Newton/Euler Equations of Motion for a Rigid Body

Using the theory of systems of particles, it can be shown that the equations of motion for rigid body motion in an inertial frame $R$ may be written as

\[
\sum_i F_i = m^R a_G \\
\sum_i (M_G) = \frac{d}{dt} (H_G)
\]

where $^R a_G$ is the acceleration of $G$ the mass center of the body, and $H_G = I_G \cdot ^R \omega_B$ is the angular momentum of the body about its mass center. Using the "derivative rule" the right hand side of the moment equation may be rewritten as follows:

\[
\sum_i (M_G) = \frac{d}{dt} (H_G) = \frac{d}{dt} (I_G \cdot ^R \omega_B) = \frac{d}{dt} (I_G \cdot ^R \omega_B) + (^R \omega_B \times H_G)
\]

or

\[
\sum_i (M_G) = (I_G \cdot ^R \omega_B) + (^R \omega_B \times H_G)
\]

Equivalent Force Systems

The above moment equation can be extended to taking moments about any point by using the concept of equivalent force systems. Systems of forces are said to be equivalent if they have the same resultant and they have the same moment about any point. The resultant $\sum F_i$ of a force system is simply the sum of all the forces.

\[
\sum F_i = \sum_i F_i
\]

The moment of the system about some point $P$ is

\[
\sum_i (M_P) = \sum_i (p_i \times F_i)
\]
Finally, note that the **moment** of the system about another point \( Q \) may be related to the moment about \( P \) as follows:

\[
\sum_i (M_Q)_i = \sum_i (q_i \times F_i) = \sum_i \left( \vec{r}_{P/Q} \times p_i \right) \times E_i = \sum_i \left( \vec{r}_{P/Q} \times \sum_i E_i \right) + \sum_i (M_P)_i
\]

or

\[
\sum_i (M_Q)_i = \sum_i (M_P)_i + \vec{r}_{P/Q} \times \left( \sum_i E_i \right)
\]

**Alternate Moment Equation**

In the above analysis, let \( Q \) be any point \( A \), and let \( P \) be the mass center \( G \). Then using the relationship between the sum of the moments about different points, we can write

\[
\sum_i (M_A)_i = \sum_i (M_G)_i + \left( \vec{r}_{G/A} \times m^R a_G \right)
\]

or

\[
\sum_i (M_A)_i = \left( I_G \cdot \vec{R}_B \right) + \left( \vec{R}_B \times H_G \right) + \left( \vec{r}_{G/A} \times m^R a_G \right) \quad (A \text{ is any point})
\]

**Special Case: Motion about a Fixed Point**

If some point \( O \) of the body is fixed, then the above equations of motion can be shown to take the form

\[
\sum_i E_i = m^R \vec{a}_G
\]

\[
\sum_i (M_O)_i = \frac{d}{dt} (H_O) = \left( I_O \cdot \vec{R}_B \right) + \left( \vec{R}_B \times H_O \right)
\]

where \( H_O = I_O \cdot \vec{R}_B \) is the **angular momentum** of the body about the **fixed point** \( O \). Note that the elements of the inertia dyadic \( I_O \) can be determined using the parallel axes theorems for moments and products of inertia.

**Note:** If the expressions used in these equations are **valid only at some instant of time**, then the equations are **algebraic**. If the expressions are **valid for all time**, then the equations are **differential equations** and may be **integrated numerically** to simulate the motion of the system.