

**MACWILLIAMS' EXTENSION THEOREM
FOR BI-INVARIANT WEIGHTS OVER
FINITE PRINCIPAL IDEAL RINGS**

MARCUS GREFERATH, THOMAS HONOLD, CATHY MC FADDEN, JAY A. WOOD,
AND JENS ZUMBRÄGEL

In memoriam Werner Heise (1944–2013)

ABSTRACT. A finite ring R and a weight w on R satisfy the *Extension Property* if every R -linear w -isometry between two R -linear codes in R^n extends to a monomial transformation of R^n that preserves w . MacWilliams proved that finite fields with the Hamming weight satisfy the Extension Property. It is known that finite Frobenius rings with either the Hamming weight or the homogeneous weight satisfy the Extension Property. Conversely, if a finite ring with the Hamming or homogeneous weight satisfies the Extension Property, then the ring is Frobenius.

This paper addresses the question of a characterization of all bi-invariant weights on a finite ring that satisfy the Extension Property. Having solved this question in previous papers for all direct products of finite chain rings and for matrix rings, we have now arrived at a characterization of these weights for finite principal ideal rings, which form a large subclass of the finite Frobenius rings. We do not assume commutativity of the rings in question.

KEY WORDS: Frobenius ring, principal ideal ring, linear code, extension theorem, Möbius function.

2010 Mathematics Subject Classification: primary 94B05; secondary 16S36, 20M25

1. INTRODUCTION

Let R be a finite ring equipped with a weight w . Two linear codes $C, D \leq {}_R R^n$ are *isometrically equivalent* if there is an isometry between them, i.e., an R -linear bijection $\varphi : C \rightarrow D$ that satisfies $w(\varphi(c)) = w(c)$ for all $c \in C$. We say that φ *preserves* the weight w .

MacWilliams in her doctoral dissertation [13] and later Bogart, Goldberg, and Gordon [2] proved that, in the case where R is a finite field and w is the Hamming weight, every isometry is the restriction of a *monomial transformation* Φ of the ambient space ${}_R R^n$. A monomial transformation of ${}_R R^n$ is simply a left linear mapping $\Phi : R^n \rightarrow R^n$ the matrix representation of which is a product of a permutation matrix and an invertible diagonal matrix. Said another way, every

Date: March 27, 2014. Accepted for publication in Journal of Combinatorial Theory, Series A 125C (2014), pp. 177-193. DOI information: 10.1016/j.jcta.2014.03.005.

This work was partially supported by Science Foundation Ireland under Grants 06/MI/006 and 08/IN.1/I1950, and by a sabbatical leave from Western Michigan University. JAW thanks the Claude Shannon Institute for its hospitality and support during a research visit in November 2011.

Hamming isometry over a finite field extends to a monomial transformation. This result is often called the *MacWilliams Extension Theorem* or the *MacWilliams Equivalence Theorem*.

With increased interest in linear codes over finite rings there arose the natural question: could the Extension Theorem be proved in the context of ring-linear coding theory? This question appeared complicated, as two different weights were pertinent: the traditional Hamming weight w_H and also a new weight w_{hom} called the *homogeneous* weight by its discoverers Constantinescu and Heise [3].

In [18] Wood proved the MacWilliams Extension Theorem for all linear codes over finite Frobenius rings equipped with the Hamming weight. In the commutative case he showed in the same paper that the Frobenius property was not only sufficient but also necessary. In the non-commutative case, the necessity of the Frobenius property was proved in [20].

Inspired by the paper of Constantinescu, Heise, and Honold [4] which used combinatorial methods to prove the Extension Theorem for homogeneous weights on \mathbb{Z}_m , Greferath and Schmidt [10] showed that the Extension Theorem is true for linear codes over finite Frobenius rings when using the homogeneous weight. Moreover, they showed that for all finite rings every Hamming isometry between two linear codes is a homogeneous isometry and vice versa.

The situation can be viewed as follows: for R a finite ring, and either the Hamming weight or the homogeneous weight, the Extension Theorem holds for all linear codes in R^n if and only if the ring is Frobenius. This is a special case of more general results by Greferath, Nechaev, and Wisbauer [9] who proved that if the codes are submodules of a quasi-Frobenius bi-module ${}_R A_R$ over any finite ring R , then the Extension Theorem holds for the Hamming and homogeneous weights. The converse of this was proved by Wood in [21].

Having understood all requirements on the algebraic side of the problem, we now focus on the metrical aspect. This paper aims to further develop a characterization of all weights on a finite (Frobenius) ring, for which the corresponding isometries satisfy the Extension Theorem.

In our discussion we will assume that the weights in question are bi-invariant, which means that $w(ux) = w(x) = w(xu)$ for all $x \in R$ and $u \in R^\times$. Our main results do not apply to weights with smaller symmetry groups such as the Lee or Euclidean weight (on $R = \mathbb{Z}_m$, except for $m \in \{2, 3, 4, 6\}$), despite their importance for ring-linear coding theory.

The goal of this paper is to give a necessary and sufficient condition that a bi-invariant weight w must satisfy in order for the Extension Theorem to hold for isometries preserving w . We are not able to characterize all such weights when the underlying ring is an arbitrary Frobenius ring, but we do achieve a complete result for *principal ideal rings*. These are rings in which each left or right ideal is principal, and they form a large subclass of the finite Frobenius rings.

The present work is a continuation and generalization of earlier work on this topic [6, 7, 8, 17, 19, 21]. As in [7, 8] the Möbius function on the partially ordered set of (principal, right) ideals is crucial for the statement and proof of our main characterization result; however, in contrast to these works we do not need the values of the Möbius function explicitly, but use its defining properties instead to achieve a more general result. Our restriction to principal ideal rings stems from our method of proof, which requires the annihilator of a principal ideal to be principal.

The main result was proved for the case of finite chain rings in [6, Theorem 3.2] (and in a more general form in [17, Theorem 16]), in the case \mathbb{Z}_m in [7, Theorem 8], for direct products of finite chain rings in [8, Theorem 22], and for matrix rings over finite fields in [21, Theorem 9.5] (see Example 4.7 below). The main result gives a concrete manifestation of [17, Proposition 12] and [19, Theorem 3.1]. Further to [8] we prove that our condition on the weight is not only sufficient, but also necessary for the Extension Theorem, using an argument similar to that in [7, 20].

Here is a short summary of the contents of the paper. In Section 2 we review the terminology of Frobenius rings, Möbius functions, and orthogonality matrices needed for the statements and proofs of our main results. In addition, we prove a result (Corollary 2.2) that says that a right-invariant weight w on R satisfies the Extension Property if the Hamming weight w_H is a correlation multiple of w .

In Section 3 we show that the Extension Property holds for a bi-invariant weight if and only if its orthogonality matrix is invertible. The main results are stated in Section 4. By an appropriate unimodular change of basis, the orthogonality matrix can be put into triangular form, with a simple expression for the diagonal entries (Theorem 4.3). The Main Result (Theorem 4.4) then says that the Extension Property holds if and only if all the diagonal entries of the orthogonality matrix are nonzero. A proof of Theorem 4.3 is given in Section 5.

This paper is written in memory of our friend, teacher, and colleague Werner Heise who, sadly, passed away in February 2013 after a long illness. Werner has been very influential in ring-linear coding theory through his discovery of the homogeneous weight on \mathbb{Z}_m (“Heise weight”) and subsequent contributions.

2. NOTATION AND BACKGROUND

In all that follows, rings R will be finite, associative and possess an identity 1. The group of invertible elements (units) will be denoted by R^\times or U . Any module ${}_R M$ will be unital, meaning $1m = m$ for all $m \in M$.

Frobenius Rings. We describe properties of Frobenius rings needed in this paper, as in [11].

The character group of the additive group of a ring R is defined as $\widehat{R} := \text{Hom}_{\mathbb{Z}}(R, \mathbb{C}^\times)$. This group has the structure of an R, R -bimodule by defining $\chi^r(x) := \chi(rx)$ and ${}^r\chi(x) := \chi(xr)$ for all $r, x \in R$, and for all $\chi \in \widehat{R}$.

The *left socle* $\text{soc}({}_R R)$ is defined as the sum of all minimal left ideals of R . It is a two-sided ideal. A similar definition leads to the *right socle* $\text{soc}(R_R)$ which is also two-sided, but will not necessarily coincide with its left counterpart.

A finite ring R is *Frobenius* if one of the following four equivalent statements holds:

- ${}_R R \cong {}_R \widehat{R}$.
- $R_R \cong \widehat{R}_R$.
- $\text{soc}({}_R R)$ is left principal.
- $\text{soc}(R_R)$ is right principal.

For a finite Frobenius ring the left and right socles coincide.

Crucial for later use is the fact that finite Frobenius rings are quasi-Frobenius and hence possess a perfect duality. This means the following: Let $L({}_R R)$ denote the lattice of all left ideals of R , and let $L(R_R)$ denote the lattice of all right ideals of R .

There is a mapping $\perp: L({}_R R) \longrightarrow L(R_R)$, $I \mapsto I^\perp$ where $I^\perp := \{x \in R \mid Ix = 0\}$ is the right annihilator of I in R . This mapping is an order anti-isomorphism between the two lattices. The inverse mapping associates to every right ideal its left annihilator.

Principal Ideal Rings. A ring R is *left principal* if every left ideal is left principal, similarly a ring is *right principal* if every right ideal is right principal. If a ring is both left principal and right principal it is a *principal ideal ring*. Nechaev in [14] proved that “a finite ring with identity in which every two-sided ideal is left principal is a principal ideal ring.” Hence every finite left principal ideal ring is a principal ideal ring. Further, as argued in [14], the finite principal ideal rings are precisely the finite direct sums of matrix rings over finite chain rings. They form a subclass of the class of finite Frobenius rings (since, for example, their one-sided socles are principal).

Möbius Function. The reader who is interested in a more detailed survey of the following is referred to [1, Chapter IV], [15], or [16, Chapter 3.6].

For a finite partially-ordered set (poset) P , we have the incidence algebra

$$\mathbb{A}(P) := \{f : P \times P \longrightarrow \mathbb{Q} \mid x \not\leq y \text{ implies } f(x, y) = 0\}.$$

Addition and scalar multiplication in $\mathbb{A}(P)$ are defined point-wise, whereas multiplication is given by the convolution:

$$(f * g)(a, b) = \sum_{a \leq c \leq b} f(a, c)g(c, b).$$

The invertible elements are exactly the functions $f \in \mathbb{A}(P)$ satisfying $f(x, x) \neq 0$ for all $x \in P$. In particular, the characteristic function of the partial order of P given by

$$\zeta : P \times P \longrightarrow \mathbb{Q}, \quad (x, y) \mapsto \begin{cases} 1 & \text{if } x \leq y, \\ 0 & \text{otherwise.} \end{cases}$$

is an invertible element of $\mathbb{A}(P)$. Its inverse is the *Möbius function* $\mu : P \times P \longrightarrow \mathbb{Q}$ recursively defined by $\mu(x, x) = 1$ and

$$\sum_{x \leq t \leq y} \mu(x, t) = 0$$

if $x < y$, and $\mu(x, y) = 0$ if $x \not\leq y$.

Weights and Code Isometries. Let R be any finite ring. By a *weight* w we mean any \mathbb{Q} -valued function $w : R \longrightarrow \mathbb{Q}$ on R , without presuming any particular properties. As usual we extend w additively to a weight on R^n by setting

$$w : R^n \longrightarrow \mathbb{Q}, \quad x \mapsto \sum_{i=1}^n w(x_i).$$

The *left* and *right symmetry groups* of w are defined by

$$\begin{aligned} G_{\text{lt}}(w) &:= \{u \in U : w(ux) = w(x), x \in R\}, \\ G_{\text{rt}}(w) &:= \{v \in U : w(xv) = w(x), x \in R\}. \end{aligned}$$

A weight w is called *left* (resp. *right*) *invariant* if $G_{\text{lt}}(w) = U$ (resp. $G_{\text{rt}}(w) = U$).

A (left) *linear code* of length n over R is a submodule C of ${}_R R^n$. A *w-isometry* is a linear map $\varphi : C \rightarrow {}_R R^n$ with $w(\varphi(x)) = w(x)$ for all $x \in C$, i.e., a mapping that preserves the weight w .

A *monomial transformation* is a bijective (left) R -linear mapping $\Phi : R^n \rightarrow R^n$ such that there is a permutation $\pi \in S_n$ and units $u_1, \dots, u_n \in U$ so that

$$\Phi(x_1, \dots, x_n) = (x_{\pi(1)}u_1, \dots, x_{\pi(n)}u_n)$$

for every $(x_1, \dots, x_n) \in R^n$. In other words, the matrix that represents Φ with respect to the standard basis of ${}_R R^n$ decomposes as a product of a permutation matrix and an invertible diagonal matrix. A $G_{\text{rt}}(w)$ -*monomial transformation* is one where the units u_i belong to the right symmetry group $G_{\text{rt}}(w)$. A $G_{\text{rt}}(w)$ -*monomial transformation* is a w -isometry of R^n .

We say that a finite ring R and a weight w on R satisfy the *Extension Property* if the following holds: For every positive length n and for every linear code $C \leq {}_R R^n$, every injective w -isometry $\varphi : C \rightarrow {}_R R^n$ is the restriction of a $G_{\text{rt}}(w)$ -monomial transformation of ${}_R R^n$. That is, every injective w -isometry φ extends to a monomial transformation that is itself a w -isometry of R^n .

Let $w : R \rightarrow \mathbb{Q}$ be a weight and let $f : R \rightarrow \mathbb{Q}$ be any function. We define a new weight wf as

$$wf : R \rightarrow \mathbb{Q}, \quad x \mapsto \sum_{r \in R} w(rx) f(r).$$

By the operation of *right correlation* $(w, f) \mapsto wf$, the vector space $V := \mathbb{Q}^R$ of all weights on R becomes a right module V_A over $A = \mathbb{Q}[(R, \cdot)]$, the rational semigroup algebra of the multiplicative semigroup (R, \cdot) of the ring (see [8]). For $r \in R$ denote by e_r the weight where $e_r(r) = 1$ and $e_r(s) = 0$ for $s \neq r$. Then $w e_r$ is simply given by $(w e_r)(x) = w(rx)$.

Denote the natural additive extension of wf to R^n by wf also.

Lemma 2.1. *Let $C \leq {}_R R^n$ be a linear code and let $\varphi : C \rightarrow R^n$ be a w -isometry, then φ is also a wf -isometry for any function $f : R \rightarrow \mathbb{Q}$.*

Proof. For all $x \in C$ we compute

$$\begin{aligned} (wf)(\varphi(x)) &= \sum_{r \in R} w(r\varphi(x)) f(r) = \sum_{r \in R} w(\varphi(rx)) f(r) \\ &= \sum_{r \in R} w(rx) f(r) = (wf)(x). \quad \square \end{aligned}$$

For a weight w consider the \mathbb{Q} -linear map $\tilde{w} : A \rightarrow V$, $f \mapsto wf$. By Lemma 2.1, if φ is a w -isometry then φ is a w' -isometry for all $w' \in \text{Im } \tilde{w}$. Note that $\text{Im } \tilde{w} = wA \leq V_A$.

Weights on Frobenius Rings. Now let R be a finite Frobenius ring. We describe two approaches that ultimately lead to the same criterion for a weight w to satisfy the Extension Property.

Approach 1. From earlier work [18] we know that the Hamming weight w_{H} satisfies the Extension Property. Combining this fact with Lemma 2.1, we immediately obtain the following result.

Corollary 2.2. *Let R be a finite Frobenius ring and let w be a weight on R such that $G_{\text{rt}}(w) = U$ and $wf = w_{\text{H}}$ for some function $f : R \rightarrow \mathbb{Q}$. Then w satisfies the Extension Property.*

In other words, if w is right-invariant and $w_{\text{H}} \in \text{Im } \tilde{w}$ then w satisfies the Extension Property.

How can we make sure that $w_{\text{H}} \in \text{Im } \tilde{w}$? One idea is to show that the \mathbb{Q} -linear map \tilde{w} is bijective: Using the natural basis $(e_r)_{r \in R}$ for V and the property $(we_r)(s) = w(rs)$ it is easy to see that \tilde{w} is described by the transpose of the matrix $(w(rs))_{r,s \in R}$. However, if the weight function w is left- or right-invariant or satisfies $w(0) = 0$ then this matrix is not invertible. Therefore we work with a “reduced” version of the map \tilde{w} .

As before, let $V := \mathbb{Q}^R$ be the vector space of all weights on R , and let V_0^U be the subspace of all weights w satisfying $w(0) = 0$ that are right-invariant. Similarly, we define the subspace ${}^U V_0$ of all weights w with $w(0) = 0$ that are left-invariant. The corresponding invariant subspaces of $A = \mathbb{Q}[(R, \cdot)]$ are A_0^U and ${}^U A_0$, where $A_0 := A/\mathbb{Q}e_0$.

If w is a weight in V_0^U then $wf \in V_0^U$ for any function $f : R \rightarrow \mathbb{Q}$, i.e., $\text{Im } \tilde{w} \leq V_0^U$. In this case we could examine the bijectivity of the \mathbb{Q} -linear map $\tilde{w} : A_0^U \rightarrow V_0^U$ (the restriction of the above map \tilde{w}). But this map does not have a nice matrix representation; setting $e_{sU} = \sum_{r \in sU} e_r$ and letting $(e_{sU})_{sU \neq 0}$ be the natural basis for A_0^U and for V_0^U , the entries of the matrix turn out to be sums of several values $w(rus)$.

However, if we work with the restriction $\tilde{w} : {}^U A_0 \rightarrow V_0^U$ instead and provided the weight w is bi-invariant (i.e., both left- and right-invariant), then this \mathbb{Q} -linear map does have a nice matrix description, namely the orthogonality matrix. This will be explained below. If this map \tilde{w} is invertible, then w satisfies the Extension Property by Corollary 2.2.

Note: Since $\text{Im } \tilde{w}$ is a submodule of V_A it follows that $w_{\text{H}} \in \text{Im } \tilde{w}$ if and only if $\text{Im } \tilde{w}_{\text{H}} \leq \text{Im } \tilde{w}$. Actually, $\text{Im } \tilde{w}_{\text{H}} = V_0^U$ (see Proposition 3.2 below), so that $w_{\text{H}} \in \text{Im } \tilde{w}$ if and only if $V_0^U \subseteq \text{Im } \tilde{w}$. This is why it is a sensible approach to investigate the surjectivity/bijectivity of the map \tilde{w} .

Approach 2. The same orthogonality matrix that appears in Approach 1 also appears in [17]. By [17, Proposition 12] (also, [19, Theorem 3.1] and [21, Section 9.2]), the invertibility of the orthogonality matrix of w implies that a w -isometry preserves the so-called *symmetrized weight composition* associated with $G_{\text{rt}}(w)$. Then, [17, Theorem 10] shows that any injective linear homomorphism that preserves the symmetrized weight composition associated with $G_{\text{rt}}(w)$ extends to a $G_{\text{rt}}(w)$ -monomial transformation. Thus, if the orthogonality matrix is invertible, any w -isometry extends to a $G_{\text{rt}}(w)$ -monomial transformation, and hence w satisfies the Extension Property.

Orthogonality Matrices. Let R be a finite Frobenius ring. There is a one-to-one correspondence between left (resp., right) principal ideals and left (resp., right) U -orbits. Each U -orbit is identified with the principal ideal of which its elements are the generators ([18, Proposition 5.1], based on work of Bass). Define for $r, s \in R \setminus \{0\}$ the functions $\varepsilon_{Rr}(x) = |Ur|^{-1}$ if $x \in Ur$, i.e., if $Rr = Rx$, and zero otherwise; similarly, let $e_{sR}(x) = e_{sU}(x) = 1$ if $xR = sR$ and zero otherwise.

Then (ε_{Rr}) and (e_{sR}) are bases for ${}^U A_0$ and V_0^U , as Rr and sR vary over all left and right nonzero principal ideals of R , respectively.

For a bi-invariant weight w , define the *orthogonality matrix* of w by $W_0 = (w(rs))_{Rr \neq 0, sR \neq 0}$. That is, the entry in the Rr, sR -position is the value of the weight w on the product $rs \in R$. The value $w(rs)$ is well-defined, because w is bi-invariant. Note that W_0 is square; this follows from work of Greferath [5] that shows the equality of the number of left and right principal ideals in a finite Frobenius ring.

Proposition 2.3. *Suppose w is bi-invariant with $w(0) = 0$. Then*

$$w \varepsilon_{Rr} = \sum_{sR \neq 0} w(rs) e_{sR}$$

for nonzero Rr , where the sum extends over all the nonzero right principal ideals sR . In particular, the matrix representing the \mathbb{Q} -linear map $\tilde{w} : {}^U A_0 \rightarrow V_0^U$, $f \mapsto wf$, with respect to the bases (ε_{Rr}) and (e_{sR}) , is the transpose of the matrix W_0 .

Proof. Since $w \in V_0^U$ we have $w \varepsilon_{Rr} \in V_0^U$, and therefore

$$w \varepsilon_{Rr} = \sum_{sR \neq 0} (w \varepsilon_{Rr})(s) e_{sR}.$$

Calculating, using that $w \in {}^U V_0$, we get:

$$(w \varepsilon_{Rr})(s) = \sum_{t \in R} w(ts) \varepsilon_{Rr}(t) = \sum_{t \in Ur} |Ur|^{-1} w(ts) = w(rs). \quad \square$$

In the algebraic viewpoint of [8], V_0^U is a right module over ${}^U A_0$. Then, W_0 is invertible if and only if w is a generator for V_0^U .

If R is a finite field and $w = w_H$, the Hamming weight on R , then W_0 is exactly the orthogonality matrix considered by Bogart, Goldberg, and Gordon [2, Section 2]. More general versions of the matrix W_0 have been utilized in [17, 19, 21].

Example 2.4. For $R = \mathbb{Z}_4$ the Lee weight w_{Lee} assigns $0 \mapsto 0$, $1 \mapsto 1$, $2 \mapsto 2$ and $3 \mapsto 1$. It is a bi-invariant weight function, as is the Hamming weight w_H on R . Based on the natural ordering of the (nonzero) principal ideals of R as $2R < R$ the orthogonality matrix for w_{Lee} is

$$W_0^{\text{Lee}} = \begin{bmatrix} 0 & 2 \\ 2 & 1 \end{bmatrix},$$

whereas the orthogonality matrix for w_H is given by

$$W_0^H = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Both of these matrices are invertible over \mathbb{Q} as observed in [5], where it was shown that the Extension Property is satisfied.

3. ORTHOGONALITY MATRICES AND THE EXTENSION THEOREM

In the present section we will show that invertibility of the orthogonality matrix of a bi-invariant weight is necessary and sufficient for that weight to satisfy the Extension Property. We split this result into two statements.

Proposition 3.1. *Let R be a finite Frobenius ring and let w be a bi-invariant weight on R . If the orthogonality matrix W_0 of w is invertible, then w satisfies the Extension Property.*

Proof. Approach 1: by Proposition 2.3 the matrix W_0 describes the \mathbb{Q} -linear map $\tilde{w} : {}^U A_0 \rightarrow V_0^U$, $f \mapsto wf$. Hence if W_0 is invertible the map \tilde{w} is bijective, and in particular $w_{\mathbb{H}} \in \text{Im } \tilde{w}$. Thus by Corollary 2.2 the weight w satisfies the Extension Property.

Approach 2: apply [17, Proposition 12] or [19, Theorem 3.1]. \square

We remark that in the foregoing discussion, \mathbb{Q} could be replaced throughout by any field K containing \mathbb{Q} , for example $K = \mathbb{C}$.

Proposition 3.2. *Let R be a finite Frobenius ring, and let w be a bi-invariant rational weight on R that satisfies the Extension Property. Then the orthogonality matrix W_0 of w is invertible.*

Proof. The proof mimics that of [20, Theorem 4.1] and [7, Proposition 7]. Assume W_0 singular for the sake of contradiction. Then there exists a nonzero rational vector $v = (v_{cR})_{cR \neq 0}$ such that $W_0 v = 0$. Without loss of generality, we may assume that v has integer entries. We proceed to build two linear codes C_+, C_- over R . Each of the codes will have only one generator. The generator for C_{\pm} is a vector g_{\pm} with the following property: for each ideal $cR \leq R_R$ with $v_{cR} > 0$ (for g_+), resp., $v_{cR} < 0$ (for g_-), the vector g_{\pm} contains $|v_{cR}|$ entries equal to c . To make these two generators annihilator-free, we append to both a trailing $1 \in R$. The typical codeword in C_{\pm} is hence of the form ag_{\pm} for suitable $a \in R$. We compare $w(ag_+)$ and $w(ag_-)$ for every $a \in R$ by calculating the difference $D(a) = w(ag_+) - w(ag_-)$. By our construction of the generators g_{\pm} , we have

$$D(a) = \sum_{cR \neq 0} w(ac) v_{cR} = (W_0 v)_{Ra} = 0,$$

for all $a \in R$. Thus $ag_+ \mapsto ag_-$ forms a w -isometry from C_+ to C_- . The codes, however, are not monomially equivalent because their entries come from different right U -orbits. \square

We summarize our findings in the following theorem.

Theorem 3.3. *A rational bi-invariant weight function on a finite Frobenius ring satisfies the Extension Property if and only if its orthogonality matrix is invertible.*

The ultimate goal is to give necessary and sufficient conditions on a bi-invariant weight w on a finite Frobenius ring R so that its orthogonality matrix W_0 is invertible. We are able to derive such a result for finite principal ideal rings.

Extended Orthogonality Matrices. Let R be a finite Frobenius ring and let w be a bi-invariant weight function with $w(0) = 0$. The orthogonality matrix for the weight w was defined as $W_0 = (w(rs))_{Rr \neq 0, sR \neq 0}$. Now define the *extended* orthogonality matrix for w as $W = (w(rs))_{Rr, sR}$. In order to examine the invertibility of W_0 we obtain a formula for $\det W$, the determinant of the matrix W . (Note that $\det W$ is well-defined up to multiplication by ± 1 , the sign depending on the particular orderings of the rows and columns of W .) First we relate $\det W$ to $\det W_0$, viewing $w(0)$ as an indeterminate.

Proposition 3.4. *The determinant $\det W_0$ is obtained from $\det W$ by dividing the latter by $w(0)$ and then setting $w(0) = 0$.*

Proof. We treat $w(0)$ as an indeterminate w_0 . Up to a sign change in $\det(W)$, we may assume that the rows and columns of W are arranged so that the first row is indexed by $R0$ and the first column is indexed by $0R$. Then W has the form

$$W = \begin{bmatrix} w_0 & w_0 & \cdots & w_0 \\ w_0 & & & \\ \vdots & & W' & \\ w_0 & & & \end{bmatrix}.$$

By subtracting the first row from every other row, we find that $\det W = w_0 \det(W' - w_0 J)$, where J is the all-one matrix. Finally the matrix W_0 equals the matrix $W' - w_0 J$ evaluated at $w_0 = 0$, so that $\det W_0 = \det(W' - w_0 J)|_{w_0=0}$. \square

Note that the extended orthogonality matrix W is not invertible for weights w satisfying $w(0) = 0$.

4. BI-INVARIANT WEIGHTS WITH INVERTIBLE ORTHOGONALITY MATRIX ON PRINCIPAL IDEAL RINGS

Let R be a finite principal ideal ring, and let w be a bi-invariant weight on R . Assume W is the extended orthogonality matrix of w . We are interested in the determinant of W and look for a way to evaluate this determinant.

We will define an invertible matrix $(Q_{cR, Rx})_{cR, Rx}$ with determinant ± 1 and multiply W by Q from the right to arrive at WQ ; then $\det(WQ) = \pm \det(W)$. The most significant advantage of considering WQ , rather than W , is that WQ will be a lower triangular matrix for which we can easily calculate the determinant.

For any finite ring define the matrix Q by

$$Q_{cR, Rx} := \mu((Rx)^\perp, cR),$$

for $cR \leq R_R$ and $Rx \leq R_R$, where μ is the Möbius function of the lattice L^* of all right ideals of R .

Lemma 4.1. *For a finite principal ideal ring R , the matrix Q is an invertible matrix with determinant ± 1 .*

Proof. We claim that the inverse of Q is given by $T_{Ra, bR} := \zeta(bR, (Ra)^\perp)$, where ζ is the indicator function of the poset L^* , meaning

$$\zeta(xR, yR) = \begin{cases} 1 & \text{if } xR \leq yR, \\ 0 & \text{otherwise.} \end{cases}$$

We compute the product TQ ,

$$(TQ)_{Ra, Rx} = \sum_{cR} \zeta(cR, (Ra)^\perp) \mu((Rx)^\perp, cR).$$

By the definition of ζ and the fact that $\mu((Rx)^\perp, cR) = 0$ unless $(Rx)^\perp \leq cR$, the expression above simplifies to

$$(TQ)_{Ra, Rx} = \sum_{(Rx)^\perp \leq cR \leq (Ra)^\perp} \mu((Rx)^\perp, cR),$$

which is 1 for $(Rx)^\perp = (Ra)^\perp$ and 0 otherwise by the definition of the Möbius function.

The matrix T is upper triangular with 1s on the main diagonal. Thus $\det T$ and hence $\det Q$ equal ± 1 . (The ± 1 allows for different orders of rows and columns.) \square

Example 4.2. Let $R := \mathbb{F}_q[x, y]/\langle x^2, y^2 \rangle$, which is a commutative local Frobenius ring. (When $q = 2^k$, R is isomorphic to the group algebra over \mathbb{F}_{2^k} of the Klein 4-group.) Here, $(Rxy)^\perp = xR + yR$ is not principal and thus the above proof does not apply; in fact, the matrix Q turns out to be singular in this case.

On the other hand, the Frobenius ring R is not a counter-example to the main result below. In fact, $\det(W_0) = \pm q w(xy)^{q+3}$ satisfies the formula in (4.1) below (up to a nonzero multiplicative constant), so that the main result still holds over R .

We are now ready to state the main theorems. The proof of the next result is contained in the final section.

Theorem 4.3. *If R is a finite principal ideal ring, then the matrix WQ is lower triangular. The diagonal entry at position (Ra, Ra) is $\sum_{dR \leq aR} w(d) \mu(0, dR)$.*

We conclude that the determinant of WQ and hence that of W is given by

$$\det(W) = \pm \det(WQ) = \pm \prod_{aR} \sum_{dR \leq aR} w(d) \mu(0, dR).$$

Applying Proposition 3.4 we find the determinant of W_0 to be

$$(4.1) \quad \det(W_0) = \pm \prod_{aR \neq 0} \sum_{0 \neq dR \leq aR} w(d) \mu(0, dR),$$

as in $\det(W)$ the term $aR = 0$ provides a factor of $w(0)$ which gets divided away, and in each remaining term the contribution from $dR = 0R$ is $w(0)$ which is set equal to 0.

This yields our main result: a characterization of all bi-invariant weights on a principal ideal ring that satisfy the Extension Property.

Theorem 4.4 (Main Result). *Let R be a finite principal ideal ring and let μ be the Möbius function of the lattice L^* of all right ideals of R . Then a bi-invariant rational weight w on R satisfies the Extension Property if and only if*

$$\sum_{0 \neq dR \leq aR} w(d) \mu(0, dR) \neq 0 \quad \text{for all } aR \neq 0.$$

The condition in Theorem 4.4 needs to be checked only for nonzero right ideals $aR \leq \text{soc}(R_R)$, since we have $\mu(0, dR) = 0$ if $dR \not\leq \text{soc}(R_R)$ (see [12, Proposition 2], for example) and since every right ideal contained in $\text{soc}(R_R)$ is principal. As a consequence, the Extension Property of w depends only on the values of w on the socle of R .

Example 4.5. For a chain ring R , the main result simply says that a bi-invariant weight function w satisfies the Extension Property if and only if it does not vanish on the socle of R (compare with [6] and [21, Theorem 9.4]). For $R = \mathbb{Z}_4$, it states that a bi-invariant weight will satisfy the Extension Property if and only if $w(2) \neq 0$.

Example 4.6. Let $R := \mathbb{Z}_m$. The nonzero ideals in $\text{soc}(\mathbb{Z}_m)$ are of the form $a\mathbb{Z}_m$ with $a \mid m$ and $m/a > 1$ square-free. The Möbius function of such an ideal is $\mu(0, a\mathbb{Z}_m) = \mu(m/a) = (-1)^r$, where $\mu(\cdot)$ denotes the one-variable Möbius function of elementary number theory and r is the number of different prime divisors of m/a . According to Theorem 4.4, an invariant weight w on \mathbb{Z}_m has the Extension Property if and only if

$$\sum_{s \mid \frac{m}{a}} w(sa) \mu\left(\frac{m}{sa}\right) = (-1)^r \sum_{s \mid \frac{m}{a}} w(sa) \mu(s) \neq 0$$

for all (positive) divisors a of m such that m/a is square-free and > 1 . We thus recover the main theorem of [7].

Example 4.7. Let $R := \text{Mat}_n(\mathbb{F}_q)$, $n \geq 2$, the ring of $n \times n$ matrices over the finite field \mathbb{F}_q with q elements, so that $U = \text{GL}_n(\mathbb{F}_q)$. The ring R is a finite principal ideal ring that is not a direct product of chain rings. For each matrix $A \in R$, the left U -orbit UA can be identified with the row space of A , and similarly, the right U -orbits correspond to the column spaces.

Let w be a bi-invariant weight on R . Its value $w(A)$ depends only on the rank of the matrix A , and therefore we can write $w([\text{rank } A]) := w(A)$. Now for $n = 2$, the main result says that w satisfies the Extension Property if and only if $w([1]) \neq 0$ and $qw([2]) \neq (q+1)w([1])$. For $n = 3$, w satisfies the Extension Property if and only if $w([1]) \neq 0$, $qw([2]) \neq (q+1)w([1])$, and $q^3w([3]) + (q^2 + q + 1)w([1]) \neq (q^2 + q + 1)qw([2])$.

It was shown in [21, Theorem 9.5] that the relevant non-vanishing sums are

$$(4.2) \quad \sum_{i=1}^s (-1)^i q^{\binom{i}{2}} \begin{bmatrix} s \\ i \end{bmatrix}_q w([i]),$$

where $\binom{i}{2}$ is the usual binomial coefficient and $\begin{bmatrix} s \\ i \end{bmatrix}_q$ is the q -binomial coefficient (Gaussian polynomial) defined as

$$\begin{bmatrix} k \\ l \end{bmatrix}_q := \frac{(1-q^k)(1-q^{k-1}) \dots (1-q^{k-l+1})}{(1-q^l)(1-q^{l-1}) \dots (1-q)}.$$

The *rank metric* $w([k]) := k$ satisfies these conditions. First we state the Cauchy binomial theorem:

$$\prod_{i=0}^{k-1} (1+xq^i) = \sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix}_q q^{\binom{j}{2}} x^j.$$

Now we write the term in (4.2) for the rank metric, changing the sign and including $i = 0$ trivially in the sum. This can then be seen as the evaluation of a derivative.

$$\sum_{i=0}^s i(-1)^{i-1} q^{\binom{i}{2}} \begin{bmatrix} s \\ i \end{bmatrix}_q = \frac{d}{dx} \sum_{i=0}^s x^i q^{\binom{i}{2}} \begin{bmatrix} s \\ i \end{bmatrix}_q \Big|_{x=-1}.$$

Applying the Cauchy binomial theorem and evaluating the derivative yields:

$$\frac{d}{dx} \prod_{i=0}^{s-1} (1+xq^i) \Big|_{x=-1} = \left(\prod_{i=0}^{s-1} (1-q^i) \right) \left(\sum_{i=0}^{s-1} \frac{q^i}{1-q^i} \right).$$

Both expressions on the right are nonzero provided q is not ± 1 , independent of s . Hence the rank metric satisfies the Extension Property for all q and n .

Example 4.8. More generally, let $R = \text{Mat}_n(S)$ be a matrix ring over a finite chain ring S . Then $\text{soc}(R_R) = \text{soc}({}_R R) = \text{Mat}_{n \times n}(\text{soc } S) \cong \text{Mat}_{n \times n}(\mathbb{F}_q)$ as a (bi-)module over the residue class field $S/\text{rad } S \cong \mathbb{F}_q$. Hence the previous example applies and characterizes all bi-invariant weights $w: R \rightarrow \mathbb{Q}$ having the Extension Property.

Example 4.9. Any finite semisimple ring is a direct product of matrix rings over finite fields and therefore a principal ideal ring. Hence, the main result also applies to this case.

5. A PROOF OF THEOREM 4.3

We perform the matrix multiplication and see that the entry of WQ in position (Ra, Rb) is given by the expression

$$(WQ)_{Ra, Rb} = \sum_{cR} W_{Ra, cR} Q_{cR, Rb} = \sum_{cR} w(ac) \mu((Rb)^\perp, cR).$$

According to the definition of the Möbius function, $\mu((Rb)^\perp, cR)$ can be nonzero only when $(Rb)^\perp \leq cR$ (or: when $cR \in [(Rb)^\perp, R]$, using interval notation on the lattice L^* of all right (necessarily principal) ideals of R). With this in mind we write

$$(5.1) \quad (WQ)_{Ra, Rb} = \sum_{cR \in [(Rb)^\perp, R]} w(ac) \mu((Rb)^\perp, cR).$$

Diagonal Entries. The diagonal terms of WQ are given by

$$(WQ)_{Ra, Ra} = \sum_{cR \in [(Ra)^\perp, R]} w(ac) \mu((Ra)^\perp, cR).$$

For an element $a \in R$ consider the left multiplication operator $L_a: R \rightarrow aR$, $t \mapsto at$. The mapping L_a is a (right) R -linear mapping with kernel $(Ra)^\perp$, and the isomorphism theorem yields an induced order isomorphism of intervals

$$\nu_a: [(Ra)^\perp, R] \rightarrow [0, aR], \quad J \mapsto aJ.$$

It follows that if $J_1, J_2 \in [(Ra)^\perp, R]$, then $\mu(J_1, J_2) = \mu(\nu_a(J_1), \nu_a(J_2)) = \mu(aJ_1, aJ_2)$.

The diagonal term simplifies to

$$\begin{aligned} (WQ)_{Ra, Ra} &= \sum_{cR \in [(Ra)^\perp, R]} w(ac) \mu((Ra)^\perp, cR) \\ &= \sum_{acR \in [0, aR]} w(ac) \mu(0, acR) \\ &= \sum_{dR \in [0, aR]} w(d) \mu(0, dR), \end{aligned}$$

where we have applied the above interval isomorphism with $J_1 = (Ra)^\perp$ and $J_2 = cR$, followed by the relabeling $acR = dR$.

Finally, observe that the formula $(WQ)_{Ra, Ra} = \sum_{dR \in [0, aR]} w(d) \mu(0, dR)$ does not depend on the choice of generator a for the left ideal Ra . Indeed, any other generator has the form ua , where u is a unit of R . Left multiplication by u induces an order isomorphism of intervals $\nu_u: [0, aR] \rightarrow [0, uaR]$, so that $\mu(0, dR) =$

$\mu(0, udR)$ for all $dR \in [0, aR]$. Since w is left-invariant, we have $w(ud) = w(d)$, and the right side of the formula is well-defined.

Lower Triangularity. Now let us return to the general form of the matrix WQ given in (5.1). We would like to prove that WQ is lower triangular, i.e., that $Rb \not\leq Ra$ will imply that $(WQ)_{Ra, Rb} = 0$. To that end, assume

$$(5.2) \quad Rb \not\leq Ra.$$

As above, the left multiplication operator L_a induces a mapping $\lambda_a : [0, R] \rightarrow [0, aR]$, which in turn induces a partition on $[0, R]$ in a natural way. We first rewrite the general expression for $(WQ)_{Ra, Rb}$ taking into account this partition.

$$(WQ)_{Ra, Rb} = \sum_{dR \in [0, aR]} w(d) \sum_{\substack{cR \in [(Rb)^\perp, R] \\ \lambda_a(cR) = dR}} \mu((Rb)^\perp, cR).$$

Our goal is to examine the inner sum and show that it vanishes for every dR in question. In other words, we will show that

$$\sum_{\substack{cR \in [(Rb)^\perp, R] \\ \lambda_a(cR) = dR}} \mu((Rb)^\perp, cR) = 0, \quad \text{for all } dR \leq aR.$$

We do this by induction on dR in the partially ordered set $[0, aR]$. Accordingly, we assume the existence of some $dR \in [0, aR]$ which is minimal with respect to the property that

$$\sum_{\substack{cR \in [(Rb)^\perp, R] \\ \lambda_a(cR) = dR}} \mu((Rb)^\perp, cR) \neq 0.$$

Consider the right ideal $K := L_a^{-1}(dR) = \sum_{acR \leq dR} cR$. For this ideal we have $(Ra)^\perp \leq K$, and moreover, $cR \leq K$ is equivalent to $acR \leq dR$. For this reason

$$\sum_{\substack{cR \in [(Rb)^\perp, R] \\ acR \leq dR}} \mu((Rb)^\perp, cR) = \sum_{cR \in [(Rb)^\perp, K]} \mu((Rb)^\perp, cR).$$

By properties of μ , the latter expression is nonzero if and only if $K = (Rb)^\perp$. This would however imply $(Rb)^\perp \geq (Ra)^\perp$ (because $(Ra)^\perp \leq K$) and hence $Rb \leq Ra$, contrary to assumption (5.2). Hence, we conclude that

$$\begin{aligned} 0 &= \sum_{\substack{cR \in [(Rb)^\perp, R] \\ acR \leq dR}} \mu((Rb)^\perp, cR) \\ &= \sum_{\substack{cR \in [(Rb)^\perp, R] \\ acR = dR}} \mu((Rb)^\perp, cR) + \sum_{\substack{cR \in [(Rb)^\perp, R] \\ acR < dR}} \mu((Rb)^\perp, cR). \end{aligned}$$

In this equation the minimality property of dR implies that the last term vanishes. This finally forces

$$\sum_{\substack{cR \in [(Rb)^\perp, R] \\ acR = dR}} \mu((Rb)^\perp, cR) = 0,$$

contradicting the minimality property of dR . Lower triangularity follows and this finishes the proof of Theorem 4.3. \square

Note that this proof heavily relies on the hypothesis that R is a finite principal ideal ring. For a general finite Frobenius ring the architecture of a proof will need to be vastly restructured. Nonetheless, we conjecture that the main result, as stated, holds over any finite Frobenius ring.

REFERENCES

1. M. Aigner, *Combinatorial theory*, Springer, Berlin, Heidelberg, New York, 1997.
2. K. P. Bogart, D. Goldberg, and J. Gordon, *An elementary proof of the MacWilliams theorem on equivalence of codes*, Information and Control **37** (1978), 19–22.
3. I. Constantinescu and W. Heise, *A metric for codes over residue class rings of integers*, Problems Inform. Transmission **33** (1997), no. 3, 208–213.
4. I. Constantinescu, W. Heise, and T. Honold, *Monomial extensions of isometries between codes over \mathbb{Z}_m* , Proc. 5th International Workshop on Algebraic and Combinatorial Coding Theory (ACCT '96), Unicorn Shumen, 1996, pp. 98–104.
5. M. Greferath, *Orthogonality matrices for modules over finite Frobenius rings and MacWilliams' equivalence theorem*, Finite Fields Appl. **8** (2002), no. 3, 323–331.
6. M. Greferath and T. Honold, *On weights allowing for MacWilliams equivalence theorem*, Proc. 4th International Workshop on Optimal Codes and Related Topics, Pamporovo, Bulgaria, 2005, pp. 182–192.
7. ———, *Monomial extensions of isometries of linear codes II: invariant weight functions on \mathbb{Z}_m* , Proc. 10th International Workshop on Algebraic and Combinatorial Coding Theory (ACCT-10), Zvenigorod, Russia, 2006, pp. 106–111.
8. M. Greferath, C. Mc Fadden, and J. Zumbärgel, *Characteristics of invariant weights related to code equivalence over rings*, Des. Codes Cryptogr. **66** (2013), no. 1-3, 145–156.
9. M. Greferath, A. Nechaev, and R. Wisbauer, *Finite quasi-Frobenius modules and linear codes*, J. Algebra Appl. **3** (2004), no. 3, 247–272.
10. M. Greferath and S. E. Schmidt, *Finite-ring combinatorics and MacWilliams' equivalence theorem*, J. Combin. Theory Ser. A **92** (2000), no. 1, 17–28.
11. T. Honold, *Characterization of finite Frobenius rings*, Arch. Math. (Basel) **76** (2001), no. 6, 406–415.
12. T. Honold and A. A. Nechaev, *Weighted modules and representations of codes*, Problems Inform. Transmission **35** (1999), no. 3, 205–223.
13. F. J. MacWilliams, *A theorem on the distribution of weights in a systematic code*, Bell System Tech. J. **42** (1963), 79–94.
14. A. A. Nechaev, *Finite principal ideal rings*, Mathematics of the USSR-Sbornik **20** (1973), no. 3, 364–382.
15. G.-C. Rota, *On the foundations of combinatorial theory. I. Theory of Möbius functions*, Z. Wahrscheinlichkeitstheorie und Verw. Gebiete **2** (1964), 340–368.
16. R. P. Stanley, *Enumerative combinatorics. Vol. 1*, Cambridge University Press, Cambridge, 1997, With a foreword by Gian-Carlo Rota, Corrected reprint of the 1986 original.
17. J. A. Wood, *Extension theorems for linear codes over finite rings*, Applied algebra, algebraic algorithms and error-correcting codes (Toulouse, 1997), Lecture Notes in Comput. Sci., vol. 1255, Springer, Berlin, 1997, pp. 329–340.
18. ———, *Duality for modules over finite rings and applications to coding theory*, Amer. J. Math. **121** (1999), no. 3, 555–575.
19. ———, *Weight functions and the extension theorem for linear codes over finite rings*, Finite fields: theory, applications, and algorithms (Waterloo, ON, 1997), Contemp. Math., vol. 225, Amer. Math. Soc., Providence, RI, 1999, pp. 231–243.
20. ———, *Code equivalence characterizes finite Frobenius rings*, Proc. Amer. Math. Soc. **136** (2008), no. 2, 699–706 (electronic).
21. ———, *Foundations of linear codes defined over finite modules: the extension theorem and the MacWilliams identities*, Codes over rings, Ser. Coding Theory Cryptol., vol. 6, World Sci. Publ., Hackensack, NJ, 2009, pp. 124–190.

MG, CM, JZ: CLAUDE SHANNON INSTITUTE, SCHOOL OF MATHEMATICAL SCIENCES, UNIVERSITY COLLEGE DUBLIN, BELFIELD, DUBLIN 4, IRELAND

E-mail address: marcus.greferath@ucd.ie, cathy.mcfadden@ucd.ie,
jens.zumbragel@ucd.ie

TH:INSTITUTE OF INFORMATION AND COMMUNICATION ENGINEERING, DEPARTMENT OF INFORMATION SCIENCE AND ELECTRONICS ENGINEERING, ZHEJIANG UNIVERSITY, ZHEDA ROAD, HANGZHOU, 310027 CHINA

E-mail address: honold@zju.edu.cn

JAW:DEPARTMENT OF MATHEMATICS, WESTERN MICHIGAN UNIVERSITY, 1903 W MICHIGAN AVE, KALAMAZOO MI 49008-5248, USA

E-mail address: jay.wood@wmich.edu