension 3. It is generated by a 3×3 matrix G_1 over $\mathbb{F}_4$ by forming all $\mathbb{F}_2$-linear combinations of the rows of G_1 (i.e., all possible sums of the rows of G_1), so that $|C_1| = 8$. Here is G_1 and the list of codewords of C_1:

$$G_1 = \begin{bmatrix} 1 & \omega & 0 \\ \omega & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \qquad \begin{matrix} 0 & 0 & 0 \\ 1 & \omega & 0 \\ \omega & 1 & 0 \\ \omega^2 & \omega^2 & 0 \\ 1 & 0 & 1 \\ 0 & \omega & 1 \\ \omega^2 & 1 & 1 \\ \omega & \omega^2 & 1 \end{matrix}.$$

Consider the following three elements of $\mathrm{GL}(3, \mathbb{F}_2)$:

$$f_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The reader will verify that f_1 and f_2 belong to $\mathrm{rMon}(C_1)$; i.e., for f_1 and f_2, there exist 3×3 monomial matrices P_1 and P_2 (whose nonzero entries are elements of $\mathrm{GL}(2, \mathbb{F}_2) \cong \Sigma_3$) such that $f_i G_1 = G_1 P_i$, $i = 1, 2$. The elements f_1 and f_2 generate $\mathrm{rMon}(C_1)$, which is isomorphic to a Klein 4-group. On the other hand, f_1 and f_3 generate $\mathrm{Isom}(C_1)$, which is isomorphic to the dihedral group of order 8.

Remark 9.2. The Magma Computational Algebra System [3] has functions that compute automorphism groups of additive codes over $\mathbb{F}_4$. However, the groups that Magma calculates are defined differently from those considered here. Their monomial transformations allow nonzero entries only from $\mathrm{GL}_{\mathbb{F}_4}(\mathbb{F}_4) = \mathbb{F}_4^\times \cong C_3$, not the more general $\mathrm{GL}_{\mathbb{F}_2}(\mathbb{F}_4) = \mathrm{GL}(2, \mathbb{F}_2) \cong \Sigma_3$. Magma does not have a function that calculates $\mathrm{Isom}(C)$.

Example 9.3. Here is another example of an additive $\mathbb{F}_4$-code C_2 of dimension 3. Its matrix G_2 and list of codewords are:

$$G_2 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega \\ \omega & \omega & 1 & 0 & \omega^2 \end{bmatrix}, \qquad \begin{matrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \omega & \omega \\ 1 & 1 & 0 & \omega^2 & \omega^2 \\ \omega & \omega & 1 & 0 & \omega^2 \\ \omega & \omega^2 & 0 & 1 & \omega \\ \omega^2 & \omega & 0 & \omega & 1 \\ \omega^2 & \omega^2 & 1 & \omega^2 & 0 \end{matrix}.$$

It is evident that every nonzero codeword has Hamming weight 4, so that every $f \in \mathrm{GL}(3, \mathbb{F}_2)$ is an isometry. Thus $\mathrm{Isom}(C_2) = \mathrm{GL}(3, \mathbb{F}_2)$. Note that $|\mathrm{GL}(3, \mathbb{F}_2)| = 168$. (The code C_2 is also self-orthogonal with respect to the Hermitian trace inner product.)

Consider the following three elements of $\mathrm{GL}(3, \mathbb{F}_2)$:

$$f_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad f_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad f_6 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

The reader is invited to verify that each of these elements belongs to $\mathrm{rMon}(C_2)$; i.e., for each f_i, there exists a 5×5 monomial matrix P_i (whose nonzero entries are elements of $\mathrm{GL}(2, \mathbb{F}_2) \cong \Sigma_3$) such that $f_i G_2 = G_2 P_i$. In fact, f_4, f_5, f_6 generate $\mathrm{rMon}(C_2)$, which is isomorphic to Σ_4, the symmetric group on 4 elements, with f_4, f_5, f_6 corresponding to the transpositions $(1,2)$, $(2,3)$, $(3,4)$, respectively. In summary, $\mathrm{rMon}(C_2) \cong \Sigma_4$, of order 24, and $\mathrm{Isom}(C_2) = \mathrm{GL}(3, \mathbb{F}_2)$, of order 168.

Additive codes over $\mathbb{F}_4$ also have an interpretation in terms of finite projective geometries, and the reader may wish to compare the results here with those in [2].

EXAMPLE 9.4. The additive $\mathbb{F}_4$-code C_3 is generated by a 3×28 matrix G_3, with individual columns being repeated with the stated multiplicities:

multiplicity	1	4	2	2	4	1	3	5	6
	1	0	0	1	1	1	1	1	1
G_3	0	1	1	ω	ω	ω	ω	0	1
	1	0	1	0	ω	1	ω^2	ω	ω

The elements of C_3 are displayed next, using the same multiplicity notation. A dividing line has been added for reference later.

multiplicity	1	4	2	2	4	1	3	5	6
	0	0	0	0	0	0	0	0	0
	1	0	0	1	1	1	1	1	1
	0	1	1	ω	ω	ω	ω	0	1
elements	1	0	1	0	ω	1	ω^2	ω	ω
	1	1	1	ω^2	ω^2	ω^2	ω^2	1	0
	0	0	1	1	ω^2	0	ω	ω^2	ω^2
	1	1	0	ω	0	ω^2	1	ω	ω^2
	0	1	0	ω^2	1	ω	0	ω^2	ω

The code C_3 has length 28, and every nonzero codeword has Hamming weight 22. This implies that every element of $\mathrm{GL}_R(M)$ is an isometry: $\mathrm{Isom}(C_3) = \mathrm{GL}(3, \mathbb{F}_2)$, of order 168. However, $\mathrm{rMon}(C_3)$ is trivial.

The monomial group $\mathrm{Mon}(C_3)$ is large. For example, permuting the positions of repeated columns (with all $\phi_i \in \mathrm{Sym}_{\mathrm{rt}}$ equal to the identity) yields monomial transformations that preserve C_3. There are $1!\,4!\,2!\,2!\,4!\,1!\,3!\,5!\,6! = 2^{16}\,3^6\,5^2 = 1,194,393,600$ such permutation monomial transformations. They all restrict to the identity id_{C_3} on C_3; i.e., they all belong to $\mathrm{Mon}_0(C_3)$. But there are more elements of $\mathrm{Mon}_0(C_3)$: for any column to the left of the dividing line above, one can choose $\phi \in \mathrm{Sym}_{\mathrm{rt}}$ that fixes 1 but interchanges ω and ω^2.

An additive code over $\mathbb{F}_9$. Write $\mathbb{F}_9 = \mathbb{F}_3[\omega]/(\omega^2 - \omega - 1)$; ω is a primitive element of $\mathbb{F}_9$. Additive codes over $\mathbb{F}_9$ correspond to the matrix module context with $R = \mathbb{F}_3$ and $A = \mathbb{F}_9$; $\mathbb{F}_9 \cong M_{1\times 2}(\mathbb{F}_3)$ as vector spaces over $\mathbb{F}_3$. The weight is the Hamming weight on $\mathbb{F}_9$.

multiplicity	5	3	6	1	1	1	2	2	2	4	3	2	6	3	7	8	9	6	4	5	2	3	1
	0	0	0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
G_4	0	1	1	0	0	0	1	1	1	-1	-1	-1	0	1	ω	ω	ω	ω	ω	ω	ω	ω	ω
	1	1	-1	0	1	-1	0	1	-1	0	1	-1	ω	ω	0	1	-1	ω	$\omega+1$	$\omega-1$	$-\omega$	$-\omega+1$	$-\omega-1$

TABLE 1. Generator matrix G_4 for an additive $\mathbb{F}_9$-code C_4.

EXAMPLE 9.5. Let C_4 be the additive code over $\mathbb{F}_9$ given by the generator matrix G_4 in Table 1 on page 39. The multiplicities of the columns are as indicated.

The additive code C_4 has dimension 3 over $\mathbb{F}_3$ and is of length 86. Every nonzero codeword has Hamming weight 72, so that every element of $\mathrm{GL}(3, \mathbb{F}_3)$ is an isometry; $\mathrm{Isom}(C_4) = \mathrm{GL}(3, \mathbb{F}_3)$ is as large as possible (with $|\mathrm{GL}(3, \mathbb{F}_3)| = 11,232$). On the other hand, $\mathrm{rMon}(C_4) = \{\pm I_3\}$ is as small as possible.

Appendix A. Additive $\mathbb{F}_4$-Codes of $\mathbb{F}_2$-Dimension 3

Let $R = \mathbb{F}_2$, $A = \mathbb{F}_4 \cong M_{1\times 2}(\mathbb{F}_2)$, $M = M_{1\times 3}(\mathbb{F}_2)$, and $\mathcal{G} = \mathrm{GL}_R(M) = \mathrm{GL}(3, \mathbb{F}_2)$. The subgroup lattice function in Magma [3] reveals that the group $\mathcal{G}$ admits subgroups of twelve isomorphism types: cyclic subgroups C_i of orders $i = 1, 2, 3, 4, 7$, the Klein 4-group V_4, the symmetric groups Σ_3 and Σ_4, the dihedral group D_8 of order 8, the alternating group A_4, a non-abelian subgroup of order 21, and the group $\mathcal{G}$ itself. (The trivial subgroup C_1 will also be denoted by I.)

Computations reveal that there are only seven isomorphism types of closed subgroups (under either action of $\mathcal{G}$ on $\mathcal{O}$ or $\mathcal{O}^\sharp$): I, C_2, V_4, Σ_3, D_8, Σ_4, and $\mathcal{G}$. There are twenty-six containments of the form $H_1 \subsetneq H_2$ using these seven isomorphism types of closed subgroups.

Examples of additive codes achieving each of the twenty-six containments $H_1 \subsetneq H_2$ are displayed in Table 2 on page 41. In that table, representatives of the 14 different nonzero elements of $\mathcal{O}^\sharp$ are displayed as column vectors over $\mathbb{F}_4$. Below them are examples of multiplicity functions η, together with the length n of the additive code determined by η, and the groups $\mathrm{rMon} = \mathrm{rMon}(\eta)$ and $\mathrm{Isom} = \mathrm{Isom}(\eta)$.

Two examples are given with $\mathrm{rMon}(\eta) = \mathrm{Isom}(\eta) = \mathcal{G}$: the first example uses every rank one orbit in $\mathcal{O}^\sharp$ exactly once, while the second example uses every rank two orbit in $\mathcal{O}^\sharp$ exactly once. These formats are consistent with our understanding of linear one-weight codes [25].

Remark A.1. The examples are produced in the following manner. Given closed subgroups $H_1 \subseteq H_2$, we seek a multiplicity function η with $\operatorname{rMon}(\eta) = H_1$ and $\operatorname{Isom}(\eta) = H_2$. The values $\eta([\lambda])$ of η are viewed as unknowns. Propositions 3.7 and 3.16 allow us to set up a system of linear equations: For every generator f of H_1, we get the linear equation $\eta f - \eta = 0$, and for every generator g of H_2, we get the linear equation $W(\eta g - \eta) = 0$. We use Maple to solve the resulting system of linear equations. There are free parameters in the solutions. We choose (by hand) numerical values of the parameters so that all the values $\eta([\lambda])$ are non-negative and reasonably small. We then verify that $\operatorname{rMon}(\eta) = H_1$ and $\operatorname{Isom}(\eta) = H_2$ by programming Maple to check Propositions 3.7 and 3.16 for every element of $\mathcal{G}$. This last step is needed because, while the generic solution of the system of linear equations has the desired symmetry by design, the chosen numerical solution may not be generic and thus may have additional symmetry. We have not pursued the problem of determining the examples of minimum length.

Appendix B. General Additive Codes

This appendix is a continuation of the discussion at the beginning of Section 8, applied to Example 2.2. Let A be a finite abelian group, written additively, and let e be the exponent of A, i.e., the smallest positive integer such that $ea = 0$ for all $a \in A$. Then A is a module over $R = \mathbb{Z}/e\mathbb{Z}$, and R is a Frobenius ring.

Consider the prime factorization of e: $e = p_1^{b_1} \cdots p_t^{b_t}$ for distinct primes $p_1, \ldots, p_t$ and positive integers $b_1, \ldots, b_t$. If we set $R_i = \mathbb{Z}/p_i^{b_i}\mathbb{Z}$, then $R \cong \bigoplus_{i=1}^t R_i$. The radical of R_i is the ideal generated by p_i, so that $R_i/\operatorname{rad}(R_i) \cong \mathbb{Z}/p_i\mathbb{Z} \cong \mathbb{F}_{p_i}$. Then $R/\operatorname{rad}(R) \cong \bigoplus_{i=1}^t \mathbb{F}_{p_i}$. In the notation of (8.1), we have $k_i = 1$, $q_i = p_i$, and $T_i \cong \mathbb{F}_{p_i}$.

Using the fundamental theorem of finitely generated abelian groups, we can write A as a direct sum of cyclic R_i-modules:

$$A \cong \bigoplus_{i=1}^t \left(\bigoplus_{j=1}^{\ell_i} \mathbb{Z}/p_i^{c_{i,j}}\mathbb{Z} \right),$$

where the integers $c_{i,j}$ satisfy $1 \leq c_{i,1} \leq c_{i,2} \leq \cdots \leq c_{i,\ell_i} = b_i$. The socle of $\mathbb{Z}/p_i^{c_{i,j}}\mathbb{Z}$ is the submodule generated by $p_i^{c_{i,j}-1}$. We then see that $\operatorname{soc}(A) \cong \bigoplus_{i=1}^t \ell_i T_i$.

Because each $k_i = 1$, $\mathrm{soc}(A)$ is not cyclic if there exists some index i with $\ell_i \geq 2$. The socle $\mathrm{soc}(A)$ is cyclic when each $\ell_i = 1$. When this happens, A is isomorphic to R.

Multiplicity function η of given columns														n	rMon	Isom
0	0	0	0	1	1	1	1	1	1	1	1	1	1			
0	1	1	1	0	0	0	1	1	1	ω	ω	ω	ω			
1	0	1	ω	0	1	ω	0	1	ω	0	1	ω	ω^2			
1	2			3										6	I	I
	1								2		1		1	5	I	C_2
1	1			2										4	C_2	C_2
3	2						1				2		4	12	I	V_4
			2		1	2	1			2				8	C_2	V_4
			2						1		1		2	6	V_4	V_4
	1		1		1	2	1	1					1	8	I	Σ_3
			1	1	1			1					1	5	C_2	Σ_3
1	1			1										3	Σ_3	Σ_3
	1		1		2	1			3	2				10	I	D_8
			2		1	1	1			1				6	C_2	D_8
			1					1					1	3	V_4	D_8
			1						1		1	1	1	5	D_8	D_8
	1	1	1		2	1			3	2				11	I	Σ_4
1	1	1			2				2	2				9	C_2	Σ_4
		2				2		1	1	2	1			9	V_4	Σ_4
1										1	1		1	4	Σ_3	Σ_4
			1	1				1					1	4	D_8	Σ_4
			1									1	1	3	Σ_4	Σ_4
5	1	3		2	2		4			2	6	4		29	I	$\mathcal{G}$
1	1	2				3	2		1	1	3	2		16	C_2	$\mathcal{G}$
		1	2	2		1			3	1	3		2	15	V_4	$\mathcal{G}$
2	1			1			1				2	2	2	11	Σ_3	$\mathcal{G}$
1	1		2	2	1		1	2					2	12	D_8	$\mathcal{G}$
		1				1			1	1	1			5	Σ_4	$\mathcal{G}$
1	1	1		1	1		1	1						7	$\mathcal{G}$	$\mathcal{G}$
			1			1			1	1	1	1	1	7	$\mathcal{G}$	$\mathcal{G}$

TABLE 2. Multiplicity functions and their groups for binary dimension 3.

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