ME 6590 Multibody Dynamics
Homework #9 - Kane’s Equations

1. Using Kane’s equations, find the equations of motion of the two degree-of-freedom system shown. The system consists of a slender bar \( B \) of length \( \ell \) and mass \( m \) that is pinned through the center of a light shaft. The rotation of the shaft about the \( Z \)-axis is described by the angle \( \phi (\dot{\phi} = \Omega) \), and the rotation of the bar \( B \) about the \( Y' \)-axis is described by the angle \( \theta (\dot{\theta} = \omega) \). A motor torque \( M_\phi \) is applied to the shaft about the \( Z \)-axis, and a motor torque \( M_\theta \) is applied to \( B \) about the \( Y' \)-axis. Use \((u_k) = (\omega'_1, \omega'_2)\) as generalized speeds, where \( \omega'_i = R_{\omega_B} \cdot e_i \) \( (i=1,2) \).

2. Spinning Top

a) Using Kane’s equations, find the equations of motion of the three degree-of freedom spinning top shown in the diagram. Assume the moments of inertia of the top about the \( e_1 \) and \( e_2 \) directions are \( I_1 = I_2 = I \), and the moment of inertia about the \( e_3 \) direction is \( I_3 \). Also, assume point \( O \) is fixed and acts like a ball-and-socket joint. Use Euler parameters to define the orientation of the top, and define the generalized speeds to be \((u_k) = (\omega'_1, \omega'_2, \omega'_3)\) the body-fixed angular velocity components, where \( \omega'_i = R_{\omega_B} \cdot e_i \) \( (i=1,2,3) \). The unit vector set \((e_1, e_2, e_3)\) is fixed in and rotates with the top.

b) Begin to derive the equations of motion using D’Alembert’s principle with Euler parameters as the generalized coordinates. Derive the equations just far enough to explain the complexities that arise resulting from the use of Euler parameters which form a dependent set of generalized coordinates. \textbf{Note:} There is no need to complete the derivation of the equations.
3. The bracket $OABC$ shown in the diagram (shaped like a “+” sign) is attached to the ground with a ball-and-socket joint at $O$. The bars $OA$ and $BC$ are identical slender bars with mass $m$ and length $L$. The orientation of the bracket is to be described using a 1-2-3 orientation angle sequence. In the configuration shown, all angles are zero so the inertial unit vectors $(N_i, i = 1,2,3)$ are aligned with the body-fixed unit vectors $(e_i, i = 1,2,3)$. The bracket moves under the action of external forces at $A$ and $B$ and its own weight at $G$. The forces at $A$ and $B$ may be written as $F_A = F_{A1} e_1 + F_{A3} e_3$ and $F_B = F_{B} e_3$. The weight force is $W = -2mg N_2$. The configuration of the bracket is described by the generalized coordinates and speeds $(q_k) = (\theta_1,\theta_2,\theta_3)$ and $(u_k) = (\omega'_1,\omega'_2,\omega'_3)$. Here, $\theta_i (i = 1,2,3)$ represent the orientation angles, and $\omega'_i (i = 1,2,3)$ represent the body-fixed angular velocity components.

Complete the following:

a) Identify the partial velocities of the mass center $G$ \( (\partial \gamma_G / \partial u_k (k = 1,2,3)) \), the partial velocities of $A$ \( (\partial \gamma_A / \partial u_k (k = 1,2,3)) \), the partial velocities of $B$ \( (\partial \gamma_B / \partial u_k (k = 1,2,3)) \), and the partial angular velocities of the bracket \( (\partial \omega / \partial u_k (k = 1,2,3)) \).

b) Find the generalized forces $F_{u_k} (k = 1,2,3)$ associated with $(u_k) = (\omega'_1,\omega'_2,\omega'_3)$.

c) Find the three equations of motion of the bracket using Kane’s equations for the set of generalized speeds $(u_k) = (\omega'_1,\omega'_2,\omega'_3)$. Express the equations in terms of the variables $(\theta_1,\theta_2,\theta_3), (\omega'_1,\omega'_2,\omega'_3)$, and $(\omega'_1,\omega'_2,\omega'_3)$.

d) Identify the kinematical differential equations for the 1-2-3 rotation sequence that must accompany the equations from part (c) in the solution process.

**Note:** Whenever necessary, use the tables to expedite your work.