ME 5550 Intermediate Dynamics
Rolling Constraints – Point Contact

When a rigid body rolls (without slipping) on a rigid surface, its motion is constrained by the surface. Consider the body $B$ rolling on the rigid surface $S$ as shown at the right. Here, we have

- $S$ : rigid surface
- $B$ : rigid body
- $C_B$ : contact point on $B$
- $C_S$ : contact point on $S$
- $R$ : fixed reference frame

The body $B$ is said to roll (without slipping) on $S$ when

$$ S \frac{\dot{v}_{C_B}}{v_B} = 0 \quad \text{or} \quad R \frac{\dot{v}_{C_B}}{v_B} = R \frac{\dot{v}_{C_S}}{v_S} $$

The velocity of other points of $B$ (e.g. $P$) may be determined by using the formula for relative velocity.

$$ R \frac{\dot{v}_P}{v_P} = R \frac{\dot{v}_{C_B}}{v_B} + R \frac{\dot{v}_{P/C_B}}{v_B} $$

or

$$ R \frac{\dot{v}_P}{v_P} = R \frac{\dot{v}_{C_S}}{v_S} + (R \omega_B \times R \frac{r_{P/C_B}}{v_B}) $$

The acceleration of point $P$ is found by direct differentiation. That is,

$$ R \frac{\ddot{a}_P}{a_P} = \frac{d}{dt} \left( R \frac{\dot{v}_P}{v_P} \right) $$

**Note:** Even though the velocities of the contact points $C_B$ and $C_S$ are equal (i.e. $\frac{\dot{v}_{C_B}}{v_B} = \frac{\dot{v}_{C_S}}{v_S}$), the accelerations of these points are *not* equal (i.e. $\frac{\ddot{a}_{C_B}}{a_{C_B}} \neq \frac{\ddot{a}_{C_S}}{a_{C_S}}$).
Rolling (without Slipping) in Two Dimensions

Rolling on a Fixed Surface

If a rigid body rolls (without slipping) on a fixed surface, the point that is in contact with the surface has zero velocity. For example, consider a circular disk $D$ that rolls on the circular surface $S$ as shown below.

Because $C'$ is in contact with the point $C$ on the fixed surface, its velocity is zero. Using this result, we can calculate the velocity of $G$ the center of the disk using the relative velocity equation.

$$v_G = v_C + v_{G/C'} = v_{G/C'} = \omega_D \times r_{G/C'} = \omega_k \times (-r e_n) = r \omega e_t = v e_t \quad \text{OR} \quad v_G = r \omega e_t = v e_t$$

The acceleration of $G$ is calculated by differentiating $v_G$.

$$a_G = \frac{d}{dt} (r \omega e_t) = \dot{r} \omega e_t + r \ddot{\omega} e_t + r \omega (\dot{k} \times e_t) = r \omega e_t + r \omega \left( \frac{v}{R + r} e_n \right)$$

or

$$a_G = r \omega e_t + r \omega \left( \frac{r^2 \omega^2}{R + r} e_n \right)$$
Rolling on a Moving Surface

If a rigid body rolls (without slipping) on a moving surface, then the velocities of the two contact points, $C$ and $C'$, must be the same (i.e. $v_{C'} = v_C$). For example, consider a circular disk $D$ that rolls on the rotating circular surface $S$ as shown in the figure at the right. As before, we can calculate the velocity of the mass center $G$ using the relative velocity equation.

\[
v_{G} = v_{C'} + v_{G/C'} = \omega_S \times l_{G/O} + \omega_D \times l_{G/C'} = \Omega k \times (-r e_n) + \omega k \times (-r e_n) = (R \Omega + r \omega) e_t
\]

or

\[
v_{G} = (R \Omega + r \omega) e_t = v e_t
\]

Again, the acceleration of $G$ is found by differentiating the velocity vector.

\[
a_{G} = \frac{d}{dt} (R \Omega + r \omega) e_t = \left( \dot{R} \Omega + r \dot{\omega} \right) e_t + \left( R \Omega + r \omega \right) e_t = \left( R \dot{\Omega} + r \dot{\omega} \right) e_t + \left( R \Omega + r \omega \right) \left( \frac{v}{R + r} e_n \right)
\]

or

\[
a_{G} = \left( R \dot{\Omega} + r \dot{\omega} \right) e_t + \left( \frac{(R \Omega + r \omega)^2}{R + r} e_n \right)
\]

**Note:** Even though the velocities of the contact points $C$ and $C'$ are equal (i.e. $v_{C'} = v_C$), the accelerations of these points are not equal (i.e. $a_{C'} \neq a_C$). However, in two dimensions, it can be shown that the components of these accelerations along the tangent direction $e_t$ are equal (i.e. $a_{C'} \cdot e_t = a_C \cdot e_t$).