ENGR 1990 Engineering Mathematics
Application of Trigonometric Functions in Mechanical Engineering: Part II

Problem: Find the coordinates of the end-point of a two link planar robot arm.

Given: The lengths of the links $OA$ and $AB$ and the angles $\theta_1$ and $\theta_2$.

Find: The $XY$ coordinates of the end point $B$.

Solution:

The coordinates of $B$ may be found by adding the coordinates of $A$ relative to $O$ and the coordinates of $B$ relative to $A$.

$$x = x_1 + x_2 = \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2)$$
$$y = y_1 + y_2 = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2)$$

Example 1:

Given: The lengths and angles of a two link planar robot are $\ell_1 = 3$ (ft), $\ell_2 = 2$ (ft), $\theta_1 = 30$ (deg), and $\theta_2 = 60$ (deg).

Find: The Cartesian coordinates $x$ and $y$ of $B$ using: a) a calculator, and b) the values listed above for commonly used angles.

Solution:

a) Using a calculator to evaluate the sine and cosine functions:

$$x = \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) = (3 \times \cos(30)) + (2 \times \cos(60)) = 2.5981 + 1 = 3.5981 \text{ (ft)}$$
$$y = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = (3 \times \sin(30)) + (2 \times \sin(60)) = 1.5 + 1.7321 = 3.2321 \text{ (ft)}$$

b) Using the values for commonly used angles:

$$x = \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) = \left(3 \times \frac{\sqrt{3}}{2}\right) + \left(2 \times \frac{1}{2}\right) = 2.5981 + 1 = 3.5981 \text{ (ft)}$$
$$y = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = \left(3 \times \frac{1}{2}\right) + \left(2 \times \frac{\sqrt{3}}{2}\right) = 1.5 + 1.7321 = 3.2321 \text{ (ft)}$$
Example 2:

**Given:** The lengths and angles of a two link planar robot are $\ell_1 = 3$ (ft), $\ell_2 = 2$ (ft), $\theta_1 = 30$ (deg), and $\theta_2 = 120$ (deg).

**Find:** The Cartesian coordinates $x$ and $y$ of $B$ using: a) a calculator, and b) the values listed above for commonly used angles.

**Solution:**

a) Using a calculator to evaluate the sine and cosine functions:

\[
\begin{align*}
  x &= \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) = (3 \times \cos(30)) + (2 \times \cos(120)) = 2.5981 - 1 = 1.5981 \text{ (ft)} \\
  y &= \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = (3 \times \sin(30)) + (2 \times \sin(120)) = 1.5 + 1.7321 = 3.2321 \text{ (ft)}
\end{align*}
\]

b) Using the values for commonly used angles: Note first that $120 = 180 - 60$ (deg), so

\[
\begin{align*}
  \cos(120) &= -\cos(60) = -\frac{1}{2} \quad \text{and} \quad \sin(120) = \sin(60) = \frac{\sqrt{3}}{2} \\
  x &= \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) = \left(3 \times \frac{\sqrt{3}}{2}\right) + \left(2 \times (-\frac{1}{2})\right) = 2.5981 - 1 = 1.5981 \text{ (ft)} \\
  y &= \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) = \left(3 \times \frac{1}{2}\right) + \left(2 \times \frac{\sqrt{3}}{2}\right) = 1.5 + 1.7321 = 3.2321 \text{ (ft)}
\end{align*}
\]
Inverse Problem: Find the angles of the links of the robot arm given the endpoint position.

Given: The $XY$ coordinates of the end point $B$ and the lengths of the links $OA$ and $AB$.

Find: The link angles $\theta_1$ and $\theta_2$.

Solution:
First, we can calculate the length $r$ using the Pythagorean Theorem.

$$r = \sqrt{x^2 + y^2}$$

Then, we can apply the law of cosines to triangle $OAB$ to find the angle $\alpha$.

$$\ell_2^2 = \ell_1^2 + r^2 - 2 \ell_1 r \cos(\alpha)$$
or

$$\alpha = \cos^{-1}\left(\frac{\ell_1^2 + r^2 - \ell_2^2}{2 \ell_1 r}\right)$$

We can apply the law of cosines again to find the angle $\beta$.

$$r^2 = \ell_1^2 + \ell_2^2 - 2 \ell_1 \ell_2 \cos(\beta) \quad \Rightarrow \quad \beta = \cos^{-1}\left(\frac{\ell_1^2 + \ell_2^2 - r^2}{2 \ell_1 \ell_2}\right)$$

Finally, the link angles may now be found by noting

a) $\tan(\theta_1 + \alpha) = y / x \quad \Rightarrow \quad \theta_1 = \tan^{-1}(y / x) - \alpha$

b) $\theta_2 - \theta_1 = \pi - \beta \quad \Rightarrow \quad \theta_2 = \pi - \beta + \theta_1$
Example 3:

Given: The $XY$ coordinates of the end point $B$ and the lengths of the links $OA$ and $AB$ are $x = 1.5$ (ft), $y = 3.5$ (ft), $\ell_1 = 3$ (ft), and $\ell_2 = 2$ (ft).

Find: The link angles $\theta_1$ and $\theta_2$.

Solution:
Following the approach outlined above,

a) \[ r = \sqrt{x^2 + y^2} = \sqrt{1.5^2 + 3.5^2} = 3.8079 \text{ (ft)} \]

b) \[ 2^2 = 3^2 + 3.8079^2 - 2 \times 3 \times 3.8079 \cos(\alpha) \]
\[
\Rightarrow \alpha = \cos^{-1}\left(\frac{3^2 + 3.8079^2 - 2^2}{2 \times 3 \times 3.8079}\right) = \begin{cases} 31.41 \text{ (deg)} \\ 0.5481 \text{ (rad)} \end{cases}
\]

c) \[ r^2 = 3^2 + 2^2 - (2 \times 3 \times 2) \cos(\beta) \Rightarrow \beta = \cos^{-1}\left(\frac{3^2 + 2^2 - 3.8079^2}{2 \times 3 \times 2}\right) = \begin{cases} 97.18 \text{ (deg)} \\ 1.6961 \text{ (rad)} \end{cases} \]

d) \[ \tan(\theta_1 + .5481) = 3.5 / 1.5 \Rightarrow \theta_1 = \tan^{-1}(3.5 / 1.5) = \begin{cases} 35.40 \text{ (deg)} \\ 0.6178 \text{ (rad)} \end{cases} \]
\[
\theta_2 - \theta_1 = \pi - \beta \Rightarrow \theta_2 = \pi - 1.6961 + 0.6178 = \begin{cases} 118.2 \text{ (deg)} \\ 2.0633 \text{ (rad)} \end{cases} \]

Check:
We can now use the calculated link angles to check the position of the endpoint. Does it match our required position?

\[
x = \ell_1 \cos(\theta_1) + \ell_2 \cos(\theta_2) \\
= (3 \times \cos(0.6178)) + (2 \times \cos(2.0633)) = 2.4455 - 0.9457 = 1.4998 \approx 1.5 \text{ (ft)} \]

\[
y = \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) \\
= (3 \times \sin(0.6178)) + (2 \times \sin(2.0633)) = 1.7377 + 1.7623 = 3.5 \text{ (ft)} \]
Elbow-down and Elbow-up Positions

Note that the above answers could be interpreted in two ways, the elbow-down position and the elbow-up position as illustrated in the following diagrams.

Note on calculator usage:

When calculating $\sin^{-1}(\theta)$, $\cos^{-1}(\theta)$ and $\tan^{-1}(\theta)$, your calculator will place the results in specific quadrants as outlined in the table. So, your calculator does not always place the angle into the correct quadrant.

Note that in the above example, we used the law of cosines (and hence $\cos^{-1}(\theta)$) to calculate the angles of the triangle $OAB$, and our calculator gave angles in the range $0 \leq \theta \leq \pi$. What if we had used the law of sines to calculate the angle $\beta$?

$$\frac{\sin(\beta)}{r} = \frac{\sin(\alpha)}{\ell_2}$$

$$\Rightarrow \quad \beta = \sin^{-1}\left(\frac{r\sin(\alpha)}{\ell_2}\right) = \sin^{-1}\left(\frac{3.8079 \times \sin(0.5481)}{2}\right) = \left\{ \begin{array}{l} 82.79 \text{ (deg)} \\ 1.4449 \text{ (rad)} \end{array} \right.$$ 

Note that this is not the correct result. As we know from our work above, the correct result is in the second quadrant. So, $\beta = \pi - 1.4449 = 1.6967 \text{ (rad)}$. This is very close to the result we found above.