Linearization of Differential Equations of Motion

The equations of motion (EOM) derived using Newton’s laws or Lagrange’s equations may be linear or nonlinear. If they are nonlinear, it may be possible to linearize the equations about some equilibrium positions. From the resulting linear equations, we can find natural frequencies and mode shapes of the system for that position. These provide the frequencies and describe the types of small deviations (motions) that the system has about that position. The equilibrium positions may be determined from Newton’s law, the principle of virtual work, or directly from the nonlinear EOM by setting all time derivatives to zero.

Linearization of Functions of a Single Variable

Given a nonlinear function \( y = f(x) \), we can expand \( f(x) \) in a Taylor series around the equilibrium position, say \( x_{eq} \)

\[
f(x_{eq} + \Delta x) = f(x_{eq}) + \Delta x \left( \frac{df}{dx} \bigg|_{x=x_{eq}} \right) + \frac{(\Delta x)^2}{2} \left( \frac{d^2 f}{dx^2} \bigg|_{x=x_{eq}} \right) + \ldots
\]

\[
\approx f(x_{eq}) + \Delta x \left( \frac{df}{dx} \bigg|_{x=x_{eq}} \right)
\]

Here \( \Delta x \) represents an excursion from the equilibrium position. If the excursions are small, then we can use the approximation as stated in the second equation. In this latter case, we can write

\[
\Delta f(x) = f(x_{eq} + \Delta x) - f(x_{eq}) = m \Delta x
\]

where

\[
m = \frac{df}{dx} \bigg|_{x=x_{eq}}
\]

This is a linear relationship between changes in \( f \) and changes in \( x \).
Linearization of Functions of Many Variables

Given a nonlinear function \( y = f(x_1, x_2, \ldots, x_n) = f(x) \), we can expand \( f(x) \) in a Taylor series around the equilibrium position, say \( x_{eq} = (x_{1, eq}, x_{2, eq}, \ldots, x_{n, eq}) \)

\[
f(x_{eq} + \Delta x) = f(x_{eq}) + \sum_{i=1}^{n} \Delta x_i \left[ \frac{\partial f}{\partial x_i} \right]_{x=x_{eq}} + \cdots
\]

\[
\approx f(x_{eq}) + \sum_{i=1}^{n} \Delta x_i \left[ \frac{\partial f}{\partial x_i} \right]_{x=x_{eq}}
\]

Here the vector \( \Delta x = (\Delta x_1, \Delta x_2, \ldots, \Delta x_n) \) represents an excursion from the equilibrium position. As before, if the excursions are small, then we can use the approximation as stated in the second equation. In this latter case, we can write

\[
\Delta f(x) = f(x_{eq} + \Delta x) - f(x_{eq}) = \sum_{i=1}^{n} m_i \Delta x_i
\]

where

\[
m_i = \left. \frac{\partial f}{\partial x_i} \right|_{x=x_{eq}}
\]

This is a linear relationship between changes in \( f \) and changes in the elements of the vector \( x \).