Dual Codes over Finite Rings—Cautions and Compromises

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Acknowledgments

Much of the material in this talk has become part of the fabric of coding theory and was developed by Delsarte, Gleason, MacWilliams, Sloane, ... . Later influences include Dinh and López-Permooth; Greferath, Nechaev, and Wisbauer; Nebe, Rains, and Sloane.
The classical case—finite fields

- On $\mathbb{F}_q^n$, the standard $\mathbb{F}_q$-valued dot product is a nondegenerate, symmetric bilinear form.
- If $C \subset \mathbb{F}_q^n$ is a linear code of dimension $k$, then the dual code is

$$C^\perp = \{ y \in \mathbb{F}_q^n : x \cdot y = 0, \text{ all } x \in C \}.$$

- We work with Hamming weights throughout.
Features of the classical case

- $C^\perp \subset \mathbb{F}_q^n$.
- $C^\perp$ is a linear code.
- $\dim C + \dim C^\perp = n$; or $|C||C^\perp| = |\mathbb{F}_q^n|$.
- $(C^\perp)^\perp = C$.
- The MacWilliams identities hold.

What happens when we use other alphabets, such as finite rings or finite modules over a finite ring?
Less structure—Additive codes

- Let $G$ be a finite abelian group.
- An additive code of length $n$ over $G$ is a subgroup $C \subset G^n$.
- Let $\beta : G \times G \to \mathbb{Q}/\mathbb{Z}$ be a nondegenerate biadditive form, and extend $\beta$ to $\beta : G^n \times G^n \to \mathbb{Q}/\mathbb{Z}$.
- For $C \subset G^n$, define
  
  $$l(C) = \{y \in G^n : \beta(y, x) = 0, \text{ for all } x \in C\},$$
  $$r(C) = \{y \in G^n : \beta(x, y) = 0, \text{ for all } x \in C\}.$$
Features of additive case

- \( l(C), r(C) \subset G^n \).
- \( l(C), r(C) \) are additive codes.
- \(|C|\|l(C)| = |C|\|r(C)| = |G^n|\).
- \( l(r(C)) = r(l(C)) = C \).
- The MacWilliams identities hold.
- If \( \beta \) is symmetric, then \( l(C) = r(C) \). Such a \( \beta \) exists for any finite \( G \).
More structure—codes over modules

- Let $R$ be a finite ring with 1.
- Let $A$ be a finite left $R$-module, $B$ a finite right $R$-module, and $E$ a finite ($R$, $R$)-bimodule.
- Let $\beta : A \times B \to E$ be a nondegenerate bilinear form. Extend to $\beta : A^n \times B^n \to E$.
- For a left linear code (submodule) $C \subset A^n$, define $r(C) = \{y \in B^n : \beta(x, y) = 0, \text{ for all } x \in C\}$.
- For a right linear code (submodule) $D \subset B^n$, define $l(D) = \{y \in A^n : \beta(y, x) = 0, \text{ for all } x \in D\}$. 
(Questionable) Features of the module case

- $r(C) \subset B^n$; $l(D) \subset A^n$.
- $r(C)$ is a right linear code; $l(D)$ is a left linear code.
- Question: Sizes?
- $C \subset l(r(C))$; $D \subset r(l(D))$. Question: Equality of double annihilators?
- Question: MacWilliams identities?
Linear codes over rings—double annihilators

- Suppose $A = B = E = R$, with $\beta$ the $R$-valued dot product.

**Theorem (M. Hall)**

There is equality of double annihilators, i.e., $l(r(C)) = C$ and $r(l(D)) = D$ for all left linear codes $C$ and right linear codes $D$, if and only if the finite ring $R$ is quasi-Frobenius.
A finite ring is *quasi-Frobenius* if it is self-injective.

Let \( R = \mathbb{F}_2[X, Y]/(X^2, XY, Y^2) \). \( R \) is not QF.

\( l(r((X))) = (X, Y) \) violates equality of double annihilators.
Sizes of annihilators

Theorem

Let $R$ be a finite quasi-Frobenius ring. Then $|C||r(C)| = |D||l(D)| = |R^n|$, for all left linear codes $C$ and right linear codes $D$, if and only if $R$ is a Frobenius ring.

- $R$ Frobenius if $R/\text{Rad } R \cong \text{Soc } R$ as one-sided modules.
Example—matrix module

- Every non-Frobenius ring contains in its socle a matrix submodule of the form $M_{k,l}(\mathbb{F}_q)$, with $k < l$.
- Let
  \[
x = \begin{pmatrix}
  1 & 0 & \ldots & 0 \\
  0 & 0 & \ldots & 0 \\
  \vdots \\
  0 & 0 & \ldots & 0
  \end{pmatrix} \in M_{k,l}(\mathbb{F}_q).
  \]
- One can show that $|Rx||r(Rx)| < |R|$.
MacWilliams identities will hold if we can relate \( \beta : A \times B \to E \) to a nondegenerate \( \beta' : A \times B \to \mathbb{Q}/\mathbb{Z} \). (Which will force \( A \) and \( B \) to be isomorphic as abelian groups.)

If \( \chi : E \to \mathbb{Q}/\mathbb{Z} \) is a homomorphism, define \( \beta' = \chi \circ \beta \).
Case of modules—special character

**Theorem**

Suppose that $\chi : E \to \mathbb{Q}/\mathbb{Z}$ has the property that $\ker \chi$ contains no nonzero left or right $R$-submodules of $E$. If $\beta : A \times B \to E$ is nondegenerate, then so is $\beta' = \chi \circ \beta : A \times B \to \mathbb{Q}/\mathbb{Z}$.

- $\beta$-annihilators for submodules agree with $\beta'$-annihilators.
- The MacWilliams identities hold in this situation.
Case of rings—MacWilliams identities

Again, let $A = B = E = R$, with $\beta$ equal to the $R$-valued dot product.

**Theorem**

There exists $\chi : R \to \mathbb{Q}/\mathbb{Z}$ with the property that $\ker \chi$ contains no nonzero one-sided ideals of $R$ if and only if $R$ is a Frobenius ring. ($\chi$ is a generating character.)