ME 6590  Multibody Dynamics
Equations of Motion for a Multibody System

The explicit form of the equations of motion of a multibody system depends on:

- choice of generalized coordinates
- choice of generalized speeds
- method used to formulate equations
- constraints on system motion

In the notes that follow, Kane’s equations are used to derive the equations of motion of an unconstrained multibody system. The choice of generalized coordinates and speeds are given below.

Generalized Coordinates and Speeds

- **Euler parameters** ($\varepsilon_{Ki}$) are used to measure changes in orientation of the bodies ($K$) relative to the inertial frame ($R$).
- **Translation variables** ($s'_{Ki}$) are used to measure displacements of the bodies ($K$) relative to their lower-numbered bodies ($L(K)$). These variables represent the lower-body-fixed components of the translation vectors of the bodies ($s_K$).
- **Absolute angular velocity components** ($\omega'_{Ki}$) are used to measure the angular velocities of the bodies ($K$) relative to the inertial frame ($R$). These are the body-fixed components of the angular velocity vectors of the bodies ($\omega_K$).

System State Vectors

Using the generalized coordinates and speeds defined above, the following system vectors can be defined

$$\{\varepsilon\}_{(4N_g)\times1} = \begin{bmatrix} \varepsilon_{i1}, \varepsilon_{i2}, \varepsilon_{i3}, \varepsilon_{i4}, \ldots, \varepsilon_{K1}, \varepsilon_{K2}, \varepsilon_{K3}, \varepsilon_{K4}, \ldots, \varepsilon_{N1}, \varepsilon_{N2}, \varepsilon_{N3}, \varepsilon_{N4} \end{bmatrix}^T$$

$$\{s'\}_{(3N_g)\times1} = \begin{bmatrix} s'_{i1}, s'_{i2}, s'_{i3}, \ldots, s'_{K1}, s'_{K2}, s'_{K3}, \ldots, s'_{N1}, s'_{N2}, s'_{N3} \end{bmatrix}^T$$

$$\{\omega'\}_{(3N_g)\times1} = \begin{bmatrix} \omega'_{i1}, \omega'_{i2}, \omega'_{i3}, \ldots, \omega'_{K1}, \omega'_{K2}, \omega'_{K3}, \ldots, \omega'_{N1}, \omega'_{N2}, \omega'_{N3} \end{bmatrix}^T$$

and
\[
\begin{align*}
\{x\}_{(7N_b)\times 1} &= \begin{bmatrix} [x_1] \\ [x_2] \end{bmatrix} = \begin{bmatrix} \mathcal{E} \\ [s'] \end{bmatrix} \\
\{y\}_{(6N_b)\times 1} &= \begin{bmatrix} [y_1] \\ [y_2] \end{bmatrix} = \begin{bmatrix} \omega' \end{bmatrix}
\end{align*}
\]  

(1)

**System Partial Angular Velocity Matrices**

- The body-fixed components of the angular velocities of the bodies (\(K\)) may be written

\[
\{\omega'_{K}\} = \begin{bmatrix} \omega'_{K,y_1} ; \omega'_{K,y_2} \end{bmatrix} \{y\} = \begin{bmatrix} \omega'_{K,y} \end{bmatrix} \{y\}
\]

(2)

where

\[
\begin{bmatrix} \omega'_{K,y_1} \end{bmatrix} \in \mathbb{R}^{3 \times (3N_b)} = \begin{bmatrix} \mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_{K-1}, \mathcal{I}_{K}, \mathcal{I}_{K+1}, \ldots, \mathcal{I}_{N_b} \end{bmatrix}
\]

\[
\begin{bmatrix} \omega'_{K,y_2} \end{bmatrix} \in \mathbb{R}^{3 \times (3N_b)} = \begin{bmatrix} \mathcal{0}, \ldots, \mathcal{I}_{K}, \mathcal{0}, \ldots, \mathcal{0} \end{bmatrix}
\]

(3)

Here \([\mathcal{I}]\) represents a \(3 \times 3\) identity matrix.

**Position Vectors of the Mass Centers of the Bodies**

- The inertial components of the position vector of the mass center of any body (\(K\)) may be written

\[
\{p_{K}\} = \{s_1\} + \left( \sum_{r=0}^{u_{K}-1} \begin{bmatrix} C_{\mathcal{L}^{r+1}(K)} \end{bmatrix} \begin{bmatrix} (q' + s')_{\mathcal{L}^{r}(K)} \end{bmatrix} \right) + [C_{K}][r']
\]

where \(u_{K}\) represents an integer such that \(\mathcal{L}^{u_{K}}(K) = 1\).

**Velocity and Partial Velocity Vectors of the Mass Centers of the Bodies**

The inertial components of the velocity vector of any body (\(K\)) may be found by differentiating the above expression as follows

\[
\begin{align*}
\{v_{K}\} &= \{\dot{p}_{K}\} = \{\dot{s}_1\} + \left( \sum_{r=0}^{u_{K}-1} \begin{bmatrix} C_{\mathcal{L}^{r+1}(K)} \end{bmatrix} \begin{bmatrix} \dot{s}'_{\mathcal{L}^{r}(K)} \end{bmatrix} \right) + \left( \sum_{r=0}^{u_{K}-1} \begin{bmatrix} \dot{C}_{\mathcal{L}^{r+1}(K)} \end{bmatrix} \begin{bmatrix} (q' + s')_{\mathcal{L}^{r}(K)} \end{bmatrix} \right) + [\hat{C}_{K}][r']
\end{align*}
\]

where

\[
[\hat{C}_{K}][r'] = [C_{K}][\hat{\omega}'_{K}][r'] = -[C_{K}][\hat{r}'_{K}][\omega'_{K}] = -[C_{K}][\hat{r}'_{K}][\omega'_{K,y}][y]
\]

and
\[
\sum_{r=0}^{u-1} \left[ C_{L^{r+1}(K)} \right] \{(q' + s')_{L^{r+1}(K)}\} = \sum_{r=0}^{u-1} \left[ C_{L^{r+1}(K)} \right] \left[ \hat{\omega}_{L^{r+1}(K)} \right] \{(q' + s')_{L^{r}(K)}\} \\
= -\sum_{r=0}^{u-1} \left[ C_{L^{r+1}(K)} \right] \left[ (\tilde{q}' + \tilde{s}')_{L^{r}(K)} \right] \{\omega_{L^{r+1}(K)}\} \\
= -\sum_{r=0}^{u-1} \left[ C_{L^{r+1}(K)} \right] \left[ (\tilde{q}' + \tilde{s}')_{L^{r}(K)} \right] \left[ \omega_{L^{r+1}(K),y} \right] \{y\}
\]

So, the inertial components of the velocity of the mass center of body \( K \) may be written

\[
\{v_K\} = \left[ v_{K,y_1}; v_{K,y_2} \right] \left\{ y_1 \right\} \left\{ y_2 \right\} \tag{4}
\]

where

\[
[v_{K,y_1}] = \begin{bmatrix}
-\left[ C_1 \right] \left[ (\tilde{q}' + \tilde{s}')_{L^{u}(K)} \right],[0],...,[0], & -\left[ C_{L^{u+1}(K)} \right] \left[ (\tilde{q}' + \tilde{s}')_{L^{u+1}(K)} \right],[0],...,[0], \\
-\left[ C_{L^{u+2}(K)} \right] \left[ (\tilde{q}' + \tilde{s}')_{L^{u+2}(K)} \right],[0],...,[0], & -\left[ C_{L^{u+2}(K)} \right] \left[ (\tilde{q}' + \tilde{s}')_{L^{u+2}(K)} \right],[0],...,[0]
\end{bmatrix}
\tag{5}
\]

\[
[v_{K,y_2}] = \begin{bmatrix}
-\left[ I \right] \left[ (\tilde{q}' + \tilde{s}')_{L^{u}(K)} \right],[0],...,[0], & -\left[ C_{L^{u+1}(K)} \right] \left[ (\tilde{q}' + \tilde{s}')_{L^{u+1}(K)} \right],[0],...,[0], \\
-\left[ C_{L^{u+2}(K)} \right] \left[ (\tilde{q}' + \tilde{s}')_{L^{u+2}(K)} \right],[0],...,[0], & -\left[ C_{L^{u+2}(K)} \right] \left[ (\tilde{q}' + \tilde{s}')_{L^{u+2}(K)} \right],[0],...,[0]
\end{bmatrix}
\tag{6}
\]

where again, \([ I ]\) is used to represent a \(3 \times 3\) identity matrix.
Time Derivatives of the Partial Angular Velocity Matrices

The results for the partial angular velocity matrices can easily be differentiated to give

\[
\begin{bmatrix}
\dot{\omega}'_{K,1} \\
\dot{\omega}'_{K,2}
\end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix}
\dot{\omega}'_{K,3}
\end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}
\]

\( \text{(7)} \)

Time Derivatives of the Partial Velocity Matrices

The time derivatives of the partial velocity matrices are found by differentiating all non-zero entries of the partial velocity matrices. Calculation of \( \dot{v}_{K,y_2} \) involves differentiation of the coordinate transformation matrices as we have done before. Calculation of \( \dot{v}_{K,y_1} \) involves differentiating the following types of terms

\[
\frac{d}{dt}\left( \begin{bmatrix}
C_{\ell^{a+1}(K)} \\
\tilde{q}'_{\ell^{a}(K)} + \tilde{s}'_{\ell^{a}(K)}
\end{bmatrix} \right) = \begin{bmatrix}
\dot{C}_{\ell^{a+1}(K)} \\
\tilde{q}'_{\ell^{a}(K)} + \tilde{s}'_{\ell^{a}(K)}
\end{bmatrix} + \begin{bmatrix}
C_{\ell^{a+1}(K)} \\
\tilde{s}'_{\ell^{a}(K)}
\end{bmatrix}
\]

Also,

\[
\frac{d}{dt}\left( \begin{bmatrix}
C_K \\
\tilde{r}'_K
\end{bmatrix} \right) = \begin{bmatrix}
\dot{C}_K \\
\tilde{r}'_K
\end{bmatrix} = \begin{bmatrix}
C_K \\
\tilde{r}'_K
\end{bmatrix} \begin{bmatrix}
\tilde{r}'_K
\end{bmatrix}
\]

Generalized Forces

Let the forces and torques acting on each body \( (K) \) of the system be replaced by an equivalent force system consisting of a single force \( F_K \) acting at the mass center \( G_K \) and a single moment \( M_K \). Then the generalized forces for the system may be calculated as follows:

\[
F_{y_i} = \sum_{K=1}^{N} \left( F_K \cdot \frac{\partial y_K}{\partial y_i} \right) + \left( M_K \cdot \frac{\partial \omega_K}{\partial y_i} \right)
\]

or, in matrix form, the column vector of generalized forces is

\[
\{F_y\}_{(6N)x1} = \sum_{K=1}^{N} \left( \begin{bmatrix}
v_{K,y} \\
\omega'_{K,y}
\end{bmatrix}^T \{F_K\} + \begin{bmatrix}
\omega'_{K,y}
\end{bmatrix}^T \{M'_K\} \right)
\]

\( \text{(8)} \)
where \( \{F_k\} \) represents the inertial components of the vector \( F_k \) and \( \{M'_k\} \) represents the body-fixed components of the vector \( M_k \).

**Kane’s Equations of Motion**

Assuming all \( 6N_B \) of the generalized speeds are independent, we can write Kane’s equations of motion for the multibody system as

\[
\sum_{k=1}^{N_B} \left( m_k a_k \cdot \frac{\partial v_{k,i}}{\partial y_i} \right) + \sum_{k=1}^{N_B} \left[ \left( \tau_k \cdot \alpha_k \right) + \left( \omega_k \times H_k \right) \right] \cdot \frac{\partial \alpha_k}{\partial y_i} = F_y, \quad (i=1, \ldots, 6N_B) \tag{9}
\]

where the generalized forces on the right side of the equation are the entries of the generalized force column vector of Equation \( (8) \). Let’s consider the terms on the left side of the equation one at a time.

\[
a_k \rightarrow \{a_k\} = \{\dot{v}_k\} = \frac{d}{dt} \left[ \{v_{k,i}\}\{y\} \right] = \left[ \{v_{k,i}\}\dot{\{y\}} + \{\dot{v}_{k,i}\}\{y\} \right]
\]

\[
\alpha_k \rightarrow \{\alpha'_k\} = \{\dot{\alpha}'_k\} = \frac{d}{dt} \left[ \{\omega'_{k,i}\}\{y\} \right] = \left[ \{\omega'_{k,i}\}\dot{\{y\}} \right]
\]

\[
\sum_{k=1}^{N_B} \left( m_k a_k \cdot \frac{\partial v_{k,i}}{\partial y_i} \right) \rightarrow \\
\sum_{k=1}^{N_B} \left( m_k \left[ v_{k,i} \right]^T \{a_k\} \right) = \sum_{k=1}^{N_B} \left( m_k \left[ v_{k,i} \right]^T \left[ v_{k,i} \right] \dot{\{y\}} + m_k \left[ v_{k,i} \right]^T \left[ \dot{v}_{k,i} \right] \{y\} \right) \tag{10}
\]

\[
\sum_{k=1}^{N_B} \left( \tau_k \cdot \alpha_k \right) \cdot \frac{\partial \alpha_k}{\partial y_i} \rightarrow \\
\sum_{k=1}^{N_B} \left( \left[ \omega'_{k,i} \right]^T \left[ I'_k \right] \{\alpha'_k\} \right) = \sum_{k=1}^{N_B} \left( \left[ \omega'_{k,i} \right]^T \left[ I'_k \right] \left[ \omega'_{k,i} \right] \{\dot{y}\} \right) \tag{11}
\]

\[
\omega_k \times H_k \rightarrow \left[ \tilde{\omega}'_k \right] \left[ I'_k \right] \{\omega'_k\} \quad \text{(body-fixed components)}
\]

\[
\sum_{k=1}^{N_B} \left( \omega_k \times H_k \right) \cdot \frac{\partial \alpha_k}{\partial y_i} \rightarrow \sum_{k=1}^{N_B} \left( \omega'_{k,i} \right)^T \left[ \tilde{\omega}'_k \right] \left[ I'_k \right] \{\omega'_k\} \tag{12}
\]

Substituting from Equations \( (8), (10), (11), \) and \( (12) \) into Equation \( (9) \) gives the matrix form of the equations of motion for the multibody system.
\[ [A] \{ \dot{y} \} = \{ f \} \quad ( [A] \text{ is called the "generalized mass matrix"}) \] (13)

where

\[ [A] = \sum_{k=1}^{N_B} \left( m_k \left[ v_{k,y} \right]^T \left[ v_{k,y} \right] + \left[ \omega'_{k,y} \right]^T \left[ I'_k \right] \left[ \omega'_{k,y} \right] \right) \] (14)

\[ \{ f \} = \sum_{k=1}^{N_B} \left[ v_{k,y} \right]^T \left( \{ F_k \} - m_k \left[ \dot{v}_{k,y} \right] \{ y \} \right) + \sum_{k=1}^{N_B} \left[ \omega'_{k,y} \right]^T \left( \{ M'_k \} - \tilde{\omega}'_k \left[ I'_k \right] \{ \omega'_k \} \right) \] (15)

Equation (13) represents \( 6N_B \) first order differential equations for the \( 13N_B \) variables defined by the \( \{ x \} \) and \( \{ y \} \) vectors of Equation (1). To form a complete set of differential equations, we must supplement Equation (13) with the following set of \( 7N_B \) first order kinematical differential equations.

\[
\begin{bmatrix}
[E'_1]^T & [0] & [0] & [0] & [0] \\
[0] & [E'_2]^T & [0] & \cdot & \cdot & [0] \\
[0] & [0] & [E'_3]^T & \cdot & \cdot & [0] \\
[0] & \cdot & \cdot & \cdot & \cdot & [0] \\
[0] & [0] & [0] & [0] & [E'_{N_B}]^T \\
\end{bmatrix}
\]

\[ \{ \dot{x}_1 \} = \{
\dot{x}' \} = \frac{1}{2} \]

\[
\begin{bmatrix}
[E'_1]^T & [0] & [0] & [0] & [0] \\
[0] & [E'_2]^T & [0] & \cdot & \cdot & [0] \\
[0] & [0] & [E'_3]^T & \cdot & \cdot & [0] \\
[0] & \cdot & \cdot & \cdot & \cdot & [0] \\
[0] & [0] & [0] & [0] & [E'_{N_B}]^T \\
\end{bmatrix}
\] (16)

and

\[ \{ \dot{x}_2 \} = \{ y_2 \} \] (17)

The matrices \( [E'_k]^T \) that are on the diagonal of Equation (16) may be found from earlier notes to be

\[
[E'_k]^T = \begin{bmatrix}
\varepsilon_{k4} & -\varepsilon_{k3} & \varepsilon_{k2} \\
\varepsilon_{k3} & \varepsilon_{k4} & -\varepsilon_{k1} \\
-\varepsilon_{k2} & \varepsilon_{k1} & \varepsilon_{k4} \\
-\varepsilon_{k1} & -\varepsilon_{k2} & -\varepsilon_{k3} \\
\end{bmatrix}
\] (18)