**ME 4710 Motion and Control**  
**PID Control of a Spring-Mass-Damper Position: Analysis Summary**

- **Fig. 1** shows a spring-mass-damper system with a force actuator for position control. The spring has stiffness $k$, the damper has coefficient $c$, the block has mass $m$, and the position of the mass is measured by the variable $x$.

- The transfer function of the SMD with the actuating force $F_a$ as input and the position $x$ as output is

$$
\frac{X(s)}{F_a} = \frac{1}{ms^2 + cs + k}
$$

- Assuming ideal actuator and sensor responses, the closed-loop position control of the SMD can be described using the following block diagram. Here, $X_d$ represents the desired position, and $G_c(s)$ represents the transfer function of the controller.

- It is assumed here that the SMD parameters are: $m = 1$ slug, $c = 8.8$ (lb-s/ft), and $k = 40$ (lb/ft). This represents an under-damped second order plant with

$$
\begin{align*}
\omega_n &= \sqrt{40} = 6.325 \text{ (rad/s)} \approx 1 \text{ (Hz)} \\
\zeta &= \frac{8.8}{2\sqrt{40}} = 0.696 \approx 0.7
\end{align*}
$$
Proportional Control

- If *proportional control* is used, then \( G_c(s) = K \), and the loop transfer function and closed-loop transfer functions are

\[
GH(s) = \frac{K}{s^2 + 8.8s + 40} \quad \text{and} \quad \frac{X(s)}{X_d(s)} = \frac{K}{s^2 + 8.8s + (40 + K)}
\]

- This is a type-zero system and will have a *finite steady-state error* for a step input. Using the final-value theorem and the closed-loop transfer function, \( x_{ss} \), the final value of \( x(t) \) to a unit step command is

\[
x_{ss} = \lim_{s \to 0} \left( s \cdot \frac{1}{s^2 + 8.8s + (40 + K)} \right) = \frac{K}{40 + K} < 1
\]

- Eq. (3) indicates that large values of \( K \) lead to smaller steady-state error; however, as seen below, they also lead to a faster, less damped responses.

- The root locus diagram for the closed-loop system for \( K \geq 0 \) and the Bode diagram for \( GH(s) \) are shown in **Fig. 2**. Note that as the value of \( K \) is increased, the closed-loop poles move straight up/down, indicating that the natural frequency is increased and the damping ratio is decreased. Also, as the value of \( K \) is increased, the phase (stability) margin is decreased.

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**Figure 2.** Root Locus Diagram and Bode Diagram for \(GH(s)\) for Proportional Control
**Fig. 3** shows step responses and Bode diagrams of the closed-loop system for proportional gains $K$ of 100, 500, and 2000. As the gain is increased the system time response is faster and less damped. The Bode diagram correspondingly shows larger bandwidths and resonant magnitudes.

Clearly, it is not possible to achieve low steady-state error and good transient response using only proportional control. As the gain is increased, the response becomes faster, but it has a lower phase margin. To remove the steady-state error and have better response, integral and/or derivative terms must be included in the controller.

Proportional-Integral (PI) Control

- If proportional-integral (PI) control is used, then
  \[
  G_c(s) = K_p + \frac{K_i}{s} = \frac{K_p(s + a)}{s} \quad (4)
  \]

- $K_p$ and $K_i$ represent the proportional and integral gains, and $a = K_i/K_p$ is the ratio of the integral and proportional gains.

- The loop and closed-loop transfer functions for this system are
  \[
  GH(s) = \frac{K_p(s + a)}{s(s^2 + 8.8s + 40)} \quad \frac{X}{X_d(s)} = \frac{K_p(s + a)}{s(s^2 + 8.8s + 40) + K_p(s + a)} \quad (5)
  \]
o **Integral control** makes the system a *type-one* system, so the *steady-state error* due to a step input is *zero*. This can be verified by using the final value theorem to show that \( x_{ss} = 1 \) when the input is a unit step function.

o The root locus diagram for the closed-loop system (with \( a = 3 \)) for \( K \geq 0 \) and the Bode diagram for \( GH(s) \) are shown in **Fig. 4**. The root locus diagram also shows the locations of the closed-loop poles for a proportional gain \( K_p = 50 \).

o Note that the integral controller has *added a third, slower pole* to the system and has *moved the asymptotes* of the complex poles closer to the imaginary axis. For low gains, the system is *slow and stable* (first order dominant).

o As the gain is *increased*, the system becomes *faster* with a *decreasing phase margin*. The Bode diagram shows that the gain could be increased somewhat above \( K_p = 25 \) without significantly decreasing the stability margin. However, further increases will decrease the phase margin.

![Figure 4. Root Locus Diagram and Bode Diagram for \((GH(s))\) for PI Control \((a=3)\)](image)

- **Fig. 5** shows step responses and Bode diagrams of the closed loop system for \( a = 3 \) and proportional gains of \( K_p = 25 \), 50, and 75. Integral control has *removed the steady-state error and improved the transient response*, but it has also *increased the system settling time*.

o Settling times can be lowered by increasing the gain. This will *increase the system bandwidth*, but it will also decrease the stability margin.
Proportional-Derivative (PD) Control

- If proportional-derivative (PD) control is used, then

\[ G_c(s) = K_p + K_Ds = K_D(s + a) \]  \hspace{1cm} (6)

- \( K_p \) and \( K_D \) represent the proportional and derivative gains, and \( a = K_p / K_D \) is the ratio of the proportional and derivative gains. The loop and closed-loop transfer functions for this system are

\[ GH(s) = \frac{K_D(s + a)}{s^2 + 8.8s + 40} \]

\[ \frac{X(s)}{X_d(s)} = \frac{K_D(s + a)}{(s^2 + 8.8s + 40) + K_D(s + a)} \]  \hspace{1cm} (7)

- Without the integral control, this is again a type-zero system, and hence will have a finite steady-state error to a unit step input. Using the final-value theorem and the closed-loop transfer function, \( x_{ss} \) the final value of \( x(t) \) to a unit step command is

\[ x_{ss} = \lim_{s \to 0} \left( \frac{s^2 + 8.8s + 40}{s^2 + 8.8s + 40 + K_D(s + a)} \right) = \frac{K_Da}{40 + K_Da} = \frac{K_p}{40 + K_p} < 1 \]  \hspace{1cm} (8)

- As with simple proportional control, the larger the proportional gain, the smaller the steady-state error.

- The root locus diagram for the closed-loop system (with \( a = 10 \)) for \( K \geq 0 \) and the Bode diagram for \( GH(s) \) are shown in Fig. 6. The root locus diagram also shows the locations of the closed-loop poles for a derivative gain \( K_D \approx 25.6 \).
- As the gain is increased the system poles become faster and more damped. The Bode diagram indicates that the phase margin never drops below 90 degrees indicating a very stable system for any gain.

![Root Locus Diagram and Bode Diagram for \((GH(s))\) for PD Control \((a = 10)\)](image)

Fig. 6. Root Locus Diagram and Bode Diagram for \((GH(s))\) for PD Control \((a = 10)\)

- Fig. 7 shows step responses and Bode diagrams of the closed loop system for \(a = 10\) and derivative gains of \(K_D = 10, 27, 50,\) and 75. The PD controller has decreased the system settling time considerably.

- However, to control the steady-state error, the derivative gain \(K_D\) must be high. This will decrease the response times and increase the bandwidth of the system and may make it susceptible to noise.

![Closed Loop Step Response and Bode Diagrams for PD Control \((a = 10)\)](image)

Figure 7. Closed Loop Step Response and Bode Diagrams for PD Control \((a = 10)\)
Proportional-Integral-Derivative Control

- If proportional-integral-derivative (PID) control is used, then

\[
G_c(s) = K_p + \frac{K_I}{s} + K_Ds = \frac{K_D(s^2 + as + b)}{s},
\]

where \( K_p, K_I, \) and \( K_D \) represent the proportional, integral, and derivative gains, \( a = \frac{K_p}{K_D} \) is the ratio of the proportional and derivative gains, and \( b = \frac{K_I}{K_D} \) is the ratio of the integral and derivative gains.

- The loop and closed-loop transfer functions for this system are

\[
GH(s) = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40)} \quad \frac{X(s)}{X_d(s)} = \frac{K_D(s^2 + as + b)}{s(s^2 + 8.8s + 40) + K_D(s^2 + as + b)}
\]

- Integral control makes the system type-one so it has zero steady-state error for a step input.

- The root locus diagram for the closed-loop system (with \( a = 15 \) and \( b = 50 \)) for \( K \geq 0 \) and the Bode diagram for \( GH(s) \) are shown in Fig. 8. The locations of the closed-loop poles for \( K_D \approx 15.8 \) are also shown. As the gain is increased, the system becomes faster without significant losses in the phase margin.

Figure 8. Root Locus Diagram and Bode Diagram for \((GH(s))\) for PID Control \((a = 15, b = 50)\)
- **Fig. 9** shows step responses and Bode diagrams of the closed-loop system for $a=15$, $b=50$, and derivative gains of $K_D=5$, 10, and 15. Using both integral and derivative control has *removed steady-state error* and *decreased system settling times* while maintaining a reasonable transient response.

![Step Response for PID Control](image1)

![Closed Loop Bode Diagram for PID Control of a Spring Mass Damper](image2)

Figure 9. Closed Loop Step Response and Bode Diagrams for PID Control ($a=15$, $b=50$)