Example 1: Resultant of a distributed load

The diagram shows a cantilevered beam with a linearly varying load intensity. The maximum load intensity is $w$ (lb/ft) at the right end of the beam. To calculate the support forces at $A$, the distributed load can be replaced by a single resultant load $R$ acting at a distance $\ell$ from the left end.

The load $R$ and distance $\ell$ are found by solving the following equations

$$R = \int_0^L wx \, dx \quad \text{and} \quad R \times \ell = \int_0^L (wx) x \, dx = \int_0^L wx^2 \, dx \quad (1)$$

The first of Eq. (1) equates the resultant $force$ $R$ with the $summation$ of the $load$ $intensity$, and the second one equates the $moment$ of $R$ about $A$ with the $sum$ of the $moments$ of the load $intensity$.

Given: $L = 10$ (ft), $w = 100$ (lb/ft)

Find: (a) the resultant force $R$; and (b) the distance $\ell$ that it acts from the support.

Solution:
(a) The resultant load is

$$R = \int_0^{10} 10x \, dx = \left(5x^2\right)_0^{10} = 5 \times 100 = 500 \text{ (lb)} \quad \text{or} \quad R = \int_0^{10} 10x \, dx = \frac{1}{2} \times 10 \times 100 = 500 \text{ (lb)}$$

(b) The distance $\ell$ is

$$500 \ell = \int_0^{10} 10x^2 \, dx = \left(\frac{10}{3} x^3\right)_0^{10} \quad \Rightarrow \quad \ell = \frac{10 \times 1000}{3 \times 5000} = \frac{2}{3} (10) = 6\frac{2}{3} \text{ (ft)}$$

So, for a linearly varying load (starting at zero), the resultant acts $\frac{2}{3}$ of the way along the distributed load. If the load started at $w$ (lb/ft) at the wall, then the resultant would be located $\frac{1}{3}$ of the way along the load.
Example 2:
Given: \( L = 10 \text{ (ft)} \), \( w = 100 \text{ (lb/ft)} \)

Find: (a) the resultant force \( R \); and (b) the distance \( \ell \) that it acts from the support.

Solution:
(a) The resultant load is
\[
R = \int_{0}^{10} 100 \, dx = 10 \times 100 = 1000 \text{ (lb)}
\]
(b) The distance \( \ell \) is
\[
1000 \ell = \int_{0}^{10} 100x \, dx = \frac{1}{2} \times 10 \times 1000 \quad \Rightarrow \quad \ell = \frac{5000}{1000} = \frac{1}{2}(10) = 5 \text{ (ft)}
\]
So, for a uniformly distributed load, the resultant acts \( \frac{1}{2} \) of the way along the load.

Example 3:
The diagram of the internal shearing force for the simply supported beam with a concentrated load is shown. The internal bending moment is related to the shearing force by the equation
\[
M(x) = \int V(x) \, dx
\]

Given: \( P = 100 \text{ (lbs)} \), \( L = 5 \text{ (ft)} \),
\( a = 3.5 \text{ (ft)} \), \( b = 1.5 \text{ (ft)} \), and
\( M(0) = M(L) = 0 \)

Find: (a) the moment diagram for the beam; and (b) \( M_{\text{max}} \) the maximum bending moment in the beam.
Solution:
(a) We can construct the moment diagram from the shear diagram. Where the shear is constant, the moment varies linearly with \( x \).

\[
V_a = \frac{bP}{L} = 1.5 \times 100/5 = 30 \text{ (lb)} \quad \text{and} \quad V_b = -\frac{aP}{L} = -3.5 \times 100/5 = -70 \text{ (lb)}
\]

\[
M(x) = \int 30 \, dx = 30x + D = 30x \quad \text{for} \quad 0 \leq x \leq 3.5 \quad \text{(recall that} \quad M(0) = 0 \quad )
\]

\[
M(x) = \int -70 \, dx = -70x + D = -70x + 350 \quad \text{for} \quad 3.5 \leq x \leq 5 \quad \left( M(3.5) = 105 \text{ (ft-lb)} \right)
\]

(b) The maximum bending moment occurs at the concentrated load. It is the area under the shear diagram from 0 to 3.5 feet.

\[
M_{\text{max}} = 30 \times 3.5 = 105 \text{ (ft-lb)}
\]
Example 4:

Given: \( L = 10 \) (ft), \( w = 100 \) (lb/ft), 
\( M(0) = M(L) = 0 \), and 
\[ V(x) = 500 - 100x \text{ (lb)} \quad (0 \leq x \leq L) \]

Find: (a) the bending moment diagram; 
and (b) the maximum bending moment.

Solution:

(a) Again, we can construct the moment 
diagram from the shear diagram. Where 
the shear is varies \textit{linearly}, the moment 
varies \textit{quadratically} with \( x \).

\[
M(x) = \int (500 - 100x) \, dx \\
= 500x - 50x^2 + D \quad (M(0) = 0) \\
= 500x - 50x^2 \text{ (ft-lb)}
\]

(b) The maximum bending moment occurs at 
the midpoint of the beam. It is the \textit{area under 
the shear diagram} from 0 to \( L/2 \).

\[
M_{\text{max}} = \frac{1}{2} \times 5 \times 500 = 1250 \text{ (ft-lb)}
\]
Example 5:

Given:  \( L = 10 \) (ft), \( w = 100 \) (lb/ft), 
\( M(L) = 0 \), and 
\[ V(x) = 1000 - 100x \text{ (lb)} \quad (0 \leq x \leq L) \]

Find: (a) the bending moment diagram; and (b) the maximum bending moment.

Solution:

(a) In this case, the shear varies \textit{linearly} with \( x \), so the moment will vary \textit{quadratically} with \( x \). Given \( M(10) = 0 \), we find

\[
M(x) = \int (1000 - 100x) \, dx \\
= 1000x - 50x^2 + D \\
= -5000 + 1000x - 50x^2 \text{ (ft-lb)}
\]

(b) The maximum moment occurs at the left end of the beam and is equal to the area under the shear diagram from 0 to \( L \).

Why?

\[
M_{\text{max}} = \frac{1}{2} \times 10 \times 1000 = 5000 \text{ (ft-lb)}
\]