ME 5550 Intermediate Dynamics
Moments and Products of Inertia and the Inertia Matrix

Moments of Inertia

A rigid body $B$ is shown in the diagram below. The unit vectors $(e_1, e_2, e_3)$ are fixed in the body and are directed along a *convenient* set of axes $(x, y, z)$ that pass through the mass center $G$. The *moments of inertia* of the body about these axes are defined as follows

\[
I_{xx}^G = \int_B (y^2 + z^2) \, dm \\
I_{yy}^G = \int_B (x^2 + z^2) \, dm \\
I_{zz}^G = \int_B (x^2 + y^2) \, dm 
\]

where $x, y, z$ are defined as the $e_i$ components of $\mathbf{r}_{P/G}$ the position vector of $P$ with respect to $G$, that is, $\mathbf{r}_{P/G} = xe_1 + ye_2 + ze_3$.

Moments of inertia of a body about a particular axis *measure the distribution of the body’s mass about that axis*. The smaller the inertia the more the mass is concentrated about the axis. Inertia values can be found either by *measurement* or by *calculation*. Calculations are based on *direct integration* or on the "body build-up" technique. In the body build-up technique, *inertias of simple shapes are added* to estimate the inertia of a composite shape. The inertias of simple shapes (about their individual mass centers) are found in *standard inertia tables*. These values are transferred to axes through the composite mass center using the *Parallel Axes Theorem for Moments of Inertia*.

**Parallel Axes Theorem for Moments of Inertia**

The inertia ($I_i^A$) of a body about an axis ($i$) through any point ($A$) is equal to the inertia ($I_i^G$) of the body about a parallel axis through the mass center $G$ plus the mass ($m$) times the distance ($d_i$) between the two axes squared. Or,

\[
I_i^A = I_i^G + m d_i^2 \quad (i = x, y, \text{ or } z)
\]

Note that moments of inertia are *always positive*. From the parallel axes theorem, it is obvious that the *minimum moments of inertia* of a body occur about axes that pass through its *mass center*.
Products of Inertia

The products of inertia of the rigid body are defined as

\[ I_{xy}^G = \int_B (xy) \, dm \quad I_{xz}^G = \int_B (xz) \, dm \quad I_{yz}^G = \int_B (yz) \, dm \]

The products of inertia of a body are measures of symmetry. If a particular plane is a plane of symmetry, then the products of inertia associated with any axis perpendicular to that plane are zero. For example, consider the thin laminate shown. The middle plane of the laminate lies in the XY-plane so that half its thickness is above the plane and half is below. Hence, the XY-plane is a plane of symmetry and

\[ I_{xz} = I_{yz} = 0 \]

Bodies of revolution have two planes of symmetry. For the configuration shown, the XZ and YZ planes are planes of symmetry. Hence, all products of inertia are zero about the X, Y, and Z axes.

Products of inertia are found either by measurement or by calculation. Calculations are based on direct integration or on the "body build-up" technique. In the body build-up technique, products of inertia of simple shapes are added to estimate the products of inertia of a composite shape. The products of inertia of simple shapes (about their individual mass centers) are found in standard inertia tables. These values are transferred to axes through the composite mass center using the Parallel Axes Theorem for Products of Inertia.

Parallel Axes Theorem for Products of Inertia

The product of inertia \( I_{ij}^A \) of a body about a pair of axes \((i, j)\) passing through any point \((A)\) is equal to the product of inertia \( I_{ij}^G \) of the body about a set of parallel axes through the mass center \(G\) plus the mass \((m)\) times the product of the coordinates \((c_i, c_j)\) of \(G\) relative to \(A\) measured along those axes.

\[ I_{ij}^A = I_{ij}^G + m \, c_i \, c_j \quad (i = x, y, \text{ or } z \text{ and } j = x, y, \text{ or } z) \]

Products of inertia may be positive, negative, or zero.
The Inertia Matrix

The inertias of a body about a set of axes (passing through some point) are often collected into a single inertia matrix. For example, the inertia matrix of a body about a set of axes through its mass center \( G \) is defined as

\[
[I_G] = \begin{bmatrix}
I_{11}^G & I_{12}^G & I_{13}^G \\
I_{21}^G & I_{22}^G & I_{23}^G \\
I_{31}^G & I_{32}^G & I_{33}^G
\end{bmatrix} =
\begin{bmatrix}
I_{xx}^G & -I_{xy}^G & -I_{xz}^G \\
-I_{xy}^G & I_{yy}^G & -I_{yz}^G \\
-I_{xz}^G & -I_{yz}^G & I_{zz}^G
\end{bmatrix}
\]

There is a different inertia matrix for each set of axes passing through a given point. There is one set of directions for each point that renders the inertia matrix diagonal. These directions are called principal directions (or principal axes) of the body for that point. In general, the principal axes are different for each point in a body. Finally, note that all inertia matrices are symmetric.

The Inertia Dyadic

The inertias of a body about a set of axes (passing through some point) may also be collected into a single inertia dyadic. For example, the inertia dyadic of a body about a set of axes through its mass center \( G \) is defined as

\[
I_G = \sum_{i=1}^{3} \sum_{j=1}^{3} I_{ij}^G \vec{e}_i \vec{e}_j
\]

where \( I_{ij}^G \) \((i, j = 1, 2, 3)\) are the elements of the inertia matrix, and the vector product \( \vec{e}_i \vec{e}_j \) is called a dyad. There are many properties that dyads satisfy. Three properties useful when using inertia dyadics are

1. \( \vec{a} \vec{b} \neq \vec{b} \vec{a} \)
2. \( \vec{c} \cdot (\vec{a} \vec{b}) = (\vec{c} \cdot \vec{a}) \vec{b} \) and \( (\vec{a} \vec{b}) \cdot \vec{c} = \vec{a}(\vec{b} \cdot \vec{c}) = (\vec{b} \cdot \vec{c}) \vec{a} \)
3. \( (\vec{a} \vec{b} + \vec{c} \vec{d}) \cdot \vec{e} = (\vec{b} \cdot \vec{e}) \vec{a} + (\vec{d} \cdot \vec{e}) \vec{c} \)

As noted above, the shorthand notation for the inertia dyadic about mass center \( G \) is \( I_G \). This notation is particularly useful when defining the angular momentum of a body.