RELATIVE ONE-WEIGHT LINEAR CODES

JAY A. WOOD

In memory of Professor F. E. P. Hirzebruch, 17 October 1927 – 27 May 2012

Abstract. Relative one-weight linear codes were introduced by Liu and Chen over finite fields. These codes can be defined just as simply for egalitarian and homogeneous weights over Frobenius bimodule alphabets. A key lemma helps describe the structure of relative one-weight codes, and certain known types of two-weight linear codes can then be constructed easily. The key lemma also provides another approach to the MacWilliams extension theorem.

1. Introduction

This paper was inspired by [17] and [18]. In those papers, the authors were concerned primarily with understanding homomorphisms between linear codes that preserve the relative generalized Hamming weights of the codes. (They turn out to be monomial transformations.) To illustrate some of their arguments, the authors defined and then used codes they called "relative constant weight codes." The authors did not describe the structure of those codes, and that served as the catalyst for this paper (see Theorem 18).

The authors of [17] and [18] worked over finite fields. In working through the structure of what we here refer to as "relative one-weight linear codes," we noticed that the arguments immediately generalize to the context of homogeneous weights over finite Frobenius rings or even over Frobenius bimodules. The key lemma (Lemma 15) is very simple and may (we hope) prove to be useful more generally. In many respects, this paper is an advertisement for the key lemma and for the functional point of view espoused by Assmus and Mattson [1].

The functional point of view is, in essence, a coordinate-free version of generator matrices. A generator matrix G of size $k \times n$ takes k information bits $x = (x_1, \ldots, x_k)$ and encodes them as a length n codeword $c = xG$ by matrix multiplication. An individual column of

Date: November 5, 2012. To appear in Designs, Codes and Cryptography. DOI 10.1007/s10623-012-9769-0.

Partially supported by a sabbatical leave from Western Michigan University.

G thus defines a linear functional (a "coordinate functional") on the k-dimensional information space. Up to permutation equivalence, what matters is the number of times a particular k-vector appears as a column of G. From the functional viewpoint, this is the number of times a particular linear functional appears as a coordinate functional.

The key lemma (Lemma 15) exploits the fact that the set of all linear functionals itself has a linear structure. If the coordinate functionals are exactly all the elements of a linear subspace E, then the weights of codewords turn out to be easy to understand (using a homogeneous weight). In particular, all the codewords annihilated by E have weight zero, while all the codewords not annihilated by E have the same nonzero weight (a particular example of a relative one-weight linear code). The key lemma deals with a coset version of this idea. Concatenation of generator matrices then allows one to superimpose weights in order to build more general examples. In this way, some known linear two-weight codes can also be described easily.

Here is a brief summary of the structure of the paper. Background material on characters, Frobenius bimodules, homogeneous and egalitarian weights, and the functional point of view is provided in Section 2. The key lemma is proved in Section 3. The structure of relative one-weight linear codes appears in Section 4, together with examples of known linear two-weight codes. Finally, in Section 5, we show how the key lemma leads to another proof of the MacWilliams extension theorem for homogeneous weights over Frobenius bimodules. The treatment here is in the spirit of, and in some sense dual to, that in [7].

Remark 1. This paper is dedicated to the memory of Professor Friedrich Ernst Peter Hirzebruch, who died while this paper was being finalized. Professor Hirzebruch was the editor of an early manuscript of mine on spinor groups. The referee's report suggested that I refer to the coding theory literature on self-dual codes. This served as my introduction to coding theory!

2. Background

This section gathers together various results that will be needed in subsequent sections.

Notational Convention. If M_1, M_2 are left modules over a ring R and $\phi : M_1 \to M_2$ is a homomorphism of left R-modules, we will write the value of ϕ at $x \in M_1$ as $x\phi$.

2.1. Characters.

Let G be a finite abelian group, written additively. A *character* π of G is a group homomorphism $\pi : G \to \mathbb{C}^\times$ from G to the multiplicative group of nonzero complex numbers. The set of all characters of G is denoted $\widehat{G} := \mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{C}^{\times})$; $\widehat{G}$ is itself a finite abelian group under pointwise multiplication of functions, with $|\widehat{G}| = |G|$.

Given a subgroup $H \subset G$, the *annihilator* of H is $(\widehat{G} : H) := \{\pi \in \widehat{G} : \pi(H) = 1\}$. It follows that $(\widehat{G} : H) \cong (G/H)\widehat{}$, so that $|(\widehat{G} : H)| = |G|/|H|$. See [22, 24, 26] for details.

Lemma 2. *Suppose H is a subgroup of G, a finite abelian group. Let $\pi \in \widehat{G}$. Then*

$$\sum_{h \in H} \pi(h) = \begin{cases} |H|, & \pi \in (\widehat{G} : H), \\ 0, & \pi \notin (\widehat{G} : H). \end{cases}$$

Proof. The case where $\pi \in (\widehat{G} : H)$ is clear. If $\pi \notin (\widehat{G} : H)$, there exists $h_0 \in H$ with $\pi(h_0) \neq 1$. Reindexing by $h_0 + h \in H$ gives

$$\sum_{h \in H} \pi(h) = \sum_{h \in H} \pi(h_0 + h) = \sum_{h \in H} \pi(h_0)\pi(h) = \pi(h_0) \sum_{h \in H} \pi(h).$$

Because $\pi(h_0) \neq 1$, the sum must vanish. □ □

2.2. **Frobenius Bimodules.** If the abelian group G is the additive group of a finite ring R with 1, then the character group $\widehat{R}$ inherits the structure of an (R, R)-bimodule. Such a bimodule $\widehat{R}$ is an example of a *Frobenius bimodule* [8, Proposition 2.7], i.e., an (R, R)-bimodule A such that $A \cong \widehat{R}$ both as left R-modules and as right R-modules.

Finite Frobenius rings are characterized by the property that $\widehat{R}$ is free as a left and as a right R-module ([11, 24]). Consequently, every finite Frobenius ring R is a Frobenius bimodule over itself.

A Frobenius bimodule A over R need not be isomorphic to $\widehat{R}$ as (R, R)-bimodules. An example is provided by a Frobenius ring that is not a *symmetric ring* (by definition, a ring R such that $\widehat{R} \cong R$ as bimodules). See the discussion in [24, Remark 3.11].

Example 3 ([24, Example 1.4 (iii)])**.** Let R be the collection of all matrices over $\mathbb{F}_q$ of the form:

$$r = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_1 & r_2 & 0 \\ 0 & 0 & r_4 & 0 \\ r_3 & 0 & 0 & r_4 \end{pmatrix}.$$

Then R is a Frobenius ring that is not symmetric.

Let A be a Frobenius bimodule over R. By dualizing, it follows that $\widehat{A} \cong R$ both as left and as right R-modules. Corresponding to $1 \in R$

under one of these isomorphisms is a character $\chi \in \widehat{A}$. This character χ has special properties, as the next proposition explains. Because of these properties, χ is called a *generating character* of A.

Proposition 4 ([26, Section 5.2]). *Let A be a finite Frobenius bimodule over a finite ring R. Then A admits a character $\chi \in \widehat{A}$ satisfying:*

- *the kernel* $\ker \chi$ *contains no nonzero left or right R-submodules of A;*
- *the character χ generates $\widehat{A}$ as a left and as a right R-module.*

Remark 5. More generally, suppose M is a finite left R-module, and suppose M admits a character $\rho \in \widehat{M}$ such that $\ker \rho$ contains no nonzero left R-submodule of M. It is shown in [27, Section 2.3] that ρ induces an R-linear embedding $M \hookrightarrow \widehat{R}$. The dual map $R \to \widehat{M}$ is surjective, and ρ generates M as a left R-module. Conversely, if $M \subset \widehat{R}$, then the restriction to M of the generating character χ of $\widehat{R}$ is a left generating character for M.

2.3. **Egalitarian and Homogeneous Weights.** Constantinescu and Heise introduced homogeneous weights over integer residue rings in [6], and these weights have been studied and generalized by a number of authors. Of particular importance for us are the papers [15] and [8], where homogeneous weights on modules are defined, and [12], where the homogeneous weight on a finite Frobenius ring is described in terms of a generating character on the ring. We will combine these ideas below.

Because some of the terminology involving homogeneous weights is inconsistent, we will first describe certain properties that are desirable in a weight. Let $\mathcal{U} = \mathcal{U}(R)$ denote the group of units of a ring R. While we will concentrate on right R-submodules, there are parallel results for left R-submodules.

Definition 6. Let A be a finite (R, R)-bimodule. A *weight* on A is a function $w : A \to \mathbb{C}$ satisfying $w(0) = 0$. A weight w has two *symmetry groups* (left and right):

$$\operatorname{Sym}_l(w) \coloneqq \{u \in \mathcal{U} : w(ua) = w(a), \text{ all } a \in A\},$$
$$\operatorname{Sym}_r(w) \coloneqq \{u \in \mathcal{U} : w(au) = w(a), \text{ all } a \in A\}.$$

A weight may also satisfy one or more of the following properties:

(1) $\operatorname{Sym}_r(w) = \mathcal{U}$.
(2) There exists a constant γ such that

$$\sum_{x \in aR} w(x) = \gamma |aR|,$$

for any nonzero cyclic right submodule $aR \subset A$.

(3) There exists a constant γ such that

$$\sum_{x \in I} w(x) = \gamma |I|,$$

for any nonzero right submodule $I \subset A$.

Remark 7. Greferath and Schmidt [9] call a weight "homogeneous" if it satisfies (1) and (2), while Honold and Nechaev [15] call a weight "homogeneous" if it satisfies (1) and (3). These two usages agree when A is a finite Frobenius bimodule, in particular, when $A = R$ is a finite Frobenius ring. Following Honold and Nechaev, we will call a weight (right) *egalitarian* if it satisfies (3). The constant γ is called the *average weight* of w.

Remark 8. A unique weight satisfying (1) and (2) with a fixed value of γ exists on any finite ring [9] and on any finite right R-module [8]. This weight is expressed in terms of the Möbius function of the poset of all principal right ideals of the ring (resp., principal right R-submodules of the module). It follows from this formula that the values of this weight are rational multiples of γ. Moreover, this weight satisfies (3) if and only if the ring R is Frobenius (resp., if and only if the socle of the module is cyclic). In particular, a finite Frobenius bimodule satisfies this condition [26, Proposition 5.3].

The next proposition generalizes a result of Heise and Honold over finite Frobenius rings [10, Theorem 2].

Proposition 9. *Suppose A is a finite Frobenius bimodule over R, equipped with a generating character χ. For any subgroup $U \subset \mathcal{U}$ and constant γ, define a weight w_U on A by*

$$w_U(a) \coloneqq \gamma \left(1 - \frac{1}{|U|} \sum_{u \in U} \chi(au) \right), \quad a \in A. \tag{2.1}$$

Then w_U is an egalitarian weight with $U \subset \operatorname{Sym}_r(w_U)$. Moreover, if $I \subset A$ is a nonzero right submodule and $a_0 \in A$, then

$$\sum_{a \in I} w_U(a_0 + a) = \gamma |I|. \tag{2.2}$$

That is, the egalitarian property applies to all cosets of nonzero right submodules of A.

If $U = \mathcal{U}$, then $w_{\mathcal{U}}$ is a homogeneous weight.

In order to verify the egalitarian property, we will need a lemma.

Lemma 10. *Let A be a finite (R,R)-bimodule, π a character of A, and I a right submodule of A. Then for any unit $u \in \mathcal{U}$,*

$$\sum_{a\in I} \pi(au) = \begin{cases} |I|, & \pi \in (\widehat{A} : I), \\ 0 & \pi \notin (\widehat{A} : I). \end{cases}$$

Proof. Because u is invertible, the map $I \to I$, given by $a \mapsto au$, is a bijection. Thus, by reindexing, $\sum_{a\in I} \pi(au) = \sum_{a\in I} \pi(a)$. The result now follows from Lemma 2. □

Proof of Proposition 9. It is easy to see that $w_U(0) = 0$ and, by reindexing, that $U \subset \operatorname{Sym}_r(w_U)$. Next, we verify (2.2). Using (2.1),

$$\begin{aligned}\sum_{a\in I} w_U(a_0 + a) &= \sum_{a\in I} \gamma \left(1 - \frac{1}{|U|}\sum_{u\in U} \chi((a_0 + a)u)\right) \\ &= \gamma|I| - \frac{\gamma}{|U|}\sum_{u\in U} \chi(a_0 u)\left(\sum_{a\in I} \chi(au)\right).\end{aligned}$$

By Lemma 10, the value of the sum $\sum_{a\in I} \chi(au)$ depends upon whether χ belongs to $(\widehat{A} : I)$. We claim that $\chi \notin (\widehat{A} : I)$. Indeed, if $\chi \in (\widehat{A} : I)$, then $\chi(I) = 1$, i.e., $I \subset \ker\chi$. But χ is a generating character of A, which implies that I must be zero, by Proposition 4. Since I is assumed to be nonzero, it follows that $\chi \notin (\widehat{A} : I)$. Then $\sum_{a\in I} \chi(au) = 0$, and the last expression above simplifies to $\gamma|I|$, as desired. □ □

2.4. **Presenting Linear Codes via Functionals.** It will be convenient to present linear codes by specifying their coordinate functionals. This approach is due to Assmus and Mattson [1] and generalizes the role of a generator matrix.

Let A be a finite (R,R)-bimodule, where R is a finite ring with 1. The bimodule A will serve as the *alphabet* for linear codes. A left *R-linear code* over A of length n is a left R-submodule $C \subset A^n$. (And one can define right R-linear codes over A as well.)

One way to present a linear code is as the image $C = M\Lambda$ of an R-linear homomorphism $\Lambda : M \to A^n$ from some finite left R-module M to A^n. The components of $\Lambda = (\lambda_1, \ldots, \lambda_n)$ are linear functionals $\lambda_i \in \operatorname{Hom}_R(M, A)$, called the *coordinate functionals* of the linear code C. Up to permutation equivalence, all that matters is the number of times, the *multiplicity* $\eta(\lambda)$, that a given element λ of $\operatorname{Hom}_R(M, A)$ appears as a coordinate functional. That is, up to permutation equivalence, a linear code is presented by a *multiplicity function* $\eta : \operatorname{Hom}_R(M, A) \to \mathbb{N}$.

Remark 11. Given a multiplicity function $\eta : \mathrm{Hom}_R(M, A) \to \mathbb{N}$, one produces a generator matrix as follows. Fix a list $m_1, \ldots, m_k$ of generators for the finite R-module M. Let

$$n \coloneqq \sum_{\lambda \in \mathrm{Hom}_R(M,A)} \eta(\lambda).$$

Define a $k \times n$ matrix G whose columns are the transposes of the vectors $(m_1\lambda, \ldots, m_k\lambda)$, repeated with multiplicity $\eta(\lambda)$, as λ varies over $\mathrm{Hom}_R(M, A)$; the columns can be in any order, so that G is well-defined only up to permutation equivalence.

Conversely, a generator matrix G defines a multiplicity function η. Refer to the underlying module of a linear code C as M. Refer to the row vectors of G as $m_1, \ldots, m_k$; these row vectors generate M. The jth column of G defines a linear functional $\lambda_j : M \to A$ via $(\sum_i r_i m_i)\lambda_j \coloneqq \sum_i r_i g_{ij}$, where the (i, j)-entry of G is g_{ij}. Now define η by $\eta(\lambda) \coloneqq |\{j : \lambda = \lambda_j\}|$; i.e., $\eta(\lambda)$ counts the number of times that λ appears as (a coordinate functional defined by) a column of G.

Remark 12. If the alphabet A is equipped with a weight w, then it may be important to understand R-linear homomorphisms between left R-linear codes that preserve the weight w. This in turn leads one to study left R-linear codes up to $\mathrm{Sym}_r(w)$-monomial equivalence, which makes use of the right action of $\mathrm{Sym}_r(w) \subset \mathcal{U}$ on the right R-module $\mathrm{Hom}_R(M, A)$. While we will not need to pursue this line of thought here, we will return to this idea in Section 5. The reader is referred to [26, Section 7] for more information.

Remark 13. Functions equivalent to the multiplicity function η have occurred in the literature under a number of different names. Some examples: the “modular representations” of Peterson [20], “projective systems” of Tsfasman and Vlăduţ [23], the multiset approach of Honold and Landjev [14, Theorem 5.1], and the “value function” of Chen and Kløve [5] and of Liu and Chen [17].

Suppose the alphabet A has a weight $w : A \to \mathbb{C}$, so that $w(0) = 0$. Given a linear code C presented via a multiplicity function $\eta : \mathrm{Hom}_R(M, A) \to \mathbb{N}$, the weights $w_\eta(x)$ of individual codewords $x \in M$ are given by the formula

$$w_\eta(x) \coloneqq \sum_{\lambda \in \mathrm{Hom}_R(M,A)} w(x\lambda)\, \eta(\lambda). \tag{2.3}$$

We invite the reader to show that this formula is consistent with the traditional definition.

3. A Key Lemma

Continue to assume that A is a finite (R,R)-bimodule, where R is a finite ring with 1, and let M be a finite left R-module. The Hom-group $\mathrm{Hom}_R(M,A)$ is a right R-module (with $m(\lambda r) = (m\lambda)r$, for $m \in M$, $\lambda \in \mathrm{Hom}_R(M,A)$, $r \in R$). For a left submodule $N \subset M$, define the *annihilator* $N^\circ \subset \mathrm{Hom}_R(M,A)$ by $N^\circ := \{\lambda \in \mathrm{Hom}_R(M,A) : N\lambda = 0\}$; N° is a right submodule of $\mathrm{Hom}_R(M,A)$. Dually, for a right submodule $E \subset \mathrm{Hom}_R(M,A)$, define the *annihilator* $E^\circ \subset M$ by $E^\circ := \{m \in M : mE = 0\}$, i.e., $m\lambda = 0$ for all $\lambda \in E$; E° is a left submodule of M.

Remark 14. While $N \subset N^{\circ\circ}$ and $E \subset E^{\circ\circ}$, equality need not hold without additional hypotheses on A. Morita duality theory tells us that equality holds when the bimodule A represents a duality functor [16, §19]. Two examples involving finite rings where this is true are:

- $A = R$ is a finite quasi-Frobenius ring;
- A is a finite Frobenius bimodule over R.

These two examples are the same when R is a finite Frobenius ring. See [24] for more details.

Here is a key lemma.

Lemma 15. *Suppose A is a finite Frobenius bimodule over a finite ring R, with a generating character χ and a right egalitarian weight w with average weight γ, as in Proposition 9. Let E be a right submodule of* $\mathrm{Hom}_R(M,A)$*, and let λ_0 be any element of* $\mathrm{Hom}_R(M,A)$*. Then, for $x \in M$,*

$$\sum_{\lambda \in \lambda_0 + E} w(x\lambda) = \begin{cases} w(x\lambda_0)|E|, & x \in E^\circ, \\ \gamma|E|, & x \notin E^\circ. \end{cases}$$

Suppose, in particular, that $\lambda_0 = 0$. Then, for $x \in M$,

$$\sum_{\lambda \in E} w(x\lambda) = \begin{cases} 0, & x \in E^\circ, \\ \gamma|E|, & x \notin E^\circ. \end{cases}$$

Proof. Fix $x \in M$. Evaluation at x defines $\check{x} : \mathrm{Hom}_R(M,A) \to A$ by $\check{x}(\lambda) := x\lambda$ for $\lambda \in \mathrm{Hom}_R(M,A)$; $\check{x}$ is a right R-linear homomorphism. The image $\check{x}(E)$ of E under $\check{x}$ is a right submodule I of A, and $I = 0$ if and only if $x \in E^\circ$. Observe that $|I| = |E|/|E \cap \ker \check{x}|$.

If $x \in E^\circ$, then

$$\sum_{\lambda \in \lambda_0 + E} w(x\lambda) = \sum_{\nu \in E} w(x\lambda_0 + x\nu) = \sum_{\nu \in E} w(x\lambda_0) = w(x\lambda_0)|E|.$$

If $x \notin E^\circ$, then $I = \check{x}(E)$ is a nonzero right submodule of A. As ν varies over E, $x\lambda_0 + x\nu$ varies over the nonzero coset $x\lambda_0 + I$ of I in A. Every element of $x\lambda_0 + I$ is hit $|E \cap \ker \check{x}|$ times. Thus, by Proposition 9,

$$\begin{aligned}\sum_{\lambda \in \lambda_0 + E} w(x\lambda) &= \sum_{\nu \in E} w(x\lambda_0 + x\nu) \\ &= |E \cap \ker \check{x}| \sum_{a \in I} w(x\lambda_0 + a) \\ &= |E \cap \ker \check{x}| \gamma |I| = \gamma |E|.\end{aligned}$$

This proves the result. □ □

4. Examples

Throughout this section, A will be a finite Frobenius bimodule over a finite ring R with 1. Assume that A admits a generating character χ and a right egalitarian weight w with average weight γ, as in Proposition 9. In particular, these hypotheses include the situation where $A = R$ is a finite Frobenius ring and w is a homogeneous weight. All linear codes will be left R-linear codes over A.

4.1. **Relative One-Weight Linear Codes.**

Definition 16. A linear code $C \subset A^n$ is a *one-weight code* if there exists $\alpha \in \mathbb{C}$ such that $w(x) = \alpha$ for all nonzero $x \in C$. A linear code is a *two-weight code* if there exist $\alpha, \beta \in \mathbb{C}$, $\alpha \neq \beta$, such that $w(x) \in \{\alpha, \beta\}$ for all nonzero $x \in C$.

Definition 17 (Liu-Chen, [17])**.** Let $C \subset A^n$ be an R-linear code, with linear subcode $C_1 \subset C$. Then C is a *relative one-weight linear code* with respect to C_1 if there exists $\alpha \in \mathbb{C}$ such that $w(x) = \alpha$ for all $x \in C \setminus C_1$, i.e., the elements of the linear code C that are not in the subcode C_1.

If N is a left R-submodule of M, there is a short exact sequence of R-modules

$$0 \to N \to M \to Q \to 0,$$

where $Q = M/N$ is the quotient module. Applying the contravariant functor $\mathrm{Hom}_R(-, A)$, we get a short exact sequence

$$0 \to \mathrm{Hom}_R(Q, A) \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(N, A) \to 0.$$

(For an arbitrary bimodule A, the Hom-sequence would generally be just left exact. But a Frobenius bimodule ${}_RA \cong {}_R\widehat{R}$ is an injective module, so the Hom-sequence is also right exact.) One sees that N° is the kernel of $\mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(N, A)$, so one may re-write the short exact sequence as $0 \to N^\circ \to \mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(N, A) \to 0$. The cosets of N° in $\mathrm{Hom}_R(M, A)$ are thus the fibers of the homomorphism $\mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(N, A)$.

Theorem 18. *Let N be a left R-submodule of M. Let η be a multiplicity function with the property that η is constant on the cosets of N° in $\mathrm{Hom}_R(M, A)$; i.e., η is constant along the fibers of $\mathrm{Hom}_R(M, A) \to \mathrm{Hom}_R(N, A)$. Then η defines a relative one-weight linear code on M with respect to N.*

Proof. Let x be any element of M with $x \notin N$. (Note that $N^{\circ\circ} = N$ by Remark 14.) We will split a sum over $\lambda \in \mathrm{Hom}_R(M, A)$ into sums over the cosets of N°. The hypothesis on η is that $\eta(\lambda_0 + \nu) = \eta(\lambda_0)$ for $\nu \in N^\circ$. We calculate:

$$\begin{aligned}
w_\eta(x) &= \sum_{\lambda \in \mathrm{Hom}_R(M,A)} w(x\lambda)\eta(\lambda) \\
&= \sum_{\text{cosets } \lambda_0} \sum_{\lambda \in \lambda_0 + N^\circ} w(x\lambda)\eta(\lambda) \\
&= \sum_{\text{cosets } \lambda_0} \sum_{\nu \in N^\circ} w(x\lambda_0 + x\nu)\eta(\lambda_0 + \nu) \\
&= \sum_{\text{cosets } \lambda_0} \eta(\lambda_0) \sum_{\nu \in N^\circ} w(x\lambda_0 + x\nu) \\
&= \sum_{\text{cosets } \lambda_0} \eta(\lambda_0)\gamma|N^\circ| = \gamma|N^\circ| \sum_{\text{cosets } \lambda_0} \eta(\lambda_0).
\end{aligned}$$

Lemma 15 was used to simplify the last line. The final formula does not depend on x (as long as $x \notin N^{\circ\circ} = N$). □ □

4.2. **Two-Weight Linear Codes.** In this subsection we show how to produce a number of known examples of linear two-weight codes using the techniques stemming from Lemma 15.

Proposition 19. *Let E be a right submodule of $\mathrm{Hom}_R(M, A)$. The multiplicity function η given by*

$$\eta(\lambda) := \begin{cases} 1, & \lambda \in E, \lambda \neq 0, \\ 0, & \lambda = 0 \text{ or } \lambda \notin E, \end{cases}$$

defines a linear code with weights

$$w_\eta(x) = \begin{cases} 0, & x \in E^\circ, \\ \gamma|E|, & x \notin E^\circ. \end{cases}$$

Corollary 20 ([25, Theorem 7.1])**.** *Let $E = \mathrm{Hom}_R(M, A)$. The multiplicity function η given by*

$$\eta(\lambda) := \begin{cases} 1, & \lambda \neq 0, \\ 0, & \lambda = 0, \end{cases}$$

defines a one-weight code on M with $w_\eta(x) = \gamma|\mathrm{Hom}_R(M, A)|$, $x \neq 0$.

Remark 21. The linear codes in the two preceding results are examples of "modular codes" as defined in [13, Definition 6]

Because (2.3) is linear in η, multiplicity functions satisfy a superposition property. Namely, if $\eta := \eta_1 + \eta_2$, then $w_\eta = w_{\eta_1} + w_{\eta_2}$. This will be illustrated in the next proposition. Adding multiplicity functions is equivalent to concatenating generator matrices.

Proposition 22. *Let N be a submodule of M. For $a, b \in \mathbb{N}$, define a multiplicity function η by*

$$\eta(\lambda) := \begin{cases} 0, & \lambda = 0, \\ a + b, & \lambda \in N^\circ, \lambda \neq 0, \\ a, & \lambda \notin N^\circ. \end{cases}$$

Then η defines a two-weight code with weights

$$w_\eta(x) = \begin{cases} 0, & x = 0, \\ a\gamma|\mathrm{Hom}_R(M, A)|, & x \in N, x \neq 0, \\ a\gamma|\mathrm{Hom}_R(M, A)| + b\gamma|N^\circ|, & x \notin N. \end{cases}$$

Proof. Let $\eta := a\eta_1 + b\eta_2$, where η_1 is the multiplicity function of Corollary 20 and η_2 is the multiplicity function of Proposition 19 with $E := N^\circ$. □ □

Remark 23. If $b := -a$ and $A = R := \mathbb{F}_q$, one has example SU1 of Calderbank-Kantor [3].

Example 24 (Camion, [4, Property 5.68])**.** Camion described a two-weight code over $\mathbb{Z}/4\mathbb{Z}$; his construction can be generalized to any finite chain ring, as follows. (For related work on MacDonald codes, see [14, Section 6.2].)

Let R be a finite chain ring. Every one-sided ideal of R is two-sided. All the ideals are principal, and they form a chain under inclusion. If the unique maximal ideal is generated by an element $\theta \in R$, then all the ideals are:

$$R = (\theta^0) \supset (\theta) \supset (\theta^2) \supset \cdots \supset (\theta^{e-1}) \supset (\theta^e) = (0),$$

for some positive integer e. Because (θ) is maximal, $R/(\theta) \cong \mathbb{F}_q$ for some prime power q. Then $|(\theta)^i| = q^{e-i}$ and $|R| = q^e$.

Define a weight w on R by

$$w(r) := \begin{cases} 0, & r = 0, \\ q, & r \in (\theta^{e-1}), r \neq 0, \\ q-1, & r \notin (\theta^{e-1}). \end{cases}$$

Then w is a homogeneous weight with $\gamma = q - 1$. For details, see [9].

Let $M := R^k$. In $M^\sharp := \mathrm{Hom}_R(M, R)$, set $E_j := M^\sharp \theta^j$. Observe that $|E_j| = q^{k(e-j)}$ and $E_j^\circ = M\theta^{e-j}$. As in Proposition 19, define two multiplicity functions by

$$\eta_0(\lambda) := \begin{cases} 1, & \lambda \in M^\sharp, \lambda \neq 0, \\ 0, & \lambda = 0; \end{cases} \qquad \eta_j(\lambda) := \begin{cases} 1, & \lambda \in E_j, \lambda \neq 0, \\ 0, & \lambda = 0 \text{ or } \lambda \notin E_j. \end{cases}$$

Set $\eta := (1/q)(\eta_0 - \eta_j)$. Then

$$w_\eta(x) = \begin{cases} 0, & x = 0, \\ q^e, & x \in (\theta^{e-j}), x \neq 0, \\ q^{e-j}(q^j - 1), & x \notin (\theta^{e-j}). \end{cases}$$

Lemma 25. *Let $R = A := \mathbb{F}_q$, and let $M := \mathbb{F}_q^m$. Suppose N_1 and N_2 are linear subspaces of $\mathbb{F}_q^m$ such that $N_1 \oplus N_2 = \mathbb{F}_q^m$.*

Define a multiplicity function η by

$$\eta(\lambda) := \begin{cases} a_1, & \lambda \in N_1^\circ, \lambda \neq 0, \\ a_2, & \lambda \in N_2^\circ, \lambda \neq 0, \\ 0, & \textit{otherwise}. \end{cases}$$

Then η defines a three-weight code with weights

$$w_\eta(x) = \begin{cases} 0, & x = 0, \\ a_2\gamma|N_2^\circ|, & x \in N_1, x \neq 0, \\ a_1\gamma|N_1^\circ|, & x \in N_2, x \neq 0, \\ a_1\gamma|N_1^\circ| + a_2\gamma|N_2^\circ|, & x \notin N_1 \cup N_2. \end{cases}$$

Proof. Observe first that $N_1^\circ \cap N_2^\circ = 0$ by the direct sum hypothesis. Let $\eta \coloneqq a_1\eta_1 + a_2\eta_2$, where η_i is the multiplicity function with $E = N_i^\circ$ from Proposition 19. □ □

Corollary 26. *Same hypotheses as in Lemma 25. Let $k_i \coloneqq \dim N_i$, $i = 1, 2$, and suppose $k_1 \leq k_2$. Define a multiplicity function η by*

$$\eta(\lambda) \coloneqq \begin{cases} 1, & \lambda \in N_1^\circ, \lambda \neq 0, \\ q^{k_2-k_1}, & \lambda \in N_2^\circ, \lambda \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Then η defines a two-weight code with weights

$$w_\eta(x) = \begin{cases} 0, & x = 0, \\ \gamma q^{k_2}, & x \in N_1 \cup N_2, x \neq 0, \\ 2\gamma q^{k_2}, & x \notin N_1 \cup N_2. \end{cases}$$

Corollary 27 (Calderbank-Kantor, [3, Example SU2])**.** *Let $R = A = \mathbb{F}_q$ and $M = \mathbb{F}_q^{2m}$. Suppose $N_1, N_2, \ldots N_\ell$ are linear subspaces of $\mathbb{F}_q^{2m}$ with $\dim N_i = m$ and $N_i \oplus N_j = \mathbb{F}_q^{2m}$ for all $i \neq j$. Assume $\cup_i N_i \neq \mathbb{F}_q^{2m}$.*

Define a multiplicity function η by

$$\eta(\lambda) \coloneqq \begin{cases} 1, & \lambda \in N_i^\circ, \text{ some } i, \\ 0, & \text{otherwise.} \end{cases}$$

Then η defines a two-weight code with weights

$$w_\eta(x) = \begin{cases} 0, & x = 0, \\ (\ell-1)\gamma q^m, & x \in \cup_i N_i, x \neq 0, \\ \ell\gamma q^m, & x \notin \cup_i N_i. \end{cases}$$

5. Theoretical Ramifications

In this section we describe how the MacWilliams extension theorem for homogeneous weights may be derived by using Lemma 15 together with poset techniques. The proof is in the spirit of Greferath [7].

Let A be a finite Frobenius bimodule over a finite ring R with 1. Equip A with a homogeneous weight w. In particular, w has maximal symmetry groups: $\mathrm{Sym}_l(w) = \mathrm{Sym}_r(w) = \mathcal{U}$, the group of units of the ring R. The weight w defines a weight on A^n via $w(a) := \sum_{i=1}^n w(a_i)$, where $a = (a_1, \dots, a_n) \in A^n$. A *monomial transformation* of A^n is a left R-linear homomorphism $T : A^n \to A^n$ of the form

$$(a_1, \dots, a_n)T = (a_{\sigma(1)}u_1, \dots, a_{\sigma(n)}u_n), \quad (a_1, \dots, a_n) \in A^n,$$

where σ is a permutation of $\{1, 2, \dots, n\}$ and $u_1, u_2, \dots, u_n \in \mathcal{U}$. Because the homogeneous weight w has maximal symmetry, every monomial transformation is a w-isometry: $w(aT) = w(a)$, for all vectors $a \in A^n$. Another consequence of maximal symmetry is that $w(ua) = w(a)$, for all $u \in \mathcal{U}$ and $a \in A^n$.

The MacWilliams extension theorem for homogeneous weights says that any linear w-isometry between R-linear codes over a finite Frobenius bimodule A is actually a monomial transformation. Here is the formal statement.

Theorem 28 (MacWilliams Extension Theorem)**.** *Let A be a finite Frobenius bimodule over a finite ring R with* 1*. Equip A with a homogeneous weight w. Suppose $C_1, C_2 \subset A^n$ are left R-linear codes over A, and suppose $f : C_1 \to C_2$ is a left R-linear isomorphism and a w-isometry. Then f extends to a monomial transformation T of A^n.*

There are forms of the extension theorem that date to [19], where the theorem was proved for isometries for the Hamming weight over finite fields. In the generality stated above, the theorem is due to Greferath, Nechaev, and Wisbauer [8].

By using the language of multiplicity functions from Subsection 2.4, the extension theorem can be recast. We will give a short summary of this recasting; a more detailed description can be found in [26].

As in Subsection 2.4, an R-linear code over A can be viewed, up to permutation equivalence, as the image of an R-linear homomorphism $M \to A^n$ that is specified by a multiplicity function $\eta : \mathrm{Hom}_R(M, A) \to \mathbb{N}$. Here, M is a finite left R-module. In the extension theorem, monomial equivalence is the appropriate notion of equivalence, not permutation equivalence. This will require us to make use of the right action of $\mathcal{U}$ on $\mathrm{Hom}_R(M, A)$, as we now explain.

Remember that A is a bimodule over R, so $\mathrm{Hom}_R(M, A)$ is a right R-module via $m(\lambda r) = (m\lambda)r$, for $m \in M$, $\lambda \in \mathrm{Hom}_R(M, A)$, and $r \in R$. Thus the group $\mathcal{U}$ acts on $\mathrm{Hom}_R(M, A)$ on the right. Denote the set of nonzero right $\mathcal{U}$-orbits of this action by $\mathcal{O}^\sharp$.

The monomial equivalence class of an R-linear code over A depends only upon how many of the coordinate functionals of the code belong to each of the $\mathcal{U}$-orbits in $\mathcal{O}^\sharp$. That is, the monomial equivalence class of a linear code is determined by a multiplicity function $\eta : \mathcal{O}^\sharp \to \mathbb{N}$.

The multiplicity function η counts the number of coordinate functionals (columns of a generator matrix) that belong to any given $\mathcal{U}$-scale class.

Given a multiplicity function $\eta : \mathcal{O}^\sharp \to \mathbb{N}$, we can rewrite (2.3) as

$$w_\eta(x) = \sum_{\lambda \in \mathcal{O}^\sharp} w(x\lambda)\,\eta(\lambda), \quad x \in M. \tag{5.1}$$

Because $\mathrm{Sym}_r(w) = \mathcal{U}$, this formula is well-defined. As mentioned above, we also have $w_\eta(ux) = w_\eta(x)$ for $u \in \mathcal{U}$, $x \in M$, because $\mathrm{Sym}_l(w) = \mathcal{U}$. The left R-module M admits a left $\mathcal{U}$-action, and let $\mathcal{O}$ be the set on nonzero left $\mathcal{U}$-orbits of this action. It is shown in [8, Section 4.2] that a homogeneous weight on A has rational values (provided the constant γ is rational). Then, (5.1) implies that w_η yields a well-defined function $\mathcal{O} \to \mathbb{Q}$.

Let $F(\mathcal{O}^\sharp, \mathbb{N})$ and $F(\mathcal{O}, \mathbb{Q})$ denote the sets of all functions $\mathcal{O}^\sharp \to \mathbb{N}$ and $\mathcal{O} \to \mathbb{Q}$, respectively. Notice that $F(\mathcal{O}^\sharp, \mathbb{N})$ is an additive monoid and $F(\mathcal{O}, \mathbb{Q})$ is a $\mathbb{Q}$-vector space. Define $W : F(\mathcal{O}^\sharp, \mathbb{N}) \to F(\mathcal{O}, \mathbb{Q})$ by $W(\eta) \coloneqq w_\eta$; then W is an additive homomorphism. If we tensor $F(\mathcal{O}^\sharp, \mathbb{N})$ by $\mathbb{Q}$, we get $F(\mathcal{O}^\sharp, \mathbb{Q})$, a $\mathbb{Q}$-vector space; W extends naturally to a $\mathbb{Q}$-linear homomorphism $W : F(\mathcal{O}^\sharp, \mathbb{Q}) \to F(\mathcal{O}, \mathbb{Q})$. By extending the coefficients for multiplicity functions from $\mathbb{N}$ to $\mathbb{Q}$, we are formally allowing coordinate functionals to have fractional or even negative values.

Here is the recasting of the extension theorem.

Theorem 29. *Let A be a finite Frobenius bimodule over a finite ring R with 1. Equip A with a homogeneous weight w that has rational values. Then the extension theorem holds for A with respect to w if and only if, for every finite left R-module M, the $\mathbb{Q}$-linear homomorphism*

$$W : F(\mathcal{O}^\sharp, \mathbb{Q}) \to F(\mathcal{O}, \mathbb{Q}), \tag{5.2}$$

is injective.

Proving the extension theorem by showing the injectivity of W in (5.2) dates to [2]. The same approach is used in [7].

Each of the $\mathbb{Q}$-vector spaces $F(\mathcal{O}^\sharp, \mathbb{Q})$ and $F(\mathcal{O}, \mathbb{Q})$ has a natural basis consisting of the indicator functions for the various $\mathcal{U}$-orbits. (That is, for a given $\mathcal{U}$-orbit, the function that is 1 on the given orbit and 0 on the other orbits.) From (5.1), it follows that the matrix representing W with respect to these bases is $(w(x\lambda))_{\mathcal{U}x, \lambda\mathcal{U}}$. This matrix is the “orthogonality matrix” referred to in the title of [7].

In order to prove the extension theorem, we will show that the $\mathbb{Q}$-linear homomorphism W in (5.2) is in fact an isomorphism by proving that $\dim_{\mathbb{Q}} F(\mathcal{O}^\sharp, \mathbb{Q}) = \dim_{\mathbb{Q}} F(\mathcal{O}, \mathbb{Q})$ and that W is surjective. In order to do this, we will analyze W using a different choice of basis for the $\mathbb{Q}$-vector space $F(\mathcal{O}, \mathbb{Q})$. We begin by quoting a result from [24].

Lemma 30 ([24, Proposition 5.1])**.** *Let M be a finite left R-module over a finite ring R with 1. Let $\mathcal{U}$ be the group of units of R. Suppose $x_1, x_2 \in M$, and consider the left R-submodules $Rx_1, Rx_2 \subset M$ and the left $\mathcal{U}$-orbits $\mathcal{U}x_1, \mathcal{U}x_2 \subset M$. Then $Rx_1 = Rx_2$ if and only if $\mathcal{U}x_1 = \mathcal{U}x_2$. There is a similar result for right R-modules.*

Proposition 31. *The indicator functions of the principal left R-submodules of M form a basis over $\mathbb{Q}$ for $F(\mathcal{O}, \mathbb{Q})$.*

Proof. Consider the partially ordered set $(\mathcal{P}, \leq)$ of all principal left R-submodules of M under inclusion. That is, $Ry \leq Rx$ when $Ry \subseteq Rx$. Denote the indicator function of a subset S by δ_S, so that $\delta_S(x) = 1$ if $x \in S$, and $\delta_S(x) = 0$ otherwise.

Each principal left R-submodule Rx, $x \in M$, is a disjoint union of the left $\mathcal{U}$-orbits inside it. By Lemma 30, these $\mathcal{U}$-orbits are in one-to-one correspondence with the principal left R-submodules of Rx. Thus,

$$\delta_{Rx} = \sum_{\mathcal{U}y \subseteq Rx} \delta_{\mathcal{U}y} = \sum_{Ry \leq Rx} \delta_{\mathcal{U}y}.$$

This equation can be inverted by using the ($\mathbb{Q}$-valued) Möbius function $\mu_{\mathcal{P}}$ of the poset $\mathcal{P}$ (see, e.g., [21] for more details):

$$\delta_{\mathcal{U}x} = \sum_{Ry \leq Rx} \mu_{\mathcal{P}}(Ry, Rx)\delta_{Ry}.$$

This shows that the δ_{Rx} span $F(\mathcal{O}, \mathbb{Q})$. Since there are exactly as many δ_{Rx} as in the basis $\delta_{\mathcal{U}x}$, the δ_{Rx} form a basis over $\mathbb{Q}$. □ □

In order to show that the homomorphism W in (5.2) is surjective, we want to show that the (new) basis elements δ_{Rx} of $F(\mathcal{O}, \mathbb{Q})$ are in the image of W. The next result, which makes use of the flexibility of using negative multiplicities afforded by $\mathbb{Q}$-coefficents, proves even more.

Corollary 32 (to Proposition 22)**.** *Let N be a submodule of M. For $a \in \mathbb{N}$, define a multiplicity function η by*

$$\eta(\lambda) := \begin{cases} 0, & \lambda = 0, \\ a\left(1 - \frac{|\mathrm{Hom}_R(M,A)|}{|N^\circ|}\right), & \lambda \in N^\circ, \lambda \neq 0, \\ a, & \lambda \notin N^\circ. \end{cases}$$

Then η defines a two-weight code with weights

$$w_\eta(x) = \begin{cases} 0, & x \notin N \text{ or } x = 0, \\ a\gamma|\mathrm{Hom}_R(M,A)|, & x \in N, x \neq 0. \end{cases}$$

Proof. Set $b := -a|\mathrm{Hom}_R(M,A)|/|N^\circ|$. Then $w_\eta(x) = 0$ for $x \notin N$. □ □

of Theorem 28. Our aim is to show that the homomorphism W in (5.2) is an isomorphism and then appeal to Theorem 29.

Take any basis element δ_{Rx} of $F(\mathcal{O},\mathbb{Q})$. In Corollary 32, set $a := 1/(\gamma|\mathrm{Hom}_R(M,A)|)$ and $N := Rx$. Then $W(\eta) = w_\eta = \delta_N = \delta_{Rx}$. This proves that W is surjective. Consequently, we have $\dim_{\mathbb{Q}} F(\mathcal{O}^\sharp,\mathbb{Q}) \geq \dim_{\mathbb{Q}} F(\mathcal{O},\mathbb{Q})$.

Now reverse the roles of M and $\mathcal{D}(M) := \mathrm{Hom}_R(M,A)$. Because A defines a Morita duality functor, $M \cong \mathrm{Hom}_R(\mathcal{D}(M),A)$, and the roles of $F(\mathcal{O},\mathbb{Q})$ and $F(\mathcal{O}^\sharp,\mathbb{Q})$ are interchanged. The dual W homomorphism maps $F(\mathcal{O},\mathbb{Q}) \to F(\mathcal{O}^\sharp,\mathbb{Q})$ and is just the transpose of the original W. By the same argument as above, the dual homomorphism W is also surjective. Thus $\dim_{\mathbb{Q}} F(\mathcal{O},\mathbb{Q}) \geq \dim_{\mathbb{Q}} F(\mathcal{O}^\sharp,\mathbb{Q})$, forcing $\dim_{\mathbb{Q}} F(\mathcal{O},\mathbb{Q}) = \dim_{\mathbb{Q}} F(\mathcal{O}^\sharp,\mathbb{Q})$. The surjective W is then an isomorphism. □ □

Acknowledgments. I thank the Department of Mathematics, Huazhong Normal University, Wuhan, China, and especially Professors Yun Fan and Hongwei Liu, for their hospitality during the summer of 2011, when much of the research for this paper was conducted. I thank the referees for their helpful comments, especially those related to the history of homogeneous weights, and for saving me from some embarassing typos. I also thank my wife Elizabeth S. Moore for her continuing support and encouragement.

References

[1] E. F. Assmus, Jr. and H. F. Mattson, Jr., *Error-correcting codes: An axiomatic approach*, Inform. and Control **6** (1963), 315–330. MR 0178997 (31 #3251)

[2] K. Bogart, D. Goldberg, and J. Gordon, *An elementary proof of the MacWilliams theorem on equivalence of codes*, Inform. and Control **37** (1978), no. 1, 19–22. MR 0479646 (57 #19067)

[3] A. R. Calderbank and W. M. Kantor, *The geometry of two-weight codes*, Bull. London Math. Soc. **18** (1986), 97–122.

[4] P. Camion, *Codes and association schemes: basic properties of association schemes relevant to coding*, Handbook of coding theory, Vol. I, II, North-Holland, Amsterdam, 1998, pp. 1441–1566. MR 1667954

[5] W. Chen and T. Kløve, *The weight hierarchies of q-ary codes of dimension* 4, IEEE Trans. Inform. Theory **42** (1996), no. 6, part 2, 2265–2272. MR 1447528 (98c:94020)

[6] Ioana Constantinescu and Werner Heise, *On the concept of code-isomorphy*, J. Geom. **57** (1996), no. 1-2, 63–69. MR 1418082 (98e:94028)

[7] M. Greferath, *Orthogonality matrices for modules over finite Frobenius rings and MacWilliams' equivalence theorem*, Finite Fields Appl. **8** (2002), no. 3, 323–331. MR 1910395 (2003d:94107)

[8] M. Greferath, A. Nechaev, and R. Wisbauer, *Finite quasi-Frobenius modules and linear codes*, J. Algebra Appl. **3** (2004), no. 3, 247–272. MR 2096449 (2005g:94099)

[9] M. Greferath and S. E. Schmidt, *Finite-ring combinatorics and MacWilliams's equivalence theorem*, J. Combin. Theory Ser. A **92** (2000), no. 1, 17–28. MR 1783936 (2001j:94045)

[10] W. Heise and T. Honold, *Homogeneous and egalitarian weights on finite rings*, Proceedings of the Seventh International Workshop on Algebraic and Combinatorial Coding Theory (ACCT-2000) (Bansko, Bulgaria), 2000, pp. 183–188.

[11] Y. Hirano, *On admissible rings*, Indag. Math. (N.S.) **8** (1997), no. 1, 55–59. MR 1617802 (99b:16034)

[12] T. Honold, *Characterization of finite Frobenius rings*, Arch. Math. (Basel) **76** (2001), no. 6, 406–415. MR 1831096 (2002b:16033)

[13] ———, *Further results on homogeneous two-weight codes*, Proceedings of Optimal Codes and Related Topics (Bulgaria), 2007.

[14] T. Honold and I. Landjev, *Linear codes over finite chain rings*, Electron. J. Combin. **7** (2000), Research Paper 11, 22. MR 1741333 (2001c:94023)

[15] T. Honold and A. A. Nechaev, *Weighted modules and representations of codes*, Problems Inform. Transmission **35** (1999), no. 3, 205–223. MR 1730800 (2001f:94016)

[16] T. Y. Lam, *Lectures on modules and rings*, Graduate Texts in Mathematics, vol. 189, Springer-Verlag, New York, 1999. MR 1653294 (99i:16001)

[17] Z. Liu and W. Chen, *Notes on the value function*, Des. Codes Cryptogr. **54** (2010), no. 1, 11–19. MR 2576877 (2010m:94157)

[18] Z. Liu, W. Chen, Z. Sun, and X. Zeng, *Further results on support weights of certain subcodes*, Des. Codes Cryptogr. **61** (2011), no. 2, 119–129.

[19] F. J. MacWilliams, *Error-correcting codes for multiple-level transmission*, Bell System Tech. J. **40** (1961), 281–308. MR 0141541 (25 #4945)

[20] W. W. Peterson, *Error-correcting codes*, The MIT Press, Cambridge, Mass. 1961.

[21] Richard P. Stanley, *Enumerative combinatorics*, The Wadsworth & Brooks/Cole Mathematics Series, vol. I, Wadsworth & Brooks/Cole Advanced Books & Software, Monterey, CA, 1986. MR 847717 (87j:05003)

[22] A. Terras, *Fourier analysis on finite groups and applications*, London Mathematical Society Student Texts, vol. 43, Cambridge University Press, Cambridge, 1999. MR 1695775 (2000d:11003)

[23] M. A. Tsfasman and S. G. Vlăduţ, *Algebraic-geometric codes*, Mathematics and its Applications (Soviet Series), vol. 58, Kluwer Academic Publishers Group, Dordrecht, 1991. MR 93i:94023

[24] J. A. Wood, *Duality for modules over finite rings and applications to coding theory*, Amer. J. Math. **121** (1999), no. 3, 555–575. MR 1738408 (2001d:94033)

[25] ———, *The structure of linear codes of constant weight*, Trans. Amer. Math. Soc. **354** (2002), no. 3, 1007–1026. MR 1867370 (2002k:94042)

[26] ———, *Foundations of linear codes defined over finite modules: the extension theorem and the MacWilliams identities*, Codes over rings (Ankara, 2008), Ser. Coding Theory Cryptol., vol. 6, World Sci. Publ., Hackensack, NJ, 2009, pp. 124–190. MR 2850303

[27] ———, *Applications of finite Frobenius rings to the foundations of algebraic coding theory*, Proceedings of the 44th Symposium on Rings and Representation Theory (Okayama University, Japan, September 25–27, 2011) (Nagoya, Japan) (O. Iyama, ed.), 2012, pp. 223–245.

Department of Mathematics, Western Michigan University, 1903 W. Michigan Ave., Kalamazoo, MI 49008-5248 USA, HTTP://HOMEPAGES.WMICH.EDU/$\SIM$JWOOD

E-mail address: jay.wood@wmich.edu