ME 4590 Dynamics of Machinery
Constraint Relaxation Method: The Meaning of Lagrange Multipliers

Previously, we noted that if a dynamic system is described using "n" generalized coordinates \( q_k \) \((k=1,\ldots,n)\), and if the system is subjected to "m" independent configuration constraint equations of the form

\[
\sum_{k=1}^{n} a_{jk} \dot{q}_k + a_{j0} = 0 \quad (j=1,\ldots,m)
\]  

then we can find the equations of motion of the system by using one of the following two forms of Lagrange's equations with Lagrange multipliers.

\[
\frac{d}{dt} \left( \frac{\partial K}{\partial \dot{q}_k} \right) - \frac{\partial K}{\partial q_k} = F_{q_k} + \sum_{j=1}^{m} \lambda_j a_{jk} \quad (k=1,\ldots,n)
\]  

or

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \left( F_{q_k} \right)_{nc} + \sum_{j=1}^{m} \lambda_j a_{jk} \quad (k=1,\ldots,n)
\]

Here, \( K \) is the kinetic energy of the system, \( F_{q_k} \) is the generalized force associated with the generalized coordinate \( q_k \), \( L \) is the Lagrangian of the system, \( V \) is the potential energy function for the conservative forces and torques, \( (F_{q_k})_{nc} \) is the generalized force associated with \( q_k \) for the non-conservative forces and torques, only, \( \lambda_j \) is the Lagrange multiplier associated with the \( j^{th} \) constraint equation, and \( a_{jk} \) \((j=1,\ldots,m; \ k=1,\ldots,n)\) are the coefficients from the constraint equations. Equations (1.1) and Equations (1.2) or (1.3) form a set of \( n+m \) differential/algebraic equations for the \( n \) generalized coordinates and the \( m \) Lagrange multipliers.

Alternatively, we can relax (or remove) some or all the constraints and replace them with force and/or torque components that are required to maintain the constraints. Then, we formulate the \( n \) Lagrange's equations in terms of the \( n \) generalized coordinates and the \( m \) constraint force (or torque) components. Together with the constraint equations, this forms a set of \( n+m \) differential/algebraic equations for the \( n \) generalized coordinates and the \( m \) constraint force and/or torque components. If all the constraints are relaxed, then Equations (1.2) and (1.3) can be written as
\[
\frac{d}{dt}\left(\frac{\partial K}{\partial \dot{q}_k}\right) - \frac{\partial K}{\partial q_k} = F_{q_k} + \left(F_{q_k}\right)_{\text{constraints}} \\
(k = 1, \ldots, n) \tag{1.4}
\]

and

\[
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \left(F_{q_k}\right)_{\text{nc}} + \left(F_{q_k}\right)_{\text{constraint}} \\
(k = 1, \ldots, n) \tag{1.5}
\]

**Example: The Simple Pendulum**

For the simple pendulum shown at the right, we will use \( q_1 = x \) and \( q_2 = y \) as the generalized coordinates, and we will relax the length constraint of the pendulum in the formulation. In this case, Lagrange's equations can be written in the form

\[
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = \left(F_{q_k}\right)_{\text{nc}} + \left(F_{q_k}\right)_{\text{constraint}} \tag{1.6}
\]

where \( L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + mgy \) and \( (F_{q_k})_{\text{nc}} = 0 \), and the contributions of the constraint force to the right hand sides of the equations are

\[
(F_x)_{\text{constraint}} = T \cdot (\dot{\gamma}y/\partial \dot{x}) = T\left(-(x/L)\dot{\dot{x}} - (y/L)\dot{\dot{y}}\right) \cdot \partial(\dot{x}\dot{\dot{x}} + \dot{y}\dot{\dot{y}})/\partial \dot{x} = -T(x/L) \tag{1.7}
\]

\[
(F_y)_{\text{constraint}} = T \cdot (\dot{\gamma}y/\partial \dot{y}) = T\left(-(x/L)\dot{\dot{x}} - (y/L)\dot{\dot{y}}\right) \cdot \partial(\dot{x}\dot{\dot{x}} + \dot{y}\dot{\dot{y}})/\partial \dot{y} = -T(y/L) \tag{1.8}
\]

Substituting into Lagrange's equations (1.4) and supplementing with the twice differentiated constrain equation give the following equations of motion

\[
\begin{align*}
mx\dddot{x} + \left(\frac{\dot{x}}{T}\right)T &= 0 \\
m\dddot{y} - mg + \left(\frac{\dot{y}}{T}\right)T &= 0 \\
x\dddot{x} + y\dddot{y} + \dot{x}^2 + \dot{y}^2 &= 0
\end{align*} \tag{1.9}
\]

Using Lagrange multipliers, we showed in previous notes that the equations for the pendulum could be written as
\[
\begin{align*}
    m\ddot{x} - \lambda x &= 0 \\
    m\ddot{y} - mg - \lambda y &= 0 \\
    \ddot{x} + \ddot{y} + \dot{x}^2 + \dot{y}^2 &= 0
\end{align*}
\] (1.10)

Comparing Equations (1.7) and (1.8), we see that the Lagrange multiplier $\lambda$ is equal to $T/L$ the force per unit pendulum length.

**Note:** In general, the Lagrange multipliers will be related to the forces and/or torques required to maintain the constraints.