ARITHMETIC PROGRESSIONS OF CONSTANT p-ADIC WEIGHT

JAY A. WOOD

Abstract. We determine the maximum length of arithmetic progressions of constant p-adic weight. A characterization of the maximum length progressions is given.

Fix a positive integer $p > 1$, not necessarily a prime. For any positive integer d, define $t = t_p(d)$ to be the smallest nonnegative integer such that $d \leq p^t$. The base p expansion of d has the form

$$d = d_0 + d_1 \cdot p + d_2 \cdot p^2 + \cdots + d_t \cdot p^t, \tag{1}$$

where each $d_i \in \{0, 1, 2, \ldots, p-1\}$. The *p-adic weight* $s_p(d)$ of d equals the rational sum of the d_i:

$$s_p(d) = \sum_{i=0}^{t} d_i.$$

The problem addressed here is the length of arithmetic progressions $d, 2d, \ldots, (k-1)d$, where every term in the progression has the same p-adic weight. For example, if $p = 10$, $d = 9$, then $9, 18, 27, \ldots, 81, 90$ is a progression of length 10 and 10-adic weight 9. Since $s_{10}(99) = 18$, the progression cannot be extended further with constant weight.

The situation when $d = 1$ is uninteresting. If $p = 2$, the progression $1, 2$ has length 2 and binary weight 1. If $p > 2$, then 1 is a progression of length 1 and p-adic weight 1. None of these progressions extend further with constant weight.

In all that follows, we assume that $d > 1$.

Lemma 1. *Suppose $1 < d \leq p^t$. Then $s_p((p^t + 1) \cdot d) = 2s_p(d)$.*

Proof. Since $d > 1$, $t \geq 1$. If $d = p^t$, then $(p^t + 1) \cdot d = p^{2t} + p^t$, and the result is clear. If $d < p^t$, then $d_t = 0$ in (1). Thus

$$(p^t + 1) \cdot d = d_0 + \cdots + d_{t-1} \cdot p^{t-1} + d_0 \cdot p^t + \cdots + d_{t-1} \cdot p^{2t-1},$$

and the result follows. □

Date: January 3, 2001.

1991 *Mathematics Subject Classification.* Primary. 11B25.

Given a positive integer d, let $k = k_p(d)$ be the smallest positive integer with the properties:

1. $s_p(m \cdot d) = s_p(d)$, $1 \le m < k$,
2. $s_p(k \cdot d) \neq s_p(d)$.

The integer k marks the first time the arithmetic progression $d, 2d, \ldots$ fails to have constant p-adic weight. Since $s_2(2 \cdot d) = s_2(d)$, we see that $k_2(d) \ge 3$. For $p > 2$, all we can say is that $k_p(d) \ge 2$.

By Lemma 1, $k_p(d) \le p^{t_p(d)}+1$, provided $d > 1$. (In fact, the inequality holds for $d = 1$ if $p > 2$. Only $k_2(1) = 3$ violates the inequality.) Our main result below is a characterization of when equality holds in this inequality.

Lemma 2. *Let m be a positive integer of the form $m = p^u \cdot q$, where p does not divide q. Assume $m < p^T$. Let*

$$m = m_u \cdot p^u + m_{u+1} \cdot p^{u+1} + \cdots + m_{T-1} \cdot p^{T-1}$$

be the base p expansion of m; $m_u > 0$. Then $p^T - m$ has base p expansion

$$p^T - m = (\widehat{m}_u + 1) \cdot p^u + \widehat{m}_{u+1} \cdot p^{u+1} + \cdots + \widehat{m}_{T-1} \cdot p^{T-1},$$

where $\widehat{m}_i = p - 1 - m_i$.

Proof. Adding the base p expansions above, we obtain

$$\begin{aligned}(m_u + \widehat{m}_u + 1) \cdot p^u &+ (m_{u+1} + \widehat{m}_{u+1}) \cdot p^{u+1} \\ &+ \cdots + (m_{T-1} + \widehat{m}_{T-1}) \cdot p^{T-1} \\ &= p^{u+1} + (p-1)p^{u+1} + \cdots + (p-1)p^{T-1} = p^T,\end{aligned}$$

as claimed. □

Theorem 3. *Let m be any positive integer, $m < p^t$. Then*

$$s_p((p^t - 1) \cdot m) = t(p-1).$$

Proof. Write $m = p^u \cdot q$, where p does not divide q. Then m has base p expansion

$$m = m_u \cdot p^u + m_{u+1} \cdot p^{u+1} + \cdots + m_{t-1} \cdot p^{t-1},$$

with $m_u > 0$. Using $T = t + u$ in Lemma 2, we obtain

$$\begin{aligned}(p^t - 1) \cdot m =& m_{t-1} \cdot p^{2t-1} + \cdots + m_{u+1} \cdot p^{t+u+1} + m_u \cdot p^{t+u} \\ &- (m_{t-1} \cdot p^{t-1} + \cdots + m_u \cdot p^u) \\ =& m_{t-1} \cdot p^{2t-1} + \cdots + m_{u+1} \cdot p^{t+u+1} + (m_u - 1) \cdot p^{t+u} \\ &+ (p-1) \cdot p^{t+u-1} + \cdots + (p-1) \cdot p^t \\ &+ \widehat{m}_{t-1} \cdot p^{t-1} + \cdots + \widehat{m}_{u+1} \cdot p^{u+1} + (\widehat{m}_u + 1) \cdot p^u.\end{aligned}$$

Since $m_i + \widehat{m}_i = p - 1$, we see that $s_p((p^t - 1) \cdot m) = t(p - 1)$, as claimed. □

It is easy to verify that $k_2(1) = k_2(2) = 3$. These special cases account for the restrictive hypotheses in the following.

Corollary 4. *For any positive integer $d > 1$, $k_p(d) \leq p^{t_p(d)} + 1$. For $d > 2$, equality holds, i.e., $k_p(d) = p^{t_p(d)} + 1$, if and only if $d = p^{t_p(d)} - 1$.*

Proof. The first part was established by Lemma 1. For the second part, let $d = p^t - 1$. It is clear that $s_p(p^t \cdot d) = s_p(d) = t(p - 1)$. Then Theorem 3 implies that $k_p(d) = p^t + 1$.

Conversely, suppose $d \neq p^t - 1$. If $d = p^t$, $p > 2$, then $s_p(2 \cdot p^t) = 2 \neq s_p(p^t) = 1$. Thus $k_p(p^t) = 2$. If $p = 2$, then $k_2(2^t) = 3$. (This explains why $d = 2$ is omitted as a special case.) For general p, if $d < p^t - 1$, then it is clear that $s_p(d) < t(p-1)$. By Theorem 3, $s_p((p^t - 1) \cdot d) = t(p-1)$. Thus $k_p(d) \leq p^t - 1$. □

Department of Mathematics and Statistics, Western Michigan University, 1903 W Michigan Ave, Kalamazoo MI 49008-5248

E-mail address: jay.wood@wmich.edu

URL: http://unix.cc.wmich.edu/jwood/