ME 3600 Control Systems
Mathematical Models of Physical Systems

- The analysis and design of control systems requires that we have *quantitative mathematical models* of the physical systems we want to control.

- A mathematical model may consist of *differential and/or algebraic equations*. The solution of these equations describes the dynamics of the system, that is, how the system responds to its expected input.

- A system may consist of a *single component*, or it may consist of *many different types of components* – mechanical, electrical, hydraulic, thermal, etc.

- A mathematical model may be *linear* or *nonlinear* depending on the system and the range of operation that is being modeled.

- If a system is nonlinear, it may be possible to *linearize* the model before applying linear system analysis. The extent to which this approach is applicable depends on the *strength and type of nonlinearities*.

- Mathematical models may be developed using *physical principles*. Using this approach, the analyst writes the differential and/or algebraic equations that are thought to describe the system dynamics.

- Laplace transforms are used to convert the *differential equations* into *transfer functions*.

- This approach is *limited* by the analyst’s ability to:
  - describe the physics of the system (especially for complex systems), and
  - estimate all important parameters.

![Figure 1. Spring-Mass-Damper System](image-url)

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Example: Spring-Mass-Damper System

Nomenclature

\( m \): mass of the block
\( k \): spring stiffness
\( c \): coefficient of the damper
\( R(t) \): external force (input)
\( \ell_u \): unstretched length of spring
\( x_{eq} \): equilibrium position of mass
\( x \): position of mass relative to the equilibrium position (output)
\( \dot{x} \): velocity of mass
\( \ddot{x} \): acceleration of mass

Static Equilibrium

Using the equations of static equilibrium, it can be shown that the equilibrium position of the system under its own weight is

\[
x_{eq} = \frac{mg}{k}.
\]  

Equation of Motion

Using Newton’s second law, it can be shown that the differential equation of motion is

\[
m \ddot{x} + c \dot{x} + k x = R(t).
\]  

Here, \( x \) is measured from the equilibrium position. Note that static forces are not present in this equation. The solution of this equation describes the forced response of the system. The free response of the system is described by solving the equation with \( R(t) \equiv 0 \).

System Parameters

The system mass \( m \), spring stiffness \( k \), and damping coefficient \( c \) are the system parameters. The first two are usually easier to measure (or estimate) than the third.
System Characteristic Equation and Response

The differential equation of motion represents a mathematical model of the system. It has three different types of solutions depending on the values of the parameters $m$, $c$, and $k$. The type (or character) of solution is determined by the roots of the characteristic equation of the system. For the spring-mass-damper system, it can be shown that the characteristic equation is

\[ s^2 + \left(\frac{c}{m}\right)s + \left(\frac{k}{m}\right) = 0 \]  

or

\[ s^2 + (2\zeta\omega_n)s + \omega_n^2 = 0 \]

where

\[ \omega_n = \sqrt{\frac{k}{m}} \]

is the natural frequency of the system, and

\[ \zeta = \frac{c}{2\sqrt{mk}} \]

is the damping ratio.

In general, the roots of the characteristic equation may be written in the form

\[ s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \]

These roots may be real or complex depending on the value of $\zeta$. The following table shows the three types of motion that are possible.

<table>
<thead>
<tr>
<th>Case</th>
<th>Type of Roots</th>
<th>Type of Motion</th>
<th>Form of Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta &lt; 1$</td>
<td>Complex conjugates</td>
<td>Under-damped</td>
<td>$x(t) = Ae^{-\zeta\omega_n t}(\cos(\omega_d t + \phi))$</td>
</tr>
<tr>
<td>$\zeta &gt; 1$</td>
<td>Real, unequal</td>
<td>Over-damped</td>
<td>$x(t) = Ae^{s_1t} + Be^{s_2t}$</td>
</tr>
<tr>
<td>$\zeta = 1$</td>
<td>Real, equal</td>
<td>Critically damped</td>
<td>$x(t) = Ae^{-\zeta\omega_n t} + B t e^{-\zeta\omega_n t}$</td>
</tr>
</tbody>
</table>