ENGR 1990 Engineering Mathematics
Application of Quadratic Equations in Electric Circuits

Example #1

Given: A 100 watt light bulb is connected in series with a resistor $R = 10$ (ohms). The applied voltage is $V = 120$ (volts). The power used by the light bulb is calculated as $P_L = V_L I$. The units of power are “volt-amps” or “watts.”

Find: The current $I$ in amperes.

Solution:
To find the current, we first apply Kirchhoff’s voltage law to the circuit.

\[ V = 120 = V_R + V_L \]  \hspace{1cm} (1)

Then, we represent the voltage drops across the resistor and the light bulb in terms of the current $I$.

Ohm’s law: $V_R = R I = 10 I$ \hspace{1cm} (2)

Power equation: $P_L = V_L I \Rightarrow V_L = P_L / I = 100 / I$ \hspace{1cm} (3)

Substituting from equations (2) and (3) into equation (1), gives

\[ 120 = (10 I) + \left(\frac{100}{I}\right) \Rightarrow 120 I = 10 I^2 + 100 \Rightarrow 10 I^2 - 120 I + 100 = 0 \]  \hspace{1cm} (4)

Quadratic formula
Using the quadratic formula, the roots of equation (4) are calculated as follows

\[ I,_{1,2} = \frac{120 \pm \sqrt{120^2 - (4 \times 10 \times 100)}}{2 \times 10} \approx \frac{120 \pm 101.98}{20} = \begin{cases} 11.1 \text{ (amps)} \\ 0.901 \text{ (amps)} \end{cases} \]  \hspace{1cm} (5)

Completing the Square
To complete the square, we first divide equation (4) by 10: $I^2 - 12 I + 10 = 0$

\[ I^2 - 12 I + \left(\frac{12}{2}\right)^2 = -10 + \left(\frac{12}{2}\right)^2 \Rightarrow I^2 - 12 I + 6^2 = -10 + 6^2 \]

\[ \Rightarrow (I - 6)^2 = 26 \Rightarrow I - 6 = \pm \sqrt{26} \approx \pm 5.09902 \]

\[ I \approx 6 \pm 5.09902 = \begin{cases} 11.1 \text{ (amps)} \\ 0.901 \text{ (amps)} \end{cases} \]
Factoring
As the roots are not integers, it is not easy factor the quadratic equation directly. However, given the above results, we recognize that

\[ I^2 - 12I + 10 \approx (I - 11.1)(I - 0.901) \]

Back to our circuit
How is it that our circuit can have two different currents?

**Answer:** Actually, it does not. The power of the light bulb is **rated** at a **specific voltage**, so only one of the currents is correct for a given light bulb.

Let’s calculate the voltage and resistance of the light bulb at each of these currents.

a) \( I = 0.901 \) (amps) \( \Rightarrow \frac{V_L}{R} = \frac{P_L}{I} = \frac{100}{0.901} = 111 \) (volts)
\( \Rightarrow \frac{V_L}{I} = \frac{111}{0.901} = 123 \) (ohms)

b) \( I = 11.1 \) (amps) \( \Rightarrow \frac{V_L}{R} = \frac{P_L}{I} = \frac{100}{11.1} = 9.01 \) (volts)
\( \Rightarrow \frac{V_L}{I} = \frac{9.01}{11.1} = 0.812 \) (ohms)

**Summary** @ \( V = 120 \) (volts) and \( R = 10 \) (ohms)

<table>
<thead>
<tr>
<th>( I ) (amps)</th>
<th>( V_R = \frac{V_R}{R} ) (volts)</th>
<th>( P_R = \frac{V_R}{I} ) (watts)</th>
<th>( V_L = \frac{V_L}{V - V_R} ) (volts)</th>
<th>( R_L = \frac{V_L}{I} ) (ohms)</th>
<th>( P_L = \frac{P_L}{V_L} ) (watts)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.901</td>
<td>9.01</td>
<td>8.12</td>
<td>111</td>
<td>123</td>
<td>100</td>
</tr>
<tr>
<td>11.1</td>
<td>111</td>
<td>1232</td>
<td>9</td>
<td>0.812</td>
<td>100</td>
</tr>
</tbody>
</table>

**Note:** The resistance of an average 100 watt light bulb at 120 volts is about 140 ohms.

**Example #2**

**Given:** The electric circuit shown has two resistors connected in **parallel**. At node \( J \) the current \( I \) splits into two parts, \( I_1 \) and \( I_2 \), and at node \( K \) the currents recombine to form \( I \). The splitting and combining of currents at a node obeys **Kirchhoff’s Current Law** which states:

The **sum** of the currents **flowing into** a node **equals** to the sum of the currents **flowing away** from a node. (In this case, \( I = I_1 + I_2 \).)
Using this law, it can be shown that the two parallel resistors \( R_1 \) and \( R_2 \) act as a single equivalent resistor \( R_{eq} \).

\[
R_{eq} = \frac{R_1 R_2}{R_1 + R_2} \quad (6)
\]

Finally, it is known that \( R_{eq} = 150 \) (ohms) and \( R_2 = R_1 + 200 \).

Find: The resistances \( R_1 \) and \( R_2 \).

Solution:
Substituting the known information into equation (6) gives

\[
R_{eq} = 150 = \frac{R_1 R_2}{R_1 + R_2} = \frac{R_1 (R_1 + 200)}{R_1 + (R_1 + 200)} = \frac{R_1^2 + 200R_1}{2R_1 + 200}
\]

Clearing fractions and combining terms gives

\[
150(2R_1 + 200) = R_1^2 + 200R_1 \quad \Rightarrow \quad R_1^2 + (200 - 300)R_1 - (150 \times 200) = 0
\]

Using the quadratic formula, we find

\[
R_1 = \frac{100 \pm \sqrt{100^2 - 4(-30000)}}{2} = \frac{100 \pm 360.55}{2} = \begin{cases} 230.28 \\ -130.28 \end{cases}
\]

As the value of resistance cannot be negative, we find \( R_1 = 230 \) (ohms) and \( R_2 = 430 \) (ohms).