Simplicial Complexes Arising from Linear Codes

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“And now for something completely different.”
—John Cleese (1969)
Simplicial complexes arising from linear codes

- Simplicial complexes, Stanley-Reisner rings.
- Alexander dual.
- Parity check matrix or generator matrix?
- Poset of subspaces of $M^\#$.
- Possible resolution of Stanley-Reisner ring.
- Good case: one-weight code.
- Examples.
- Effect of puncturing.
- Effect of higher multiplicities.
Setting for this lecture

- Linear codes over a finite field, $\mathbb{F}_2$ in examples.
- This is work in progress.
Simplicial complexes

- Let $E$ be a finite set, say $E = \{1, 2, \ldots, n\}$.
- An abstract **simplicial complex** $\Delta$ is a collection of subsets of $E$ that is closed under taking subsets. I.e., if $\sigma \in \Delta$ and $\tau \subseteq \sigma$, then $\tau \in \Delta$.
- Elements of $\Delta$ are called **faces**, and maximal faces (under inclusion) are called **facets**.
Polynomial ring

- Let $\mathbb{K}$ be any field, $E = \{1, 2, \ldots, n\}$.
- Polynomial ring $S = \mathbb{K}[x_1, \ldots, x_n]$.
- Notation: for $\sigma \subseteq E$, write $x^\sigma = \prod_{i \in \sigma} x_i$. ($x^\emptyset = 1$.)
- Fine grading: $S$ is $\mathbb{N}^n$-graded by exponents.
- Coarse grading: $S$ is $\mathbb{N}$-graded by total degree.
- Can then have finely-graded or coarsely-graded modules over $S$. 
Stanley-Reisner ring

- Given a simplicial complex $\Delta$, the **Stanley-Reisner ideal** $I_\Delta \subseteq S$ is generated by $\{x^\sigma : \sigma \notin \Delta\}$.
- The **Stanley-Reisner ring** is $R_\Delta = S/I_\Delta$.
- One goal: determine minimal free resolution of $R_\Delta$ as a finely- or coarsely-graded $S$-module.
- Field of “combinatorial commutative algebra.”
Alexander dual

- Complement: if $\sigma \subseteq E$, define $\bar{\sigma} = E \setminus \sigma$.
- Given a simplicial complex $\Delta$, define its **Alexander dual**:
  \[ \Delta^\vee = \{ \bar{\sigma} : \sigma \not\in \Delta \}. \]
- If $D_\Delta = \{ \bar{\sigma} : \sigma \in \Delta \}$, then $\Delta^\vee = \{ \tau : \tau \not\in D_\Delta \}$.
- Also, $D_{\Delta^\vee} = \{ \bar{\tau} : \tau \in \Delta^\vee \} = \{ \sigma : \sigma \not\in \Delta \}$, which provides the exponents for generators of $I_\Delta$. 
Simplicial complexes arising from linear codes

Simplicial complex from parity check matrix

- Suppose a linear code $C \subseteq \mathbb{F}_q^n$ is given by a parity check matrix $H$. If $\dim C = k$, then $H$ is an $(n - k) \times n$ matrix, and $c \in C$ if and only if $Hc^T = 0$.

- Let $E = \{1, 2, \ldots, n\}$, thought of as the position numbers of the columns of $H$.

- Define $\Delta_H = \{\sigma \subseteq E : \sigma$-columns of $H$ are linearly independent\}.

- In fact, $\Delta_H$ is a matroid.
Using generator matrix instead

- If $C$ has generator matrix $G$, then $G$ has size $k \times n$. The columns of $G$ represent coordinate functionals $\lambda_i \in M^\# = \text{Hom}_{F_q}(M, F_q)$. Think $C$ as image of $\Lambda : M \rightarrow F_q^n$.

- Define $\Delta_G = \{\bar{\tau} : \text{\tau-columns of $G$ span $M^\#$}\}$.

- Then observe, for later use, that $\Delta_G^\vee = \{\tau : \text{\tau-columns of $G$ do not span $M^\#$}\}$. 
The following statements are equivalent:

1. \( \sigma \in \Delta_H \).
2. \( \sigma \)-columns of \( H \) are linearly independent.
3. If \( c \in \mathbb{F}_q^n \) has support in \( \sigma \) and \( Hc^T = 0 \), then \( c = 0 \).
4. If \( c \in C \) has support in \( \sigma \), then \( c = 0 \).
5. If \( x \in M \) has \( x\lambda_i = 0 \) for \( i \in \bar{\sigma} \), then \( x = 0 \).
6. \( (\text{Span}\{\lambda_i : i \in \bar{\sigma}\})^\circ = 0 \); i.e., \( \bar{\sigma} \)-columns span \( M^\# \).
7. \( \sigma \in \Delta_G \).
Recall that the Alexander dual of $\Delta_G$ was
$$\Delta_G^\vee = \{ \tau : \text{$\tau$-columns of } G \text{ do not span } M^{\#} \}. $$
If $\tau \in \Delta_G^\vee$, then what space do the $\tau$-columns span?
For every proper subspace $L \subseteq M^{\#}$, define
$$\tau_L = \{ i : \lambda_i \in L \}.$$ 
As $L$ varies over the maximal proper subspaces of $M^{\#}$, the $\tau_L$ include all the facets of $\Delta_G^\vee$.
Then the $\bar{\tau}_L$, $L$ maximal, provide the exponents for the generators of $I_{\Delta}$. 

Poset of subspaces of $M^{\#}$
Example 1

- One weight code of dimension 3 over $\mathbb{F}_2$ has generator matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\end{bmatrix}
\]

- There are seven 2-dimensional subspaces $L \subseteq M^\#$, and seven 1-dimensional subspaces. The $\tau_L$ are: 246, 145, 347, 123, 257, 167, 356; 1, 2, 3, 4, 5, 6, 7; and $\emptyset$. 
Possible resolution of Stanley-Reisner ring

- Notation: for $\sigma \subseteq E$, write $S(-\sigma)$ for a free finely-graded $S$-module isomorphic to $Sx^\sigma$.
- It seems to be the case that the following is a (non-minimal) free resolution of $R_{\Delta G}$:

\[
\begin{align*}
0 & \leftarrow R_{\Delta G} \leftarrow S \leftarrow \bigoplus_{L \text{ codim 1}} S(-\bar{\tau}_L) \leftarrow \\
& \quad \cdots \leftarrow \bigoplus_{L \text{ codim } d} S(-\bar{\tau}_L)^{(d)} \leftarrow \\
& \quad \cdots \leftarrow \bigoplus_{L \text{ codim } k} S(-\bar{\tau}_L)^{(k)} \leftarrow 0.
\end{align*}
\]
Good case: one-weight code

- Johnsen and Verdure show that the complex above is a minimal free resolution of $R_{\Delta G}$ when $C$ is a linear one-weight code.
- This involves a careful analysis of the subcodes of a one-weight code and the use of Hochster’s formula for the Betti numbers of a minimal resolution in terms of the reduced homology of certain subcomplexes.
Example 1 again (a)

- One weight code of dimension 3 over $\mathbb{F}_2$ has generator matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1
\end{bmatrix}
\]

- There are seven 2-dimensional subspaces $L \subseteq M^\#$, and seven 1-dimensional subspaces. The $\tau_L$ are: 246, 145, 347, 123, 257, 167, 356; 1, 2, 3, 4, 5, 6, 7; and $\emptyset$. 
Example 1 (b)

- The respective $\tau_L$ have cardinalities 4, 6, 7, respectively.
- The data suggest, and Macaulay 2 confirms, a minimal coarse resolution:

$$0 \leftarrow R_\Delta \leftarrow S \leftarrow S(-4)^7 \leftarrow S(-6)^{14} \leftarrow S(-7)^8 \leftarrow 0.$$
Example 2 (a)

- Now consider the code of dimension 3 obtained by puncturing column 7:

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 & 0
\end{bmatrix}
\]

- The \( \tau_L \) are: 246, 145, 34, 123, 25, 16, 356; 1, 2, 3, 4, 5, 6, \( \emptyset \); \( \emptyset \). (Delete any 7 from previous listing.)
Example 2 (b)

- These data would suggest a (non-minimal) coarse resolution:

\[
\begin{align*}
0 & \leftarrow R_\Delta \leftarrow S \leftarrow S(-3)^4 \oplus S(-4)^3 \\
& \leftarrow S(-5)^{12} \oplus S(-6)^2 \leftarrow S(-6)^8 \leftarrow 0.
\end{align*}
\]

- The minimal coarse resolution from Macaulay 2:

\[
\begin{align*}
0 & \leftarrow R_\Delta \leftarrow S \leftarrow S(-3)^4 \oplus S(-4)^3 \\
& \leftarrow S(-5)^{12} \leftarrow S(-6)^6 \leftarrow 0.
\end{align*}
\]
Example 3 (a)

- This time, duplicate the last column in the one-weight code:

\[
\begin{bmatrix}
0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
\end{bmatrix}
\]

- Now the $\tau_L$ are: 246, 145, 3478, 123, 2578, 1678, 356; 1, 2, 3, 4, 5, 6, 78; and $\emptyset$. (Anytime there is a 7, also include an 8.)
Example 3 (c)

- These data would suggest a coarse resolution:

\[ 0 \leftarrow R_\Delta \leftarrow S \leftarrow S(-4)^3 \oplus S(-5)^4 \]
\[ \leftarrow S(-6)^2 \oplus S(-7)^{12} \leftarrow S(-8)^8 \leftarrow 0. \]

- This agrees with what one gets from Macaulay 2.
Effect of puncturing

- If a column is removed (punctured), say column \( j \), then the number of columns is smaller. Call the original code \( C \) and the punctured code \( C' \).
- Set \( E' = E \setminus \{ j \} \). Then \( \bar{\tau}'_L = \tau_L \cap E' \).
- Note that \( \bar{\tau}'_L = E' \setminus \tau'_L = \bar{\tau}_L \cap E' \).
- Thus \( |\bar{\tau}'_L| = |\bar{\tau}_L| \) when \( j \in \tau_L \), and \( |\bar{\tau}'_L| = |\bar{\tau}_L| - 1 \) when \( j \not\in \tau_L \).
- This explains the shifts in degrees in Example 2.
Effect of higher multiplicities

- Now duplicate column $j$. Set $E' = E \cup \{j^*\}$.
- If $j \in \tau_L$, then $\tau'_L = \tau_L \cup \{j^*\}$. If $j \notin \tau_L$, then $\tau'_L = \tau_L$.
- Thus $|\tilde{\tau}'_L| = |\tilde{\tau}_L|$ when $j \in \tau_L$, and $|\tilde{\tau}'_L| = |\tilde{\tau}_L| + 1$ when $j \notin \tau_L$.
- This explains the shifts in degrees in Example 3.
Interpretation of coarse grading degrees

- At homological degree $i$, the smallest coarse grading degree is the generalized Hamming weight for $C$ in dimension $i$. (Chen) That is, among the subcodes of $C$ of dimension $i$, the smallest support length.

- A subcode $D \subseteq C$ is determined by its annihilator $L \subseteq M^\#$: codewords vanishing on $\tau_L$ belong to $D$. Such codewords have support contained in $\bar{\tau}_L$. 
Codes over rings

- Most of the ideas presented should make sense for linear codes over rings or even over modules.
- One twist: in the proposed free resolution, the modules in homological degree $i$ corresponded to subspaces $L \subseteq M^\#$ of codimension $i$. For codes over rings or modules, there may not be a way to assign degrees or dimensions to $L \subseteq \text{Hom}_R(M, A)$.
- Perhaps there is a more general limit coming from viewing the terms in the complex as a functor on the poset of submodules of $\text{Hom}_R(M, A)$. 
Category of linear codes

- In 1998, Ed Assmus proposed a category of linear codes. Morphisms are defined as homomorphisms that do not increase the Hamming distance.
- Is $C \mapsto \Delta_C$ a functor from the category of linear codes to the category of simplicial complexes? If not, is there a way to fix it?
Thank you

- Thanks again to Professor Hongwei Liu for the invitation and for his generous hospitality.
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