ME 5550 Intermediate Dynamics
Lagrange's Equations – Example System II

In previous notes for Example System II, we found $\omega_B$ the angular velocity of the bar and $H_G$ the angular momentum of B resolved in bar-fixed directions $B : (\xi_1, \xi_2, \xi_3)$ to be

$$\omega_B = (-\Omega S_\theta) \xi_1 + \omega n_2 + (\Omega C_\theta) \xi_3$$

and

$$H_G = I_G \cdot \omega_B = \frac{m l^2}{2} [\omega n_2 + \Omega C_\theta \xi_3]$$

In these notes, we will assume the frame $F$ is light and that the $M_\phi(t)$ is applied to $F$ by the ground and torque $M_\theta(t)$ is applied to $B$ by $F$.

Assuming the degrees of freedom of the system are described by the generalized coordinates (angles) $\phi (\dot{\phi} = \Omega)$ and $\theta$, the equations of motion of the system can be found using Lagrange's equations.

$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) - \frac{\partial L}{\partial \phi} = F_\phi$$
$$\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) - \frac{\partial L}{\partial \theta} = F_\theta$$

(1)

Lagrangian

Assuming the datum is located at the level of the mass center $G$, the Lagrangian is simply the kinetic energy of the system

$$L = K = \frac{1}{2} m v_G^2 + \frac{1}{2} \omega_B \cdot H_G = \frac{1}{2} m d^2 \ddot{\phi}^2 + \frac{1}{24} m l^2 \left( \dot{\theta}^2 + C_\theta^2 \dot{\theta}^2 \right)$$

Generalized Forces

The generalized forces associated with the driving torques are

$$F_\theta = \left( M_\theta n_2 \cdot \frac{\partial \omega_B}{\partial \dot{\theta}} \right) + \left( -M_\theta n_2 \cdot \frac{\partial \omega_F}{\partial \dot{\theta}} \right) + \left( M_\theta k \cdot \frac{\partial \omega_F}{\partial \dot{\theta}} \right) = M_\theta$$

Reference Frames

$F : (n_1, n_2, k)$

$B : (\xi_1, n_2, \xi_3)$
\[ F_\phi = \left( M_\phi n_2 \cdot \frac{\partial^R \omega_B}{\partial \phi} \right) + \left( -M_\phi n_2 \cdot \frac{\partial^R \omega_F}{\partial \phi} \right) + \left( M_\phi k \cdot \frac{\partial^R \omega_F}{\partial \phi} \right) = M_\phi \]

Derivatives of Lagrangian

\[ \frac{\partial L}{\partial \dot{\phi}} = m d^2 \ddot{\phi} + \frac{1}{12} m L^2 \dot{\phi} C_\theta^2 \]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\phi}} \right) = \left( \frac{1}{12} m L^2 C_\theta^2 + m d^2 \right) \ddot{\phi} - \frac{1}{6} m L^2 \dot{\phi} S_\theta C_\theta \]
\[ \frac{\partial L}{\partial \dot{\theta}} = \frac{1}{12} m L^2 \dot{\theta} \]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{\theta}} \right) = \frac{1}{12} m L^2 \ddot{\theta} \]
\[ \frac{\partial L}{\partial \phi} = 0 \]
\[ \frac{\partial L}{\partial \theta} = -\frac{1}{12} m L^2 \phi^2 S_\theta C_\theta \]

Equations of Motion

Substituting the above results into Lagrange's equations (1.1) give the equations of motion

\[ \left( m d^2 + \frac{1}{12} m L^2 C_\theta^2 \right) \ddot{\phi} - \left( \frac{1}{6} m L^2 S_\theta C_\theta \right) \dot{\phi} = M_\phi(t) \]
\[ \left( \frac{1}{12} m L^2 \right) \ddot{\theta} + \left( \frac{1}{12} m L^2 S_\theta C_\theta \right) \phi^2 = M_\theta(t) \]