ME 3600 Control Systems
Characteristics of Open-Loop and Closed-Loop Systems

Important Control System Characteristics

- Sensitivity of system response to parametric variations
- Transient and steady-state response
- Steady-state error
- Response of system to disturbances (disturbance response)

Parametric Sensitivity: Concept

- Mathematically speaking, whenever a dynamic system is subjected to one specific input, its response is always the same. For example, the step response of a system as calculated by MATLAB is always the same (unless the transfer function is altered).

- For real systems, however, this is not the case. Each time a real system is subjected to an input, its response will vary. The variations may be small random fluctuations producing the same average response, or they may be large fluctuations producing very different responses. For example, the effects of friction and damping may vary during day-to-day operation of the system.

- To compare the effects of these variations on the response of open-loop and closed-loop systems, consider the systems shown in Figure 1. In each system, the transfer function $G(s)$ has been changed to $G(s) + \Delta G(s)$.

Figure 1. Open-Loop and Closed-Loop Systems with Plant Variations

- Open-loop system: $Y(s) + \Delta Y(s) = (G(s) + \Delta G(s))R(s) = G(s)R(s) + \Delta G(s)R(s)$

$$\Delta Y(s) = \Delta G(s)R(s)$$  (1)

The plant changes are seen to be passed directly to the system output.
○ Closed-loop system: For this system, it can be shown that

\[
\Delta Y(s) \approx \left( \frac{\Delta G(s)}{(1 + GH(s))^2} \right) R(s)
\]

(2)

From this result it clear that the amount of \( \Delta G(s) \) that is passed to the output depends on the magnitude of the loop (or open-loop) transfer function \( GH(s) \). The larger the magnitude of \( GH(s) \), the less changes in \( G(s) \) will affect the system response.

**Sensitivity: Calculation**

○ The sensitivity of a system with transfer function \( T(s) \) to changes in a parameter \( \alpha \) is defined as

\[
S^T_\alpha = \frac{\alpha}{T} \left( \frac{\partial T}{\partial \alpha} \right)
\]

(3)

○ If the magnitude of \( S^T_\alpha \) is between zero and one \( (0 < |S^T_\alpha| < 1) \), then the effects of \( \Delta G(s) \) will be lowered (i.e. 10% changes in \( G(s) \) will result in less than 10% changes in the response). However, if the magnitude of \( S^T_\alpha \) is greater than one \( (|S^T_\alpha| > 1) \), then the effects of \( \Delta G(s) \) will be magnified (i.e. 10% changes in \( G(s) \) will result in greater than 10% changes in the response).

○ To gain some general insight into the issue of sensitivity for a simple closed loop system (Fig. 2), consider the sensitivity of the system transfer function \( T(s) = \frac{G}{1 + GH} \) to bulk changes in the transfer functions \( G(s) \) or \( H(s) \).

\[
S^T_G = \frac{G}{T} \left( \frac{\partial T}{\partial G} \right) \quad \text{and} \quad S^T_H = \frac{H}{T} \left( \frac{\partial T}{\partial H} \right)
\]

Using these definitions, it can be shown that

\[
S^T_G = \frac{1}{1 + GH} \quad \text{and} \quad S^T_H = \frac{-GH}{1 + GH}
\]

(4)
The first of Eq (4) indicates that as $|GH(s)|$ is increased, the effects on the response of the system to changes in $G(s)$ are lowered. (This is the same conclusion that was drawn from Eq (2).) However, the second of Eq (4) indicates that as $|GH(s)|$ is increased, the effects on the response of the system to changes in $H(s)$ are passed directly to the output ($S_H^T \approx 1$).

Note also that for an open loop system with transfer function $T(s) = G(s)$, 

$$S_T^G = S_G^G = \frac{G}{G\left(\frac{\partial G}{\partial G}\right)} = 1.$$ 

Control of Transient Response

In previous notes the block diagram for the open-loop response of an armature controlled DC motor was given. If the electrical response of the motor is much faster than the speed changes, then the time dependence of the circuitry can be ignored. Under these conditions, the block diagram reduces to that shown in Fig 3.

Fig 3. Block Diagram of an Armature-Controlled DC Motor

As before, the input to the motor is the armature voltage $V_a(s)$ and the output is the angular velocity (speed) of the motor. Using block diagram reduction, the open-loop transfer for this system is found to be

$$\frac{\omega(s)}{V_a(s)} = \frac{K^*}{s + a^*} \left\{ \begin{array}{c} K^* = K_{ma}/R_aJ \\ a^* = (R_a c + K_b K_{ma})/R_aJ \end{array} \right.$$ (5)

The parameters $K^*$ and $a^*$ are constants that depend on characteristics of the motor, the inertial load, and the damping coefficient. Note that the value of $a^*$ determines how quickly the motor responds when a step increase in voltage is applied to the motor.
To study the effects of feedback on transient response, consider proportional, closed-loop control of the DC motor as shown in Fig 4. The input to the system is the desired angular velocity $\omega_d(s)$ and the output of the system is the actual angular velocity $\omega(s)$. The parameter $K_t$ is the calibration constant of the tachometer that relates changes in angular velocity to changes in voltage. The signal $E(s)$ represents a tachometer voltage error, and the parameter $K_a$ is the proportional gain.

\[ \omega_d(s) \quad \text{+} \quad E(s) \quad \text{+} \quad V_a(s) \quad \text{+} \quad K^* \quad \frac{s + a^*}{s + a} \quad \omega(s) \]

Fig 4. Proportional Control of a DC Motor Using a Tachometer

Using block diagram reduction, the transfer function of the closed-loop system is

\[
\frac{\omega(s)}{\omega_d(s)} = \frac{K_t \hat{K}}{s + \hat{a}} \quad \begin{cases} \hat{K} = K_a K^* \\ \hat{a} = a^* + K_a K^* K_t \end{cases}
\]

(6)

The parameter $\hat{a}$ determines the speed of response of the closed-loop, speed control system. The value of $\hat{a}$ can be increased by increasing the proportional gain $K_a$. Note, however, that if the value of $K_a$ is increased too much, the voltage input to the motor may become too large, potentially causing harm to the motor.

Control of Steady-State Error

Consider again the closed-loop speed control system of Fig 4. To track the error in the system as it responds to a commanded speed change, the error signal $E(s)$ is taken to be the output of the system as shown in Fig 5.

\[ \omega_d(s) \quad \text{+} \quad E(s) \quad \text{+} \quad \frac{K_a K^* K_t}{s + a^*} \]

Fig 5. Error of a DC Motor Speed Control System
Using block diagram reduction, the system error transfer function is found to be

\[
\frac{E}{\omega_d}(s) = \frac{K_i(s + a^*)}{s + a^* + K_aK_i} = \frac{K_i(s + a^*)}{s + \hat{a}}
\]  

(7)

Using the final value theorem, the steady-state error to a unit step, speed change command is found to be

\[
e_{ss} = \lim_{s \to 0} \left( E \cdot \frac{s}{s_d} \cdot \frac{1}{s} \right) = \frac{K_i \hat{a}^*}{\hat{a}}
\]  

(8)

This result shows that the amount of steady-state error is affected by the proportional control gain \( K_a \). As the value of \( K_a \) is increased, the value of \( \hat{a} \) is increased and \( e_{ss} \) the steady-state error is decreased.

Control of Disturbance Response

Consider the block diagram of an armature-controlled DC motor with a disturbance torque \( T_D(s) \) as shown in Figure 6. It is assumed that the disturbance torque reduces the torque generated by the motor under ideal conditions.

Fig 6. Block Diagram of an Armature-Controlled DC Motor with Disturbance

To study the effect of the disturbance on the response of this system, the disturbance transfer function must be found. One way to do this is to move the disturbance to the left-most summing block as shown in Fig 7.

Fig 7. Block Diagram of a DC Motor with only Disturbance Input
The disturbance transfer function for the open-loop system is then identified to be

\[
\frac{\omega}{T_D}(s) = -\frac{R_a K^*}{K_{ma} K_a} \frac{s + \hat{a}}{s + a^*}
\]  
(open loop system)  

Following this same approach, the disturbance transfer function of the closed-loop, speed control system of Fig 4 can be found. In that case, to move the disturbance to the left-most summing block, the disturbance must be additionally moved over the proportional gain block as shown in Fig 8.

Using block diagram reduction, the disturbance transfer function for the closed-loop speed control system is found to be

\[
\frac{\omega}{T_D}(s) = -\frac{R_a \hat{K}}{K_{ma} K_a} \frac{s + \hat{a}}{s + \hat{a}} = -\frac{1}{J}  
\]  
(closed-loop, speed control system)  

The steady-state angular velocity change of the motor to a unit step disturbance torque is found using the final value theorem.

\[
(\omega_{ss})_{T_D} = \lim_{s \to 0} \left( \chi \cdot \frac{\omega}{T_D} \cdot \frac{1}{s} \right) = -\frac{1}{J\hat{a}}  
\]  
(closed-loop, speed control system)  

These last two results indicate that as the proportional gain $K_a$ is increased, the disturbance response decays faster and the steady-state angular velocity change is decreased. Both are positive effects.