ME 2580 Dynamics  
Principle of Work and Energy  

Recall that Newton’s second law for a particle may be written as \( \vec{F} = m \vec{a} \), where \( \vec{F} \) is the resultant force acting on the particle and \( \vec{a} \) is its acceleration. Recall also that the force and the acceleration are generally not tangent to it’s path, except in the case of rectilinear motion.  

We can develop an integrated form of Newton’s second law that relates the work done on the particle to the change in its kinetic energy as follows:

**Derivation:**

Taking the scalar product of both sides of Newton’s law with the velocity \( \vec{v} \) gives

\[
\vec{F} \cdot \vec{v} = m (\vec{a} \cdot \vec{v}) = m \left( \frac{d\vec{v}}{dt} \cdot \vec{v} \right) = \frac{1}{2} m \frac{d}{dt} (\vec{v} \cdot \vec{v})
\]

**Integrating** both sides of this equation with respect to time gives

\[
\int_{t_1}^{t_2} (\vec{F} \cdot \vec{v}) dt = \frac{1}{2} m \int_{t_1}^{t_2} \frac{d}{dt} (\vec{v} \cdot \vec{v}) dt = \frac{1}{2} m (\vec{v} \cdot \vec{v}) |_{t_1}^{t_2} = K_2 - K_1 = \Delta K.E.
\]

where

\[
\int_{t_1}^{t_2} (\vec{F} \cdot \vec{v}) dt = \int_{t_1}^{t_2} \left( \vec{F} \cdot \frac{d\vec{r}}{dt} \right) dt = \oint \vec{F} \cdot d\vec{r} = U_{1\rightarrow 2} \quad \text{Work done by forces moving particle from position 1 to position 2}
\]

Note that in the last equation, \( \vec{F} \cdot \vec{v} \) represents the power of the resultant force \( \vec{F} \).  

The above results can be generalized to a system of particles to give

\[
\sum U_{1\rightarrow 2} = \sum \Delta K.E. \quad \text{or} \quad \sum K_1 + \sum U_{1\rightarrow 2} = \sum K_2
\]

where \( \sum U_{1\rightarrow 2} \) represents the total work done by all forces acting on the system, and \( \sum \Delta K.E. \) represents the sum of the changes in kinetic energies of all the particles in the system.
Work Done by Common Forces

Forces Tangent to Path:

As a particle moves from position 1 to position 2, the work done by a force $F_t$ that is tangent to the path of motion can be written as

$$U_{1\rightarrow 2} = \oint F_t \cdot ds = \int_{s_i}^{s_f} (F_t \cdot \hat{t}) \cdot ds = \int F_t ds$$

If the magnitude $F_t$ is constant, then the integration process gives $U_{1\rightarrow 2} = F_t (s_2 - s_1) = F_t \Delta s$.

Weight Forces:

As a particle moves from position 1 to position 2, the work done by its weight force $W = -mg \hat{j}$ can be written as

$$U_{1\rightarrow 2} = \int W \cdot dr = \int (-mg \hat{j}) \cdot (dx \hat{i} + dy \hat{j}) = -mg \int_{y_1}^{y_2} dy$$

$$U_{1\rightarrow 2} = -mg \int_{y_1}^{y_2} dy = -mg (y_2 - y_1) = -mg \Delta y$$

The work is positive when the particle moves down and negative when it moves up.

Linear Spring Forces:

As a particle moves from position 1 to position 2, the work done by a spring force $F_s = -k(r - r_o) \hat{r}$ can be written as

$$U_{1\rightarrow 2} = \oint F_s \cdot dr = \int (-k(r - r_o) \hat{r}) \cdot (dr \hat{r} + rd\theta \hat{\theta})$$

$$U_{1\rightarrow 2} = \int_{r_1}^{r_2} -k(r - r_o) dr = \int_{e_1}^{e_2} -k e de \quad \{e = (r - r_o) = \text{elongation}\}$$

$$U_{1\rightarrow 2} = -\frac{1}{2} k (e_2^2 - e_1^2)$$