The Extension Problem for General Weights

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Extension problem for general weights

- Report on work of Dyshko for Lee and Euclidean weights (2017)
- Generalized to weights over modules
- Fourier transforms and linear independence of characters
- Recursive argument involving posets
Dyshko’s work

- Sergii Dyshko has proved EP for the Lee and Euclidean weights over any $\mathbb{Z}/N\mathbb{Z}$ (2017)
- Part of his proof was a general criterion for any weight over $\mathbb{Z}/N\mathbb{Z}$ to have EP.
- Dyshko’s ideas can be generalized to module alphabets.
Set up: the alphabet

- $R$ finite ring, $A$ finite left $R$-module
- Assume $A$ is pseudo-injective and has cyclic socle.
- Pseudo-injective: for any submodule $B \subseteq A$ and injective homomorphism $f : B \rightarrow A$, $f$ extends to isomorphism $A \rightarrow A$.
- Cyclic socle: implies that $A$ injects into $\hat{R}$ and that $\hat{A}$ is a cyclic right $R$-module, with generating character $\chi$. 
Set up: the weight

- Let $w$ be a weight on $A$; $w : A \to \mathbb{C}$, $w(0) = 0$.
- Symmetry groups

\[
G_{lt} = \{ u \in \mathcal{U}(R) : w(ua) = w(a), a \in A \}, \\
G_{rt} = \{ \phi \in \text{GL}_R(A) : w(a\phi) = w(a), a \in A \}.
\]
The problem

- Determine conditions on $w$ that imply $w$ has EP.
- EP: for $C \subseteq A^n$ and linear $w$-isometry $f : C \to A^n$, $f$ extends to a $G_{rt}$-monomial transformation of $A^n$. 
Some matrices

For any left $R$-submodule $B \subseteq A$, define a matrix $Q^B = (Q^B_{\phi,u})$:

$$Q^B_{\phi,u} = \sum_{b \in B} w(b\phi)\chi(ub),$$

where $\phi \in \text{Stab}(B)\backslash \text{GL}_R(A)/\text{G}_{rt}$ and $u \in \text{Stab}(\chi|_B)\backslash \mathcal{U}(R)/\text{G}_{lt}$.

Here, $\text{Stab}(B) = \{\phi \in \text{GL}_R(A) : b\phi = b, b \in B\}$ is the point-wise stabilizer of $B$. 
Main result

- Condition: for each nonzero left $R$-submodule $B \subseteq A$:

  the matrix $Q^B$ has zero left nullspace.  \hfill (1)

Theorem

If (1) is satisfied, then $w$ has EP.
Isometry condition

- Let $C \subseteq A^n$ be the image of $\Lambda : M \to A^n$, $\Lambda = (\lambda_1, \ldots, \lambda_n)$, with information module $M \cong C$.
- Set $N = \Lambda f : M \to A^n$, $N = (\nu_1, \ldots, \nu_n)$.
- $f$ being a $w$-isometry means
  \[
  \sum_{i=1}^{n} w(x \lambda_i) = \sum_{j=1}^{n} w(x \nu_j), \quad x \in M.
  \]
- Goal: show that the numbers of $\lambda_i$ and $\nu_j$ in a given $G_{rt}$-orbit are equal.
- Method: take Fourier transform and set up linear equations in these numbers.
Fourier transform calculation

- Each $\lambda_i, \nu_j \in \text{Hom}_R(M, A)$.
- For arbitrary $\sigma \in \text{Hom}_R(M, A)$, what is the Fourier transform of $f_\sigma : M \rightarrow \mathbb{C}$, $x \mapsto w(x\sigma)$?
- For $\pi \in \hat{M}$,

$$\hat{f}_\sigma(\pi) = \sum_{x \in M} \pi(x) f_\sigma(x) = \sum_{x \in M} \pi(x) w(x\sigma).$$

- Write in terms of sum over values $x\sigma \in \text{im} \sigma$. 
For $\pi \in \hat{M}$,

$$\hat{f}_\sigma(\pi) = \sum_{a \in \text{im } \sigma} \sum_{x : x\sigma = a} \pi(x)w(a) = \sum_{a \in \text{im } \sigma} w(a) \sum_{x : x\sigma = a} \pi(x).$$

Because $\sigma$ is a homomorphism, $\sum_{x : x\sigma = a} \pi(x)$ is a sum over a coset of $\ker \sigma$.

Let $x_a \in M$ be one element with $x_a\sigma = a$. Then every $x \in M$ with $x\sigma = a$ has the form $x = x_a + k$ with $k \in \ker \sigma$. 

Re-write sum
Simplify character sum

\[
\sum_{x : x\sigma = a} \pi(x) = \sum_{k \in \ker \sigma} \pi(x_a + k) \\
= \pi(x_a) \sum_{k \in \ker \sigma} \pi(k) \\
= \begin{cases} 
|\ker \sigma| \pi(x_a), & \pi \in (\hat{M} : \ker \sigma), \\
0, & \pi \not\in (\hat{M} : \ker \sigma). 
\end{cases}
\]
Summary of calculation

- For $\pi \in \hat{M}$,

$$\hat{f}_\sigma(\pi) = \begin{cases} |\ker \sigma| \sum_{a \in \im \sigma} w(a) \pi(x_a), & \pi \in (\hat{M} : \ker \sigma), \\ 0, & \pi \notin (\hat{M} : \ker \sigma). \end{cases}$$

- When $\pi \in (\hat{M} : \ker \sigma)$, the value of $\pi(x_a)$ depends only on $a$: $\pi$ descends to well-defined character on $M / \ker \sigma \cong \im \sigma$.

- When $\pi \in (\hat{M} : \ker \sigma)$, write $(\mathcal{F}_{\im \sigma} w)(\pi) = \sum_{a \in \im \sigma} w(a) \pi(x_a) = \sum_{x \in M / \ker \sigma} w(x \sigma) \pi(x).$
Dual maps

- For $\sigma \in \text{Hom}_R(M, A)$, $\sigma : M \to A$, there is the dual map $\hat{\sigma} : \hat{A} \to \hat{M}$ with image $\text{im} \hat{\sigma}$.

- Remember that $\hat{A}$ is a cyclic right $R$-module, so $\text{im} \hat{\sigma}$ is also cyclic.

- $\text{im} \hat{\sigma} = (\hat{M} : \ker \sigma)$.

- If $\psi \in \hat{A}$, $x \in \ker \sigma$, $\hat{\sigma}(\psi)(x) = \psi(x\sigma) = \psi(0) = 1$.

- If $\pi \in (\hat{M} : \ker \sigma)$, $\pi$ descends to a well-defined character on $M/\ker \sigma \cong \text{im} \sigma \subseteq A$. Any lift $\tilde{\pi}$ of $\pi$ under $\hat{A} \to (\text{im} \sigma)\hat{\cong}$ has $\hat{\sigma}(\tilde{\pi}) = \pi$. 
Let the indicator function of a subset $S \subseteq \hat{M}$ be $\delta_S$: value 1 on $S$, value 0 elsewhere.

Isometry condition: $\sum_{i=1}^{n} w(x\lambda_i) = \sum_{j=1}^{n} w(x\nu_j)$.

Fourier transform: an equation of functions on $\hat{M}$:

$$\sum_{i=1}^{n} |\ker \lambda_i| (\mathcal{F}_{im} \lambda_i w) \delta_{im} \hat{\lambda}_i = \sum_{j=1}^{n} |\ker \nu_j| (\mathcal{F}_{im} \nu_j w) \delta_{im} \hat{\nu}_j.$$
Picking a maximal submodule

- The set of cyclic right $R$-submodules of the character module $\hat{M}$ is partially ordered by set inclusion.
- Among the submodules $\text{im } \hat{\lambda}_i, \text{im } \hat{\nu}_j \subseteq \hat{M}$, choose one that is maximal under set inclusion. Refer to it as $\text{im } \hat{\sigma}$, with $\sigma \in \text{Hom}_R(M, A)$.
- Recall that $\text{im } \hat{\sigma}$ is a cyclic right $R$-module. Denote by $\mathcal{U}(\text{im } \hat{\sigma})$ the set of all generators of $\text{im } \hat{\sigma}$. We restrict the Fourier transform equation to $\mathcal{U}(\text{im } \hat{\sigma}) \subseteq \hat{M}$. 
Exploiting the Fourier transform

- If \( \pi \in \mathcal{U}(\text{im } \hat{\sigma}) \), evaluating the Fourier transform equation at \( \pi \) yields nonzero terms only when \( \pi \in \text{im } \hat{\lambda}_i \) or \( \pi \in \text{im } \hat{\nu}_j \).
- Because \( \pi \) generates \( \text{im } \hat{\sigma} \), this means \( \text{im } \hat{\sigma} \subseteq \text{im } \hat{\lambda}_i \) or \( \text{im } \hat{\sigma} \subseteq \text{im } \hat{\nu}_j \).
- But \( \text{im } \hat{\sigma} \) was chosen to be maximal, so \( \text{im } \hat{\sigma} = \text{im } \hat{\lambda}_i \) or \( \text{im } \hat{\sigma} = \text{im } \hat{\nu}_j \).
- Thus \( (\hat{M} : \text{ker } \sigma) = (\hat{M} : \text{ker } \lambda_i) \) or \( (\hat{M} : \text{ker } \sigma) = (\hat{M} : \text{ker } \nu_j) \); so \( \text{ker } \sigma = \text{ker } \lambda_i \) or \( \text{ker } \sigma = \text{ker } \nu_j \).
Let $A$ be pseudo-injective. For $\sigma, \tau \in \text{Hom}_R(M, A)$,\begin{itemize}
\item \text{ker } \sigma = \text{ker } \tau \text{ if and only if } \sigma = \tau \phi \text{ for some } \phi \in \text{GL}_R(A).
\item \text{If } \sigma = \tau \phi, \text{ then } x\sigma = 0 \text{ iff } x\tau = 0, \text{ as } \phi \text{ is invertible.}
\item \text{If ker } \sigma = \text{ker } \tau, \sigma, \tau \text{ descend to well-defined injective maps } \bar{\sigma}, \bar{\tau} : M/\text{ker } \tau \to A. \text{ Set } B = \text{im } \bar{\tau} \subseteq A. \text{ Then } \bar{\tau}^{-1}\bar{\sigma} : B \to A \text{ is injective.}
\item \text{By pseudo-injectivity, } \bar{\tau}^{-1}\bar{\sigma} \text{ extends to } \phi \in \text{GL}_R(A); \text{ then } \bar{\sigma} = \bar{\tau}\phi \text{ and } \sigma = \tau\phi.
\end{itemize}
Summary of exploitation

- Evaluating the Fourier transform equation at \( \pi \in \mathcal{U}(\text{im } \hat{\sigma}) \) yields

\[
\sum_{\lambda_i \in \sigma \ \text{GL}_R(A)} (\mathcal{F}_{\text{im } \lambda_i \ W})(\pi) = \sum_{\nu_j \in \sigma \ \text{GL}_R(A)} (\mathcal{F}_{\text{im } \nu_j \ W})(\pi).
\]

- The factors of \( |\ker \lambda_i| = |\ker \sigma| = |\ker \nu_j| \) cancel.
Next steps

- Write equations in terms of $G_{rt}$-orbits, not just $GL_R(A)$-orbits.
- Vary $\pi \in U(\text{im } \hat{\sigma})$: get different equations for different $G_{lt}$-orbits.
- How does the Fourier transform equation depend on these orbits?
Dependency on orbits

- Remember that for each \( \pi \in \mathcal{U}(\text{im}\, \hat{\sigma}) \), we have

\[
\sum_{\lambda_i \in \sigma \text{ GL}_R(A)} (\mathcal{F}_\text{im}\, \lambda_i\, w)(\pi) = \sum_{\nu_j \in \sigma \text{ GL}_R(A)} (\mathcal{F}_\text{im}\, \nu_i\, w)(\pi).
\]

- The right \( \text{GL}_R(A) \)-orbit of \( \sigma \) is a disjoint unit of \( \text{G}_{rt} \)-orbits, parametrized by elements of \( \text{Stab}(\sigma) \setminus \text{GL}_R(A)/\text{G}_{rt} \).

- The generators \( \mathcal{U}(\text{im}\, \hat{\sigma}) \) equal the right \( \mathcal{U} \)-orbit of \( \pi \), which is a disjoint union of \( \text{G}_{lt} \)-orbits, parametrized by elements of \( \text{Stab}(\pi) \setminus \mathcal{U}/\text{G}_{lt} \).
What does \((\mathcal{F}_{\text{im}} \lambda_i w)(\pi^u)\) depend on?

- Fix \(\sigma\) and \(\pi \in \mathcal{U}(\text{im} \hat{\sigma})\). Let \(\xi \in \text{Stab}(\sigma)\), \(\phi \in \text{GL}_R(A)\), \(\psi \in G_{rt}\), \(s \in \text{Stab}(\pi)\), \(u \in \mathcal{U}\), and \(v \in G_{lt}\). Then (with \(y = vx\)),

\[
(\mathcal{F}_{\text{im}} \sigma \xi \phi \psi w)(\pi^{suv}) = \sum_{x \in M/\ker \sigma} w(x \sigma \xi \phi \psi) \pi^{suv}(x)
\]

\[
= \sum_{x \in M/\ker \sigma} w(x \sigma \phi) \pi^u(vx)
\]

\[
= \sum_{y \in M/\ker \sigma} w(v^{-1} y \sigma \phi) \pi^u(y)
\]

\[
= (\mathcal{F}_{\text{im}} \sigma \phi w)(\pi^u).
\]
Re-write the Fourier transform equation

- Remember that for each $\pi \in \mathcal{U}(\text{im } \hat{\sigma})$, we have

$$\sum_{\lambda_i \in \sigma \text{ GL}_R(A)} (\mathcal{F}_{\text{im}} \lambda_i \mathcal{W})(\pi) = \sum_{\nu_j \in \sigma \text{ GL}_R(A)} (\mathcal{F}_{\text{im}} \nu_i \mathcal{W})(\pi).$$

- Break up sum into pieces that depend on the $G_{rt}$-orbits of $\sigma$.

- Get an equation for each generator of the form $\pi^u$, as $\pi^u$ varies over different $G_{lt}$-orbits of $\pi$. 
Counting functions

- For each $\tau \in \text{Stab}(\sigma) \backslash \text{GL}_R(A)/G_{rt}$, set
  $$\beta(\tau) = |\{i : \lambda_i \in \sigma \tau G_{rt}\}| - |\{j : \nu_j \in \sigma \tau G_{rt}\}|.$$

- At $\pi^u$, $u \in \text{Stab}(\pi) \backslash \mathcal{U}/G_{lt}$, Fourier transform equation becomes
  $$\sum_{\tau} \beta(\tau)(\mathcal{F}_{\text{im}} \sigma \tau \mathcal{w})(\pi^u) = 0.$$

- View as matrix equation with rows given by $\tau$ and columns given by $u$. 
Bringing in condition (1)

- Condition (1) had $Q_{\phi,u}^B = \sum_{b \in B} w(b\phi)\chi(ub)$.
- Recall: $(\mathcal{F}_{\text{im} \sigma_T w})(\pi^u) = \sum_{x \in M/\ker \sigma} w(x\sigma_T)\pi^u(x)$.
- Use $(\hat{M} : \ker \sigma) \cong (M/\ker \sigma) \hat{\cong} \cong (\text{im} \sigma) \hat{\cong}$: $\pi \in (\hat{M} : \ker \sigma) \leftrightarrow \rho \in (\text{im} \sigma) \hat{\cong}$, with $\pi(x) = \rho(x\sigma)$ or $\rho(b) = \pi(x_b)$ where $x_b \sigma = b$.
- Then $(\mathcal{F}_{\text{im} \sigma_T w})(\pi^u) = \sum_{b \in \text{im} \sigma} w(b\tau)\rho(ub)$.
- This is condition (1) for $B = \text{im} \sigma$.
- Generator $\chi$ of $\hat{A}$ projects to a generator of any $\hat{B}$. 

Extension Problem for general weights
Applying condition (1)

- By condition (1), we have $\beta(\tau) = 0$ for all $\tau$. I.e., 
  \[ |\{ i : \lambda_i \in \sigma \tau G_{rt} \}| = |\{ j : \nu_j \in \sigma \tau G_{rt} \}|, \text{ any } \tau. \]

- Choose a matching: for any of these $j$, there is an $i = P(j)$ and $\phi_j \in G_{rt}$ such that $\nu_j = \lambda_{P(j)} \phi_j$.

- Then $w(x \nu_j) = w(x \lambda_{P(j)} \phi_j) = w(x \lambda_{P(j)})$, $x \in M$.

- Subtract these terms from the isometry condition, and proceed recursively.

- From remaining im $\hat{\lambda}_i$, im $\hat{\nu}_j$, choose one that is maximal, etc. Repeat.
Dyshko’s result on Lee and Euclidean weights

- Consider $R = \mathbb{Z}/N\mathbb{Z}$ with Lee or Euclidean weight.
- By some clever estimates, Dyshko shows that (a permutation of) the matrix $Q^B$ is diagonally dominant, hence invertible.
- Uses fact that for $ab = c$, $\mathbb{Z}/a\mathbb{Z} \hookrightarrow \mathbb{Z}/c\mathbb{Z}$, $x \mapsto bx$, the restriction of the Lee weight of $\mathbb{Z}/c\mathbb{Z}$ to $b(\mathbb{Z}/a\mathbb{Z})$ is $b$ times the Lee weight of $\mathbb{Z}/a\mathbb{Z}$. (It’s $b^2$ for Euclidean weight.)
- I won’t go into the details.