Proportional Control

Proportional control of a simple hydraulic actuator is shown below. The system has two parameters, $K$ and $p$. The parameter $K$ is the proportional gain, and the parameter $p$ represents how quickly the actuator gets to full speed.

Problem: Select the parameters $K$ and $p$ so the closed loop system has:

a) As fast a response as possible with less than or equal to 5% overshoot, and
b) A settling time, $T_s \leq 4$ (sec).

Solution:

1. The closed loop transfer function is second order, $\frac{Y(s)}{R(s)} = \frac{K}{s^2 + ps + K}$. For as fast a response as possible with less than or equal to 5% overshoot, choose $\zeta = 0.7$.

2. For a settling time, $T_s \leq 4$ (sec), set $T_s = \frac{4}{\zeta \omega_n} \leq 4$ (sec), or for the slowest response, set $[\zeta \omega_n = 1]$. Given $\zeta = 0.7$, then $\omega_n = 1.4286$ (rad/s).

3. Hence, the system parameters are $p = 2\zeta \omega_n = 2$ and $K = \omega_n^2 = 2.04$.

4. The error transfer function of this system is $\frac{E(s)}{R(s)} = \frac{s(s + p)}{s^2 + ps + K}$. As a type 1 system clearly there is zero steady state error for a step input, and the steady state error for a unit ramp input is

$$e_{ss} = \lim_{s \to 0} \left( s \cdot \frac{1}{s^2} \cdot \frac{E(s)}{R(s)} \right) = \frac{p}{K}.$$

5. Note that since this is a type 1 second order system, this also represents ITAE optimal step response.
6. Step and Ramp responses:

Proportional/Derivative Control

The system from above is shown here with $p = 2$ and a $PD$ controller with a zero at $s = -5$.

Problem:

a) Find the values of $K$ for which the closed loop system has a damping ratio of $\zeta = 0.7$.

b) Find $e_{ss}$ the steady state error of the system for a ramp input.

c) Plot the step and ramp responses of the system for the values of $K$ found in part (a).

Solution:

1. The closed loop transfer function of this system is second order with a zero. Note that the gain $K$ now effects the stiffness and damping of the closed loop system.

$$\frac{Y(s)}{R(s)} = \frac{K(s + 5)}{s^2 + (K + 2)s + 5K}$$

2. To find the values of $K$ for which the closed loop system has a damping ratio of $\zeta = 0.7$, we set $2\zeta \omega_n = K + 2$ and $\omega_n^2 = 5K$. Solving these two equations simultaneously gives $K = 0.8$ and $K = 5$. 

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3. The error transfer function of this system is
\[
E(s) = \frac{s(s + 2)}{R(s) + \frac{s^2 + (K + 2)s + 5K}{s}}
\] As a type 1 system clearly there is zero steady state error for a step input, and the steady state error for a unit ramp input is
\[
e_{ss} = \lim_{s \to 0} \left( s \cdot \frac{1}{s^2} \cdot \frac{E(s)}{R(s)} \right) = \frac{2}{5K}
\]

4. Step and Ramp responses:

Proportional/Integral Control

The system from above is shown here with \( p = 2 \) and a PI controller with a zero at \( s = -0.3 \).

Problem: a) Find the values of \( K \) for which the complex poles of the closed loop system have a damping ratio of \( \zeta = 0.7 \).

b) Find \( e_{ss} \) the steady state error of the system for a ramp input.

c) Plot the step and ramp responses of the system for the values of \( K \) found in part (a).
Solution:

1. The closed loop transfer function of this system is \textit{third order with a zero}.

\[
\frac{Y(s)}{R(s)} = \frac{K(s + 0.3)}{s^3 + 2s^2 + Ks + 0.3K}
\]

2. This system is of 3\textsuperscript{rd} order, so it is more difficult to find the values of $K$ for which the complex poles of the closed loop system have a damping ratio of $\zeta = 0.7$. After some trial and error, the values of $K$ are found to be $K = 1.27$ and $K = 1.9$. Complex poles have a damping ratio of $\zeta = 0.7$ when their real and imaginary parts are equal. (More about this later…)

3. The error transfer function of this system is

\[
\frac{E(s)}{R(s)} = \frac{s^2(s + 2)}{s^3 + 2s^2 + Ks + 0.3K}
\]

As a type 2 system clearly there is \textit{zero steady state error for both step and ramp inputs}.

4. This transfer function has the \textit{form to be optimized for ramp input}. Using the location of the zero of the controller as the second parameter, we set

\[
s^3 + 2s^2 + Ks + Kz = s^3 + 1.75\omega_n s^2 + 3.25\omega_n^2 s + \omega_n^3
\]

Comparing coefficients, for optimal response, we need $K = 4.245$ and $z = 0.3374$.

5. Step and Ramp responses:
6. ITAE Optimal Ramp Response and Corresponding Step Response: