THE EXTENSION THEOREM FOR THE LEE AND EUCLIDEAN WEIGHTS OVER $\mathbb{Z}/p^k\mathbb{Z}$

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Dedicated to Jacques Wolfmann, our teacher, colleague, and friend.

ABSTRACT. The Lee and Euclidean weights have the extension property over the local rings $\mathbb{Z}/p^k\mathbb{Z}$, p prime. The non-vanishing of certain Fourier coefficients is established by expressing the coefficients in terms of generalized Bernoulli numbers and making use of knowledge of the locations of zeros of Dirichlet L-functions.

1. OVERVIEW

In this paper we show that the Lee and Euclidean weights over $\mathbb{Z}/p\mathbb{Z}^k$, p prime, have the extension property. The extension property is equivalent to the non-vanishing of certain Fourier coefficents of the weight functions with respect to even multiplicative characters modulo p^k. The non-vanishing of the Fourier coefficents follows from knowledge of which negative integers are zeros of Dirichlet L-functions.

2. EXTENSION PROPERTY

Let K be a finite field, and let $\mathrm{H}\colon K \to \mathbb{N}$ be the indicator of $K^\times$; that is, $\mathrm{H}(x) = 1$ if $x \neq 0$ and $\mathrm{H}(0) = 0$ otherwise. For a positive integer n, the space K^n is a metric space with the distance $(x, y) \mapsto \mathrm{w}_\mathrm{H}(y - x)$ where $\mathrm{w}_\mathrm{H}(x) = \sum_{i=1}^{n} \mathrm{H}(x_i)$ is the *Hamming weight* of x. A linear map $f\colon C \to K^n$ that preserves the Hamming weight over a subspace C of K^n, that is

$$\mathrm{w}_\mathrm{H}(f(x)) = \mathrm{w}_\mathrm{H}(x),$$

for every $x \in C$, is called an H-*isometry* over C. Let $\{e_i : i = 1, \ldots, n\}$ be the canonical basis of K^n. It is well known that an H-isometry over K^n maps $e_i \mapsto \lambda_i e_{\sigma(i)}$, where σ is a permutation of $\{1, 2, \ldots, n\}$ and the λ_i's are in $K^\times$. Such a map is a *monomial map* whose scalars are the λ's.

Theorem 2.1 (MacWilliams, [5, 6]). *An* H*-isometry over $C \subseteq K^n$ extends to an* H*-isometry over K^n.*

It is natural to examine the above extension property when H is replaced by another weight function or K is replaced by a finite ring (or both). For example, the Hamming weight and the homogeneous weight have the extension property over finite Frobenius rings [3, 8]. In this paper, we will prove the analogue of Theorem 2.1 for the Lee and Euclidean weights over the local rings $\mathbb{Z}/p^k\mathbb{Z}$, p prime. Some cases of this result are already known: $p^k = 2^k$, $p^k = 3^k$, and primes p (with $k = 1$) of

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the form $p = 1 + 2q$, $p = 1 + 4q$, q prime [1, 4]. The case of a general odd prime p with $k = 1$ was announced in [2].

Let $R = \mathbb{Z}/p^k\mathbb{Z}$ for some prime p and positive integer k. Let ω be any weight on R; i.e., $\omega : R \to \mathbb{C}$ and $\omega(0) = 0$. This weight on R extends to a weight w_ω on R^n by $\mathrm{w}_\omega(x) = \sum_{i=1}^n \omega(x_i)$, for $x = (x_1, \ldots, x_n) \in R^n$. If $C \subseteq R^n$ is an R-submodule and $f : C \to R^n$ is a homomorphism of R-modules, we say that f is *ω-preserving* over C when, for all $x \in C$,

$$\mathrm{w}_\omega(f(x)) = \mathrm{w}_\omega(x).$$

The weight ω may have some symmetry. Define the *symmetry group* U of ω by $U = \{u \in \mathcal{U}(R) : \omega(ur) = \omega(r) \text{ for all } r \in R\}$, where $\mathcal{U}(R)$ is the group of units of R. A module homomorphism $T : R^n \to R^n$ is a *U-monomial transformation* of R^n when there exist units $u_i \in U$ and a permutation σ of $\{1, \ldots, n\}$ such that

$$T(x_1, \ldots, x_n) = (u_1 x_{\sigma(1)}, \ldots, u_n x_{\sigma(n)}),$$

for all $x = (x_1, \ldots, x_n) \in R^n$. The restriction to C of any U-monomial transformation is ω-preserving over C.

Conversely, we say that ω has the *extension property* over R when any ω-preserving homomorphism $f : C \to R^n$ over any submodule $C \subseteq R^n$ extends to a U-monomial transformation of R^n. That is, there exists a U-monomial transformation $T : R^n \to R^n$ such that $f(x) = T(x)$ for all $x \in C$.

The Lee weight $\textsc{l}$ and Euclidean weight $\textsc{e}$ are defined on $R = \mathbb{Z}/p^k\mathbb{Z}$ by

$$\textsc{l}(t) = \begin{cases} t, & 0 \le t \le p^k/2, \\ p^k - t, & p^k/2 < t < p^k; \end{cases} \qquad \textsc{e}(t) = \begin{cases} t^2, & 0 \le t \le p^k/2, \\ (p^k - t)^2, & p^k/2 < t < p^k. \end{cases}$$

Note that $\textsc{e} = \textsc{l}^2$ and the symmetry group for each is $U = \{\pm 1\}$.

Theorem 2.2. *The Lee and Euclidean weights have the extension property over $\mathbb{Z}/p^k\mathbb{Z}$, p prime.*

3. A Criterion for Extension

As in the previous section, we assume $R = \mathbb{Z}/p^k\mathbb{Z}$ with p prime and ω is a weight on R with symmetry group U. The group U acts on R by ring multiplication; denote the set of nonzero orbits by $(R/U)^\times$. Form a matrix W_ω with rows and columns indexed by $(R/U)^\times$; the entry in position rU, sU is defined to be $\omega(rs)$, i.e., ω evaluated on the product $rs \in R$. Because of the symmetry of ω, this value is well-defined.

Theorem 3.1 ([9, Theorem 3.1])**.** *The weight ω has the extension property if the matrix W_ω is invertible over $\mathbb{C}$.*

One way to apply Theorem 3.1 is to show that the determinant Δ_ω of W_ω is a nonzero element of $\mathbb{C}$. For example, when $p^k = 7$, the determinants for the Lee weight $\textsc{l}$ and the Euclidean weight $\textsc{e}$ are:

$$-\Delta_{\textsc{l}} = -\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = 18 = 2 \times 3^2, \qquad -\Delta_{\textsc{e}} = -\begin{vmatrix} 1 & 4 & 9 \\ 4 & 9 & 1 \\ 9 & 1 & 4 \end{vmatrix} = 686 = 2 \times 7^3.$$

When $k = 1$, i.e., when $R = \mathbb{F}_p$ is a prime field, the well-known Dedekind determinant formula expresses the determinant Δ_ω as a product of Fourier coefficients.

This formula was generalized to the context of finite commutative chain rings in [10], as we now explain for $R = \mathbb{Z}/p^k\mathbb{Z}$.

The group of units $\mathcal{U} = \mathcal{U}(R)$ acts on $R = \mathbb{Z}/p^k\mathbb{Z}$ by ring multiplication. The nonzero orbits are $\mathcal{O}_i = \mathrm{orb}_{\mathcal{U}}(p^i) = Rp^i \setminus Rp^{i+1}$, for $i = 0, \dots, k-1$, so that $|\mathcal{O}_i| = (p-1)p^{k-i-1}$. The stabilizer subgroup $\mathcal{S}_i = \mathrm{Stab}_{\mathcal{U}}(p^i) \subseteq \mathcal{U}$ of $p^i \in \mathcal{O}_i$ has order p^i and has the form $\mathcal{S}_i = 1 + Rp^{k-i}$.

Denote the group of complex characters of $\mathcal{U}$ by $\widehat{\mathcal{U}}$; a *character* of $\mathcal{U}$ is a group homomorphism $\chi : \mathcal{U} \to \mathbb{C}^\times$ to the multiplicative group of nonzero complex numbers. For any subgroup $H \subseteq \mathcal{U}$, denote the *annihilator* of H in $\widehat{\mathcal{U}}$ by

$$(\widehat{\mathcal{U}} : H) = \{\chi \in \widehat{\mathcal{U}} : H \subseteq \ker \chi\}.$$

A pair $(\chi, \mathcal{O}_i)$ of a character $\chi \in \widehat{\mathcal{U}}$ and a $\mathcal{U}$-orbit $\mathcal{O}_i$ is *admissible* if $\chi \in (\widehat{\mathcal{U}} : \mathcal{S}_i)$. Note that $\mathcal{S}_0 = \{1\}$, so $(\chi, \mathcal{O}_0)$ is admissible for every $\chi \in \widehat{\mathcal{U}}$. For $\chi \in \widehat{\mathcal{U}}$, define i_χ to be the largest value of i, $0 \le i \le k-1$, so that $(\chi, \mathcal{O}_i)$ is admissible. For example, the trivial character $\chi_0 = 1$ has $i_{\chi_0} = k-1$.

For $\chi \in (\widehat{\mathcal{U}} : U)$, define the following character sum:

$$\breve{\omega}(\chi, i_\chi) = \sum_{u \in \mathcal{U}/U\mathcal{S}_{i_\chi}} \omega(up^{i_\chi})\chi(u). \tag{1}$$

Observe that this character sum is well-defined. The next result gives a factorization of the determinant Δ_ω of W_ω into a product of character sums, thereby providing a criterion for ω to have the extension property.

Theorem 3.2 ([10, Theorem 7]). *Suppose ω is a weight on $\mathbb{Z}/p^k\mathbb{Z}$, p prime, with symmetry group U. Then there is a nonzero integer C (depending only on the ring $\mathbb{Z}/p^k\mathbb{Z}$ and the symmetry group U, not the weight ω) such that*

$$\Delta_\omega = C \prod_{\chi \in (\widehat{\mathcal{U}}:U)} \breve{\omega}(\chi, i_\chi)^{1+i_\chi}.$$

The weight ω has the extension property if $\breve{\omega}(\chi, i_\chi) \neq 0$ for all $\chi \in (\widehat{\mathcal{U}} : U)$.

Remark 3.3. When $k = 1$, i.e., when $R = \mathbb{F}_p$, there is only one nonzero orbit $\mathcal{O}_0 = \mathcal{U} = \mathbb{F}_p^{\times}$. Thus $i_\chi = 0$ for all $\chi \in \widehat{\mathcal{U}}$. Moreover, the character sum (1) is just the Fourier coefficient of ω on the quotient group $\mathcal{U}/U$. In this way, one recovers the Dedekind determinant formula from Theorem 3.2.

In Lemma 5.1, the value i_χ will be compared to the conductor of χ, when χ is viewed as a Dirichlet character.

4. Dirichlet Characters

In this section we give a short summary of the properties of Dirichlet characters. More details can be found in [7, Chapters 3 and 4], for example.

Fix a positive integer m and consider the integer residue ring $R = \mathbb{Z}/m\mathbb{Z}$. The group of units $\mathcal{U}_m$ consists of the residue classes modulo m of integers that are relatively prime to m. Let $\chi \in \widehat{\mathcal{U}}_m$ be a character of $\mathcal{U}_m$. Extend χ to a function (a *Dirichlet character* modulo m) $\chi : \mathbb{Z} \to \mathbb{C}$ by

$$\chi(n) = \begin{cases} \chi(n \bmod m), & \text{when } n \text{ is relatively prime to } m, \\ 0, & \text{when } n \text{ is not relatively prime to } m. \end{cases}$$

For any χ, $\chi(-1) = \pm 1$ and $\chi(-n) = \chi(-1)\chi(n)$. We say that χ is *even* if $\chi(-1) = 1$ and *odd* if $\chi(-1) = -1$.

If f is a positive divisor of m, there is a natural quotient homomorphism $\mathbb{Z}/m\mathbb{Z} \twoheadrightarrow \mathbb{Z}/f\mathbb{Z}$ which induces a surjective homomorphism $\mathcal{U}_m \twoheadrightarrow \mathcal{U}_f$. The dual homomorphism on character groups $\widehat{\mathcal{U}}_f \hookrightarrow \widehat{\mathcal{U}}_m$ is injective. If a character $\chi \in \widehat{\mathcal{U}}_m$ is not in the image of $\widehat{\mathcal{U}}_f \hookrightarrow \widehat{\mathcal{U}}_m$ for any positive divisor f of m, $f \neq m$, we say that χ is a *primitive* character. Otherwise, we say that χ is an *imprimitive* character. When χ is imprimitive, there is a unique divisor $f \neq m$ of m and a primitive character $\chi^0 \in \widehat{\mathcal{U}}_f$ such that χ^0 maps to χ under $\widehat{\mathcal{U}}_f \hookrightarrow \widehat{\mathcal{U}}_m$. In this case we say that χ (and χ^0) have *conductor* f.

Given a Dirichlet character χ modulo m, the *Dirichlet L-function* associated to χ is

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

The series converges absolutely for complex numbers s with real part $\Re(s) > 1$. The Dirichlet L-function generalizes the Riemann ζ-function $\zeta(s)$ and, like $\zeta(s)$, satisfies a functional equation, which, for conductor $f > 1$, implies that $L(s,\chi)$ is an entire function of s.

For a Dirichlet character χ of conductor f, the *generalized Bernoulli numbers* $B_n(\chi)$ determined by χ are defined by

$$\sum_{a=1}^{f} \frac{\chi(a) t e^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_n(\chi) \frac{t^n}{n!}.$$

In particular, there are formulas for $B_1(\chi)$ and $B_2(\chi)$, $\chi \neq 1$:

$$B_1(\chi) = \frac{1}{f} \sum_{a=1}^{f} a\chi(a), \qquad B_2(\chi) = \frac{1}{f} \sum_{a=1}^{f} (a^2 - af)\chi(a).$$

Finally, we quote a few facts about values of Dirichlet L-functions to which we will refer later [7, Chapter 4].

Proposition 4.1. (i) *The values of Dirichlet L-functions at negative integers satisfy*

$$L(1-n,\chi) = -B_n(\chi)/n, \quad n \geq 1.$$

(ii) *For integers $n \geq 1$, if χ is even, $\chi \neq 1$, then $L(1-n,\chi) = 0$ if and only if n is odd. If χ is odd, then $L(1-n,\chi) = 0$ if and only if n is even.*

5. Character Sums for the Lee Weight

In order to prove Theorem 2.2 we will appeal to Theorem 3.2 by showing that the character sums $\breve{\omega}(\chi, i_\chi)$ of (1) do not vanish for $\omega = \textsc{l}$ or $\omega = \textsc{e}$ and for all $\chi \in (\widehat{\mathcal{U}} : U)$. Recall that $R = \mathbb{Z}/p^k\mathbb{Z}$, p prime, with $\mathcal{U} = \mathcal{U}_{p^k}$ its group of units and $U = \{\pm 1\}$. We begin by understanding admissible pairs $(\chi, \mathcal{O}_i)$ in the language of Dirichlet characters.

Any $\chi \in (\widehat{\mathcal{U}}_{p^k} : U)$, when viewed as a Dirichlet character modulo p^k, is an even character. Recall that $(\chi, \mathcal{O}_i)$ is admissible if $\chi \in (\widehat{\mathcal{U}}_{p^k} : \mathcal{S}_i)$, where $\mathcal{S}_i$ is the stabilizer subgroup of $p^i \in \mathbb{Z}/p^k\mathbb{Z}$, and i_χ is the largest value of i, $0 \leq i \leq k-1$, such that $(\chi, \mathcal{O}_i)$ is admissible. The trivial character $\chi_0 = 1$ has $i_{\chi_0} = k-1$.

Lemma 5.1. *Let $\chi \in (\widehat{\mathcal{U}}_{p^k} : U)$, $\chi \neq 1$. Then χ has conductor $f = p^{k-i_\chi}$.*

Proof. This reflects the fact that i_χ is the largest integer such that $(\chi, \mathcal{O}_{i_\chi})$ is admissible. Recall that $(\widehat{\mathcal{U}}_{p^k} : \mathcal{S}_i) \cong (\mathcal{U}_{p^k}/\mathcal{S}_i)^\frown$ and $\mathcal{U}_{p^k}/\mathcal{S}_i \cong \mathcal{U}_{p^{k-i}}$. If $(\chi, \mathcal{O}_i)$ is admissible, then χ is the image of some χ^0 under $\widehat{\mathcal{U}}_{p^{k-i}} \hookrightarrow \widehat{\mathcal{U}}_{p^k}$. The character χ^0 will be imprimitive unless i is maximal, i.e., $i = i_\chi$. Thus the conductor is $f = p^{k-i_\chi}$. □

The next lemma computes the character sum $\check{\omega}(\chi, i_\chi)$, for $\omega = \mathrm{L}$ or $\omega = \mathrm{E}$, in terms of a similar character sum for the conductor. It will be important to keep track of which ring $\mathbb{Z}/p^j\mathbb{Z}$ is being used, so let L_{p^j} and E_{p^j} be the Lee and Euclidean weights on $\mathbb{Z}/p^j\mathbb{Z}$.

Lemma 5.2. *Suppose $\chi \in (\widehat{\mathcal{U}}_{p^k} : U)$, $\chi \neq 1$. Write χ^0 for the primitive character in $\widehat{\mathcal{U}}_{p^{k-i_\chi}}$ that maps to χ under $\widehat{\mathcal{U}}_{p^{k-i_\chi}} \hookrightarrow \widehat{\mathcal{U}}_{p^k}$. Then*

$$\check{\mathrm{L}}_{p^k}(\chi, i_\chi) = p^{i_\chi} \cdot \check{\mathrm{L}}_{p^{k-i_\chi}}(\chi^0, 0),$$
$$\check{\mathrm{E}}_{p^k}(\chi, i_\chi) = p^{2i_\chi} \cdot \check{\mathrm{E}}_{p^{k-i_\chi}}(\chi^0, 0).$$

Proof. Just pull out the common factor of p^{i_χ}, resp., p^{2i_χ}, in (1). □

Character sums of the form $\check{\mathrm{L}}_{p^k}(\chi, 0)$ and $\check{\mathrm{E}}_{p^k}(\chi, 0)$ for primitive characters are the subject of the next section.

6. Fourier Analysis

In this section, we continue to use the notation from Section 5, remembering that p is a prime. Suppose $\chi \in (\widehat{\mathcal{U}}_{p^k} : U)$, $\chi \neq 1$, is a primitive even character. Then $i_\chi = 0$, $\mathcal{S}_0 = \{1\}$, and the character sum $\check{\omega}(\chi, 0) = \sum_{u \in \mathcal{U}_{p^k}/U} \omega(u)\chi(u)$ is exactly the Fourier coefficient $\widehat{\omega}(\chi)$ over the group $G_k = \mathcal{U}_{p^k}/U$.

We identify the dual $\widehat{G}_k$ of G_k with the group of even (multiplicative) Dirichlet characters modulo p^k, and we take as representatives of G_k the integers j, $0 < j < p^k/2$, that are relatively prime to p. (When $p = 2$, any such j must be odd, so the strict inequality indeed gives the correct representatives.) The Fourier coefficient at χ of a function g on G_k then takes the form

$$\widehat{g}(\chi) = \sum_{x \in G_k} g(x)\chi(x) = \sum_{j < p^k/2} g(j)\chi(j).$$

We isolate several technical steps for later use.

Lemma 6.1. *Let g be a function on $\mathbb{Z}/p^k\mathbb{Z}$ and χ be an even Dirichlet character modulo p^k, with $p^k > 2$. Then*

$$\sum_{j=1}^{p^k} g(j)\chi(j) = \sum_{j < p^k/2} \left(g(j) + g(p^k - j)\right)\chi(j).$$

Proof. Split the interval of summation into $0 < j < p^k/2$ and $p^k/2 < j < p^k$. (Note that $\chi(p^k) = 0$.) For $p^k/2 < j < p^k$, reindex with $m = p^k - j$ and use the fact that $\chi(m) = \chi(j)$ because χ is an even character modulo p^k. □

Lemma 6.2. *Suppose p is an odd prime and g is a function on G_k. Let h be defined by $h(t) = g(2t)$, $t \in G_k$. Then for any character $\chi \in \widehat{G}_k$,*

$$\widehat{h}(\chi) = \bar{\chi}(2)\widehat{g}(\chi).$$

Proof. Because p is odd, 2 is a unit in G_k. When calculating Fourier transforms, we change coordinates $s = 2t$ in G_k and use the fact that χ is a multiplicative character:

$$\begin{aligned}\widehat{h}(\chi) &= \sum_{t\in G_k} h(t)\chi(t) = \sum_{t\in G_k} g(2t)\chi(t) = \sum_{s\in G_k} g(s)\chi(2^{-1}s)\\ &= \sum_{s\in G_k} g(s)\chi(2^{-1})\chi(s) = \bar{\chi}(2)\widehat{g}(\chi).\end{aligned}$$

□

Lemma 6.3. *Suppose $p = 2$ and g is an even function on $\mathbb{Z}/2^k\mathbb{Z}$, $k \geq 3$; i.e., $g(t) = g(2^k - t)$ for $0 \leq t \leq 2^{k-1}$. Let h be defined by $h(t) = g(2t)$, $t \in \mathbb{Z}/2^k\mathbb{Z}$. Then for any primitive even character χ modulo 2^k, $\widehat{h}(\chi) = 0$.*

Proof. Because χ is an even character modulo 2^k, $\chi(2^k - t) = \chi(-t) = \chi(t)$ for all t. Using $t = 2^{k-1} - s$, we see that $\chi(2^{k-1} + s) = \chi(2^{k-1} - s)$ for all s.

The hypothesis that χ is primitive implies that $\chi(2^{k-1} + 1) = -1$. Indeed, the subgroup $\{1, 2^{k-1} + 1\} \subset \mathcal{U}_{2^k}$ is the kernel of the natural projection $\mathcal{U}_{2^k} \twoheadrightarrow \mathcal{U}_{2^{k-1}}$. If $\chi(2^{k-1} + 1) = 1$, then χ descends to a well-defined character modulo 2^{k-1}, which contradicts χ being primitive. By evenness, $\chi(2^{k-1} - 1) = -1$, as well.

Remember that $\chi(t)$ is nonzero only for odd values of t. Then, for s odd and $k \geq 3$, $(2^{k-2} - s)(2^{k-1} - 1) \equiv 2^{k-1} - 2^{k-2} + s \equiv 2^{k-2} + s \mod 2^k$. (Note that $s2^{k-1} \equiv 2^{k-1} \mod 2^k$ for any odd s.) It now follows that $\chi(2^{k-2} + s) = -\chi(2^{k-2} - s)$ for all s.

In contrast, $h(2^{k-2} - s) = g(2^{k-1} - 2s) = g(2^{k-1} + 2s) = h(2^{k-2} + s)$ for all s. The result now follows by symmetry considerations. □

The next result makes use of the particular form of the Lee and Euclidean weights. We note that $\chi = 1$ is the only even character modulo 2 or 4.

Proposition 6.4. *Let χ be a primitive even character modulo p^k, $\chi \neq 1$. Then $\widehat{\mathrm{L}}_{p^k}(\chi) = 0$ if and only if $\widehat{\mathrm{E}}_{p^k}(\chi) = 0$.*

Proof. Because we do not compare different rings, simply write L and E for the Lee and Euclidean weights. Let j, $0 < j < p^k/2$, be a representative of G_k. If $j < p^k/4$, then $2j < p^k/2$, and $\mathrm{L}(2j) = 2\mathrm{L}(j)$. If $p^k/4 < j < p^k/2$, then $p^k/2 < 2j < p^k$, so that $\mathrm{L}(2j) = p^k - 2\mathrm{L}(j)$. Thus the product $(\mathrm{L}(2j) - 2\mathrm{L}(j))\left(\mathrm{L}(2j) + 2\mathrm{L}(j) - p^k\right)$ vanishes for every $j \in G_k$. This yields a quadratic relation, valid for all $j \in G_k$: $\mathrm{L}(2j)^2 - 4\mathrm{L}(j)^2 = (\mathrm{L}(2j) - 2\mathrm{L}(j))\, p^k$. Because $\mathrm{E} = \mathrm{L}^2$, this relation takes the form, for all $j \in G_k$:

$$\mathrm{E}(2j) - 4\mathrm{E}(j) = (\mathrm{L}(2j) - 2\mathrm{L}(j))\, p^k. \tag{2}$$

Calculating the Fourier coefficent at χ of both sides of (2), and making use of Lemmas 6.2 and 6.3, we obtain the following relations:

$$\widehat{\mathrm{E}}(\chi) = 2^{k-1}\widehat{\mathrm{L}}(\chi) \qquad (p = 2) \tag{3}$$

$$(\bar{\chi}(2) - 4)\widehat{\mathrm{E}}(\chi) = p^k(\bar{\chi}(2) - 2)\widehat{\mathrm{L}}(\chi) \qquad (p \neq 2) \tag{4}$$

Because character values are roots of unity, the factors involving $\bar{\chi}(2)$ are nonzero, and the result follows. □

Lemma 6.5. *Let χ be a primitive even character modulo p^k, $\chi \neq 1$. Then $\widehat{1}(\chi) = 0$ and $B_1(\chi) = 0$, where 1 is the constant function 1 on G_k.*

Proof. The sum of any non-trivial character over its group vanishes. Thus $\widehat{1}(\chi) = 0$. As for $B_1(\chi)$, Lemma 6.1 implies

$$p^k B_1(\chi) = \sum_{j=1}^{p^k} j\chi(j) = \sum_{j<p^k/2} (j + (p^k - j))\chi(j) = p^k \cdot \widehat{1}(\chi) = 0. \qquad \square$$

Proposition 6.6. *Suppose χ is a primitive even character modulo p^k, $\chi \neq 1$. If $\widehat{\mathrm{L}}(\chi) = 0$, then $B_2(\chi) = 0$.*

Proof. If $\widehat{\mathrm{L}}(\chi) = 0$, then $\widehat{\mathrm{E}}(\chi) = 0$, by Proposition 6.4. Use $B_1(\chi) = 0$ and Lemma 6.1 to see:

$$\begin{aligned} p^k B_2(\chi) &= \sum_{j=1}^{p^k} (j^2 - p^k j)\chi(j) = \sum_{j=1}^{p^k} j^2\chi(j) = \sum_{j<p^k/2} (j^2 + (p^k - j)^2)\chi(j) \\ &= \sum_{j<p^k/2} j^2\chi(j) + \sum_{j<p^k/2} p^{2k}\chi(j) - \sum_{j<p^k/2} 2p^k j\chi(j) + \sum_{j<p^k/2} j^2\chi(j) \\ &= 2 \cdot \widehat{\mathrm{E}}(\chi) + p^{2k} \cdot \widehat{1}(\chi) - 2p^k \cdot \widehat{\mathrm{L}}(\chi) = 0. \qquad \square \end{aligned}$$

7. Proof of Theorem 2.2

We can now prove the main result.

Proof of Theorem 2.2. By Theorem 3.2 it suffices to show that $\check{\mathrm{L}}_{p^k}(\chi, i_\chi) \neq 0$ and $\check{\mathrm{E}}_{p^k}(\chi, i_\chi) \neq 0$, for all $\chi \in (\widehat{\mathcal{U}}_{p^k} : U)$.

If $\chi = 1$, then $i_\chi = k - 1$, and the character sums are nonzero because all the nonzero terms ($\mathrm{L}(up^{k-1})$ or $\mathrm{E}(up^{k-1})$) in the sum are positive.

For the sake of contradiction, suppose $\chi \neq 1$ is an even character with $\check{\mathrm{L}}_{p^k}(\chi, i_\chi) = 0$. (The argument when $\check{\mathrm{E}}_{p^k}(\chi, i_\chi) = 0$ is similar because of Proposition 6.4.) By Lemma 5.2, we have $\check{\mathrm{L}}_{p^{k-i_\chi}}(\chi^0, 0) = 0$ for a primitive even character χ^0 modulo p^{k-i_χ}. But $\check{\mathrm{L}}_{p^{k-i_\chi}}(\chi^0, 0) = \widehat{\mathrm{L}}_{p^{k-i_\chi}}(\chi^0)$, so the latter also vanishes.

By Proposition 6.6, $B_2(\chi^0) = 0$. This contradicts Proposition 4.1. Indeed, the vanishing of $B_2(\chi^0)$ implies $L(1-2, \chi^0) = L(-1, \chi^0) = 0$, while χ^0 being even and nontrivial implies $L(-1, \chi^0) \neq 0$. $\square$

8. A Relationship Between the Two Determinants

In the case where $k = 1$, so that $R = \mathbb{F}_p$, there is a nice relation between the two determinants Δ_{L} and Δ_{E}.

Theorem 8.1. *Let $R = \mathbb{F}_p$, p an odd prime. If 2 has order r in G_p, then*

$$(2^r + 1)^{\frac{p-1}{2r}}\ \Delta_{\mathrm{E}} = p^{\frac{p-1}{2}}\ \Delta_{\mathrm{L}}.$$

Proof. Let ω be either L or E. When $k = 1$, every character χ has $i_\chi = 0$, so that Theorem 3.2 yields

$$\Delta_\omega = C \prod_{\chi \in \widehat{G}_p} \widehat{\omega}(\chi).$$

By taking the product of (4) as χ varies over $\widehat{G}_p$, we obtain

$$\Delta_{\mathrm{E}} \prod_{\chi\in\widehat{G}_p} (\bar\chi(2)-4) = p^{(p-1)/2}\Delta_{\mathrm{L}} \prod_{\chi\in\widehat{G}_p} (\bar\chi(2)-2).$$

If ζ is a primitive rth root of 1 in $\mathbb{C}$, the polynomial t^r-1 factors:

$$t^r-1=\prod_{j=0}^{r-1}(t-\zeta^j).$$

Given the hypothesis that 2 has order r in G_p, $\bar\chi(2)$ is an rth root of 1, for any character $\chi\in\widehat{G}_p$. Indeed, the map $\widehat{G}_p\to\mathbb{C}^\times$, $\chi\mapsto\bar\chi(2)$, is a group homomorphism with image the group of rth roots of 1. Each rth root of 1 is the image of $|G_p|/r$ (the size of the kernel) characters. Thus

$$\prod_{\chi\in\widehat{G}_p} (t-\bar\chi(2)) = (t^r-1)^{|G_p|/r}.$$

By evaluating this expression at $t=4$ and $t=2$ and dividing, we have

$$\frac{\prod_{\chi\in\widehat{G}_p}(4-\bar\chi(2))}{\prod_{\chi\in\widehat{G}_p}(2-\bar\chi(2))} = \frac{(4^r-1)^{|G_p|/r}}{(2^r-1)^{|G_p|/r}} = (2^r+1)^{|G_p|/r},$$

from which the result follows. □

There is a similar, but more complicated, relation in the prime power case.

Theorem 8.2. *Let $R=\mathbb{Z}/p^k\mathbb{Z}$, with p an odd prime. Suppose 2 has order r_j in G_{p^j}, $j=1,\dots,k$. Then*

$$\Delta_{\mathrm{E}}\prod_{j=1}^{k}(2^{r_j}+1)^{\frac{(p-1)p^{j-1}}{2r_j}} = p^{k(p^k-1)/2}\Delta_{\mathrm{L}}. \tag{5}$$

Proof. The proof is similar to that of Theorem 8.1, using the general case of Theorem 3.2 and Lemma 5.2. □

Remark 8.3. When 2 is a primitive root modulo p^k, then $r_j=|G_j|=p^{j-1}(p-1)/2$. In that case, (5) becomes

$$\Delta_{\mathrm{E}}\prod_{j=1}^{k}(2^{p^{j-1}(p-1)/2}+1) = p^{k(p^k-1)/2}\Delta_{\mathrm{L}}.$$

For many other primes, the orders r_j satisfy $r_{j+1}=pr_j$, which also simplifies the exponents on the left side of (5). However, the pattern $r_{j+1}=pr_j$ is not true for all primes, e.g., the two known Wieferich primes, $p=1093$ and $p=3511$. For $p=1093$, $r_1=r_2=182$; for $p=3511$, $r_1=r_2=1755$.

Finally, when $p=2$, Theorem 3.2 and (3) imply the following relation.

Theorem 8.4. *Let $R=\mathbb{Z}/2^k\mathbb{Z}$. Then*

$$\Delta_{\mathrm{E}} = 2^{(k-1)2^{k-1}}\Delta_{\mathrm{L}}.$$

While we suspect that the MacWilliams extension theorem holds for the Lee and Euclidean weights for all integer residue rings $\mathbb{Z}/m\mathbb{Z}$, we do not have a proof in that generality. One of the missing ingredients is a useful criterion for the extension property that generalizes Theorem 3.2.

Added in proof: Serhii Dyshko, "The extension theorem for the Lee and Euclidean weight codes over integer residue rings" (under review), has proved the extension property for the Lee and Euclidean weights for all integer residue rings $\mathbb{Z}/m\mathbb{Z}$.

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