**Simple Angular Velocity**

The rigid body $B$ shown in the diagram below rotates about the $Z$-axis. The $XYZ$ reference frame is a fixed frame, while the $xyz$ reference frame is fixed in (and rotates with) the body. The $XYZ$ reference frame is represented by the unit vector set $R: (i, j, k)$, and the $xyz$ reference frame is represented by the unit vector set $B: (e_1, e_2, k)$. Note that each unit vector set is a *right-handed* set, that is $i \times j = k$ and $e_1 \times e_2 = k$.

The unit vectors fixed in the body $B$ can be differentiated by using the concept of *angular velocity*. It can be shown that

$$\frac{Rde_i}{dt} = R\omega_B \times e_i \quad (i = 1, 2)$$

where $\frac{Rde_i}{dt}$ represents the derivative of the unit vector $e_i$ in the reference frame $R$, and $R\omega_B = \dot{\theta}k$ is the angular velocity of the body $B$ in the reference frame $R$.

**Aside:**

$$\frac{Rde_1}{dt} = \frac{Rd}{dt} (C_\theta i + S_\theta j)$$

$$= \dot{\theta}(-S_\theta i + C_\theta j)$$

$$= \dot{\theta}e_2$$

$$= \dot{\theta}(k \times e_1)$$

$$= R\omega_B \times e_1$$
Differentiation of Unit Vectors – General Case

Consider now a rigid body \( B \) moving in three dimensional space. In general, given a set of unit vectors \( (e_1, e_2, e_3) \) fixed in \( B \), it can be shown that

\[
\frac{d e_i}{dt} = \omega_B \times e_i \quad (i = 1, 2, 3)
\]

where, as before, \( \frac{d e_i}{dt} \) represents the derivative of the unit vector \( e_i \) in the reference frame \( R \), and \( \omega_B \) is the angular velocity of the body \( B \) in the reference frame \( R \). What we are presently lacking is a means of calculating \( \omega_B \), unless the body has simple angular motion.

Simple Angular Acceleration

The angular acceleration of \( B \) in \( R \) is found by differentiating the angular velocity vector. That is,

\[
\alpha_B = \frac{d}{dt} (\omega_B) = \ddot{\theta} k
\]
**Kinematics of Fixed Axis Rotation**

Consider the rigid body $B$ shown in the diagram below. The fixed reference frame is represented by the unit vector set $R: (i, j, k)$, and the rotating reference frame is represented by the unit vector set $B: (\xi_1, \xi_2, \xi_3)$. All points of $B$ travel in a circular path around the fixed axis. The **velocity** and **acceleration** of any point within the body can be determined by differentiating (with respect to time) its position vector $r_{A}$ relative to any point on the fixed axis.

For example, the velocity of point $A$ may be calculated as follows

$$v_{A} = \frac{R}{dt} (ae_{1} + be_{5}) = a \frac{R}{dt} e_{1} + b \frac{R}{dt} e_{5}$$

$$= a(R \omega_{B} \times e_{1}) + b(R \omega_{B} \times e_{5})$$

$$= R \omega_{B} \times (ae_{1} + be_{5})$$

$$= R \omega_{B} \times r_{A}$$

Performing the cross product in the last equation gives $v_{A} = a\dot{\theta}e_{2}$. Note that the velocity is **tangent** to the circular path. Similarly, the acceleration of $A$ may be calculated as follows

$$\dot{a}_{A} = \frac{R}{dt} (\\dot{v}_{A}) = \frac{R}{dt} (R \omega_{B} \times r_{A})$$

$$= (R \alpha_{B} \times r_{A}) + (R \omega_{B} \times R v_{A})$$

Performing the operations in this last equation gives $\dot{a}_{A} = -a\dot{\theta}^{2}e_{1} + a\ddot{\theta}e_{2}$. Note the acceleration has both **normal** and **tangential** components.